REAL TREES

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Abstract. We survey the definition and some elementary properties of real trees. There are no new results, as far as we know. One purpose is to give a number of different definitions and show the equivalence between them. We discuss also, for example, the four-point inequality, the length measure and the connection to the theory of Gromov hyperbolic spaces. Several examples are given.

1. Introduction

This is a survey of various equivalent definitions of real trees and some properties of them, mostly with proofs. We do not think that there are any new results. Most of the paper considers deterministic real trees, but we include also some brief comments on random real trees. For further results, see for example [7], [11], [16], and the references there.

1.1. **Definition.** There are several different but equivalent definitions of real trees (also called $\mathbb{R}$-trees). We collect several of them as follows. We define below conditions (T1) and (T2a)–(T2j) on a metric space (T, d); we will show that assuming (T1), the conditions (T2a)–(T2j) are all equivalent. We then make the following definition. (Which we state already here, although it is not yet justified.)

Definition 1.1. A *real tree* (or $\mathbb{R}$-*tree*) is a non-empty metric space $T = (T, d)$ that satisfies condition (T1) and one (and thus all) of (T2a)–(T2j).

Remark 1.2. Some authors assume also that the metric space T is complete. We will not do so. See further Remark 6.6 below. Note also that in many applications, T is assumed to be compact; again we do not assume this. □

Another equivalent, and related, definition is given in [11, Definition 3.15]. A characterization of a different kind of real trees is given in Theorem 6.1.

1.2. **Some notation.** Throughout, $T = (T, d)$ is a (non-empty) metric space. We often write $d_{x,y}$ for $d(x, y)$.

$B(x, r) := \{y : d(x, y) < r\}$ denotes the open ball with centre $x \in T$ and radius $r > 0$.

If $\psi_1 : [0, a] \to T$ and $\psi_2 : [0, b] \to T$ are continuous maps with $\psi_1(a) = \psi_2(0)$, their *concatenation* $\psi_1 * \psi_2 : [0, a + b] \to T$ is defined by

$$\psi_1 * \psi_2(t) := \begin{cases} \psi_1(t), & 0 \leqslant t \leqslant a, \\ \psi_2(t - a), & a \leqslant t \leqslant a + b. \end{cases} \tag{1.1}$$

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The concatenation is clearly a continuous map $[0, a+b] \to T$.

If $s, t \in \mathbb{R}$, then $s \wedge t := \min\{s,t\}$ and $s \vee t := \max\{s,t\}$. (These operations have priority over addition and subtraction.)

2. The conditions

In this section we state the conditions on a metric space $T = (T, d)$, beginning with the central (T1).

(T1) For any $x, y \in T$, there exists a unique isometric embedding $\varphi_{x,y}$ of the closed interval $[0, d_{x,y}] \subset \mathbb{R}$ into T such that $\varphi_{x,y}(0) = x$ and $\varphi_{x,y}(d_{x,y}) = y$.

Assume that (T1) holds. We then denote the image $\varphi_{x,y}([0, d_{x,y}]) \subseteq T$ by $[x, y]$; thus $[x, y]$ is a connected compact subset of T, homeomorphic with $[0, 1]$ if $x \neq y$. We similarly define $[x, y) = [x, y] \setminus \{y\}$, $(x, y] = [x, y] \setminus \{x\}$, $(x, y) = [x, y] \setminus \{x, y\}$. (If $x = y$, then $[x, x] = \{x\}$, and $[x, x) = (x, x] = (x, x) = \emptyset$.)

Obviously, $\varphi_{y,x}(t) = \varphi_{x,y}(d_{x,y} - t)$ and $[y, x] = [x, y]$.

Furthermore, still assuming (T1), let $x, y, z \in T$. Since $\varphi_{x,y}$ and $\varphi_{x,z}$ are isometries,

$$[x, y] \cap [x, z] = \bigl\{\varphi_{x,y}(t) : t \in [0, d_{x,y} \wedge d_{x,z}],\ \varphi_{x,y}(t) = \varphi_{x,z}(t)\bigr\}. \tag{2.1}$$

We define, noting that the maximum exists (i.e., the supremum is attained) by continuity,

$$\Delta(x, y, z) := \max\bigl\{t \in [0, d_{x,y} \wedge d_{x,z}] : \varphi_{x,y}(t) = \varphi_{x,z}(t)\bigr\}, \tag{2.2}$$

$$\gamma(x, y, z) := \varphi_{x,y}(\Delta(x, y, z)) = \varphi_{x,z}(\Delta(x, y, z)) \in [x, y] \cap [x, z]. \tag{2.3}$$

Further properties of these objects are given in Section 3.

We turn to the conditions (T2a)–(T2j). These are stated for a metric space $T = (T, d)$ such that (T1) holds, so we can use the notations just introduced.

(T2a) For any $x, y \in T$ and any $z \in (x, y)$, x and y are in different components of $T \setminus \{z\}$.

(T2b) For any $x, y, z \in T$, $[y, z] \subseteq [x, y] \cup [x, z]$.

(T2c) For any $x, y, z \in T$, $[x, y] \cap [x, z] \cap [y, z] \neq \emptyset$.

(T2d) For any $x, y, z \in T$, $\gamma(x, y, z) \in [y, z]$.

(T2e) For any injective continuous map $\psi : [0, 1] \to T$,

$$d\bigl(\psi(0), \psi(t)\bigr) + d\bigl(\psi(t), \psi(1)\bigr) = d\bigl(\psi(0), \psi(1)\bigr), \qquad t \in [0, 1]. \tag{2.4}$$

(T2f) For any injective continuous map $\psi : [0, 1] \to T$, $\psi([0, 1]) \subseteq [\psi(0), \psi(1)]$.

(T2g) For any injective continuous map $\psi : [0, 1] \to T$, $\psi([0, 1]) = [\psi(0), \psi(1)]$.

(T2h) Any injective continuous map $\psi : [0, 1] \to T$ equals $\varphi_{x,y}$ up to parametrization, where $x = \psi(0)$ and $y = \psi(1)$; i.e., $\psi = \varphi_{x,y} \circ h$ for some strictly increasing homeomorphism $[0, 1] \to [0, d_{x,y}]$.

(T2i) For any injective continuous map $\psi : [0, 1] \to T$, $\psi([0, 1]) \supseteq [\psi(0), \psi(1)]$.

(T2j) For any continuous map $\psi : [0, 1] \to T$, $\psi([0, 1]) \supseteq [\psi(0), \psi(1)]$.

As said in the introduction, we have the following equivalences.

Theorem 2.1. *Assume that $T = (T, d)$ is a metric space such that* (T1) *holds. Then* (T2a)–(T2j) *are all equivalent.*

The proof is given in Section 4.

Remark 2.2. Condition (T1) alone is not sufficient. Examples of spaces satisfying (T1) without being real trees are the Euclidean space $\mathbb{R}^d$, $d \geqslant 2$, and any convex subset of $\mathbb{R}^d$ of dimension $\geqslant 2$; for example the unit disc. $\square$

3. Consequences of (T1)

In this section we assume (T1) (and sometimes further conditions), and show some lemmas used in the proof of Theorem 2.1.

Lemma 3.1. *Suppose that* (T1) *holds. Then T is connected, pathwise connected, and locally pathwise connected. Hence, if $V \subset T$ is an open subset of T, then V is a union of open (pathwise) connected components.*

Proof. T is obviously pathwise connected by (T1). Thus, T is connected.

Furthermore, T is locally pathwise connected, since every open ball $B(x,r)$ is pathwise connected. (Every $y \in B(x,r)$ is connected to x by the path $[x,y] \subseteq B(x,r)$.) $\square$

In particular, if $z \in T$, then the components of $T \setminus \{z\}$ are open and pathwise connected. These (path) components are called the *branches* at z; see also Section 8.

Lemma 3.2. *Suppose that* (T1) *holds. If $x, y \in T$ and $z, w \in [x,y]$, then $[z,w] \subseteq [x,y]$, and, furthermore,*

$$\varphi_{z,w}(t) := \varphi_{x,y}(d_{x,z}+t), \qquad 0 \leqslant t \leqslant d_{z,w}. \tag{3.1}$$

Proof. By symmetry, we may assume $d_{x,z} \leqslant d_{x,w}$. Since $\varphi_{x,y}$ is an isometry with $\varphi_{x,y}(0) = x$, we have $z = \varphi_{x,y}(d_{x,z})$ and $w = \varphi_{x,y}(d_{x,w})$; furthermore, $d(z,w) = |d_{x,z} - d_{x,w}| = d_{x,w} - d_{x,z}$. Let

$$\varphi(t) := \varphi_{x,y}(d_{x,z}+t), \qquad 0 \leqslant t \leqslant d_{x,w} - d_{x,z} = d_{z,w}. \tag{3.2}$$

Then φ is an isometry and it follows that $\varphi = \varphi_{z,w}$. The result follows. $\square$

Lemma 3.3. *Suppose that* (T1) *holds. Then, for any $x, y \in T$,*

$$[x,y] = \{z : d(x,z) + d(z,y) = d(x,y)\}. \tag{3.3}$$

Proof. If $z \in [x,y]$, then by definition $z = \varphi_{x,y}(s)$ for some $s \in [0, d_{x,y}]$. Since $\varphi_{x,y}$ is an isometry, we have $d(x,z) = d\big(\varphi_{x,y}(0), \varphi_{x,y}(s)\big) = s$ and $d(z,y) = d\big(\varphi_{x,y}(s), \varphi_{x,y}(d_{x,y})\big) = d(x,y) - s$. Hence,

$$d(x,z) + d(z,y) = s + \big(d(x,y) - s\big) = d(x,y). \tag{3.4}$$

Conversely, suppose that $z \in T$ with $d(x,z) + d(z,y) = d(x,y)$. Define $\varphi : [0, d_{x,y}] \to T$ as the concatenation $\varphi := \varphi_{x,z} * \varphi_{z,y}$, see (1.1). Then, φ is a continuous map $[0, d_{x,y}] \to T$ with $\varphi(0) = x$ and $\varphi(d_{x,y}) = y$. Furthermore, φ is an isometry on $[0, d_{x,z}]$ and on $[d_{x,z}, d_{x,y}]$. It follows that if $s \in [0, d_{x,z}]$ and $t \in [d_{x,z}, d_{x,y}]$, then, by the triangle inequality,

$$\begin{aligned} d\big(\varphi(s), \varphi(t)\big) &\leqslant d\big(\varphi(s), \varphi(d_{x,z})\big) + d\big(\varphi(d_{x,z}), \varphi(t)\big) \\ &= \big(d_{x,z} - s\big) + \big(t - d_{x,z}\big) = t - s. \end{aligned} \tag{3.5}$$

On the other hand, if we have strict inequality $d\bigl(\varphi(s),\varphi(t)\bigr) < t-s$ for some s,t with $0 \leqslant s \leqslant t \leqslant d_{x,y}$, then, similarly,

$$\begin{aligned} d(x,y) &\leqslant d\bigl(x,\varphi(s)\bigr) + d\bigl(\varphi(s),\varphi(t)\bigr) + d\bigl(\varphi(t),y\bigr) \\ &= s + d\bigl(\varphi(s),\varphi(t)\bigr) + d_{x,y} - t \\ &< s + (t-s) + (d_{x,y} - t) = d_{x,y} = d(x,y), \end{aligned} \tag{3.6}$$

a contradiction.

Consequently, φ is an isometry, and thus $\varphi = \varphi_{x,y}$, by the uniquness assumption in (T1). Hence, $z = \varphi(d_{x,z}) = \varphi_{x,y}(d_{x,z}) \in [x,y]$. ☐

Lemma 3.4. *Suppose that* (T1) *holds. If* $x,y \in T$ *and* $z,w \in [x,y]$ *with* $z \in [x,w]$*, then* $w \in [z,y]$.

Proof. Since $z \in [x,w]$, we have $d_{x,z} \leqslant d_{x,w}$. Hence, by Lemma 3.3,

$$d_{y,w} = d_{x,y} - d_{x,w} \leqslant d_{x,y} - d_{x,z} = d_{y,z}. \tag{3.7}$$

Hence, by Lemma 3.2,

$$w = \varphi_{y,x}(d_{y,w}) = \varphi_{y,z}(d_{y,w}) \in [y,z]. \tag{3.8}$$

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Lemma 3.5. *Suppose that* (T1) *holds. Then, for any* $x,y,z \in T$,

$$\bigl\{t \in [0, d_{x,y} \wedge d_{x,z}] : \varphi_{x,y}(t) = \varphi_{x,z}(t)\bigr\} = [0, \Delta(x,y,z)] \tag{3.9}$$

and

$$[x,y] \cap [x,z] = [x, \gamma(x,y,z)] = \bigl\{\varphi_{x,y}(t) : t \in [0, \Delta(x,y,z)]\bigr\}. \tag{3.10}$$

Proof. Let

$$J := \bigl\{t \in [0, d_{x,y} \wedge d_{x,z}] : \varphi_{x,y}(t) = \varphi_{x,z}(t)\bigr\}. \tag{3.11}$$

Thus, the definition (2.2) says $\Delta(x,y,z) = \max J$. If $t \in J$, let $w := \varphi_{x,y}(t) \in [x,y] \cap [x,z]$. Then $[x,w] \subseteq [x,y] \cap [x,z]$ by Lemma 3.2. Hence, if $0 \leqslant s \leqslant t$, then

$$\varphi_{x,y}(s) = \varphi_{x,w}(s) = \varphi_{x,z}(s) \tag{3.12}$$

and consequently, $s \in J$. This shows that J is an interval, and (3.9) follows.

Finally, (3.10) follows from (3.9), using (2.1), (2.3) and (3.12). ☐

Lemma 3.6. *Suppose that* (T1) *and* (T2d) *hold. Then, for any* $x,y,z \in T$,

$$[x,y] \cap [x,z] \cap [y,z] = \{\gamma(x,y,z)\} \tag{3.13}$$

and

$$\Delta(x,y,z) = \tfrac{1}{2}\bigl(d(x,y) + d(x,z) - d(y,z)\bigr). \tag{3.14}$$

In particular, Δ *is a continuous function* $T^3 \to \mathbb{R}$*, and* $\gamma(x,y,z)$ *is a symmetric function of* x,y,z.

Proof. By the definition (2.3) and the assumption (T2d),

$$\gamma(x,y,z) \in [x,y] \cap [x,z] \cap [y,z]. \tag{3.15}$$

Let $w \in [x,y] \cap [x,z] \cap [y,z]$. Then, by Lemma 3.3,

$$d_{x,w} + d_{y,w} = d_{x,y}, \tag{3.16}$$

$$d_{x,w} + d_{z,w} = d_{x,z}, \tag{3.17}$$

$$d_{y,w} + d_{z,w} = d_{y,z}, \tag{3.18}$$

and consequently

$$2d_{x,w} = d_{x,y} + d_{x,z} - d_{y,z}. \tag{3.19}$$

Hence, $d_{x,w}$ is uniquely determined by x, y, z, and thus so is $w = \varphi_{x,y}(d_{x,w})$. Consequently, (3.15) implies (3.13). Furthermore, (2.3) implies $\Delta(x, y, z) = d(x, \gamma(x, y, z))$ and thus (3.14) follows from (3.19). □

Lemma 3.7. *Suppose that* (T1) *and* (T2d) *hold, and let* $x, y, z \in T$.

(i) *Then*

$$z \in [x, y] \iff \Delta(z, x, y) = 0. \tag{3.20}$$

(ii) *If* $z \notin [x, y]$ *and* $d(w, z) < \Delta(z, x, y)$, *then*

$$\Delta(x, y, w) = \Delta(x, y, z). \tag{3.21}$$

Proof. (i): Immediate by Lemma 3.3 and (3.14).

(ii): Let, using the symmetry of γ in Lemma 3.5, $u := \gamma(x, y, z) = \gamma(z, x, y)$ and $v := \gamma(x, z, w)$. Then $v \in [z, w]$ and thus, using the assumption,

$$d(z, v) \leqslant d(z, w) < \Delta(z, x, y) = d(z, u). \tag{3.22}$$

Since $u, v \in [x, z]$, this implies

$$d(x, u) = d(x, z) - d(u, z) < d(x, z) - d(v, z) = d(x, v). \tag{3.23}$$

Hence, if $d(x, u) < t < d(x, v)$, then, by (2.1),

$$\varphi_{x,w}(t) = \varphi_{x,z}(t) \neq \varphi_{x,y}(t) \tag{3.24}$$

On the other hand, $u \in [x, y]$ too, and thus

$$\varphi_{x,w}(d_{x,u}) = \varphi_{x,z}(d_{x,u}) = u = \varphi_{x,y}(d_{u,}). \tag{3.25}$$

Hence, $\Delta(x, y, w) = d_{x,u} = \Delta(x, y, z)$ by Lemma 3.5. □

4. Proof of Theorem 2.1

(T2a) $\Longrightarrow$ (T2j): If (T2a) holds and $\psi : [0, 1] \to T$ is a continuous map, let $x := \psi(0)$ and $y := \psi(1)$. If $z \in [x, y]$ but $z \notin \psi([0, 1])$, then ψ is a curve connecting x and y in $T \setminus \{z\}$, which contradicts (T2a).

(T2j) $\Longrightarrow$ (T2b): Define $\psi := \varphi_{y,x} * \varphi_{x,z}$, see (1.1). This is a continuous map $[0, d_{y,x} + d_{x,z}] \to T$ and thus $\psi_1(t) := \psi\bigl(t(d_{y,x} + d_{x,z})\bigr)$ is a continuous map $\psi_1 : [0, 1] \to T$. We have $\psi_1([0, 1]) = [y, x] \cup [x, z]$, $\psi_1(0) = y$ and $\psi_1(1) = z$. Hence (T2j) yields (T2b).

(T2b) $\Longrightarrow$ (T2c): Let $A := [y, z] \cap [x, y]$ and $B := [y, z] \cap [x, z]$. These are closed subsets of $[y, z]$ and $A \cup B = [y, z]$ by assumption; furthermore, A and B are both nonempty since $y \in A$ and $z \in B$. Since $[y, z]$ is homeomorphic to $[0, 1]$, it is connected. Consequently, $A \cap B \neq \emptyset$.

(T2c) $\Longrightarrow$ (T2d): By (2.3), it remains to show that $\gamma(x, y, z) \in [y, z]$.

Let $w \in [x,y]\cap[x,z]\cap[y,z]$. By Lemma 3.5, then $w \in [x,\gamma(x,y,z)]$. Since $w,\gamma(x,y,z) \in [x,y]$, Lemma 3.4 shows that $\gamma(x,y,z) \in [w,y]$. Hence, using also $w \in [y,z]$ and Lemma 3.2,

$$\gamma(x,y,z) \in [w,y] = [y,w] \subseteq [y,z]. \tag{4.1}$$

(T2d) $\Longrightarrow$ (T2a): Let $z \in (x,y)$. Then $z = \varphi_{x,y}(d_{x,z})$ with $0 < d_{x,z} < d_{x,y}$. Partition $T \setminus \{z\} = U_1 \cup U_2 \cup U_3$ where

$$U_1 := \{w \in T \setminus \{z\} : \Delta(x,y,w) < d_{x,z}\}, \tag{4.2}$$

$$U_2 := \{w \in T \setminus \{z\} : \Delta(x,y,w) = d_{x,z}\}, \tag{4.3}$$

$$U_3 := \{w \in T \setminus \{z\} : \Delta(x,y,w) > d_{x,z}\}. \tag{4.4}$$

By Lemma 3.6, $\Delta(x,y,w)$ is a continuous function of w, and thus U_1 and U_3 are open subsets of $T \setminus \{z\}$ (and of T). Furthermore, U_2 is open by Lemma 3.7.

Hence, $T \setminus \{z\} = U_1 \cup U_2 \cup U_3$ is a partition into three disjoint open sets. Each connected component of $T \setminus \{z\}$ has to be a subset of one of these, and since $x \in U_1$ and $y \in U_3$, x and y are in different components. (Although not needed, it is easy to see that U_1 and U_3 are connected, while U_2 may be empty, connected or disconnected with any number of components, finite or infinite.)

(T2j) $\Longrightarrow$ (T2i): Trivial.

(T2i) $\Longrightarrow$ (T2f): Suppose that (T2f) fails, and let $\varphi : [0,1] \to T$ be an injective continuous map such that $\varphi(u) \notin [x,y]$ for some $u \in (0,1)$, where $x = \varphi(0)$, $y = \varphi(1)$. Since $[x,y]$ is a compact, and thus closed, subset of T, the set $U := \{t \in (0,1) : \varphi(t) \notin [x,y]\}$ is an open subset of $(0,1)$. Hence, the component of U containing u is an open interval $(a,b) \subseteq (0,1)$.

Let $z := \varphi(a)$ and $w := \varphi(b)$, and note that $z,w \in [x,y]$. Since $a < b$ and φ is injective, we have $z \neq w$. By Lemma 3.2, $[z,w] \subseteq [x,y]$. The map $\psi(t) := \varphi(a + (b-a)t)$ is an injective continuous map $[0,1] \to T$ such that $\psi(0) = z$, $\psi(1) = w$ and

$$\psi([0,1]) \cap [z,w] = \varphi([a,b]) \cap [z,w] \subseteq \varphi([a,b]) \cap [x,y] = \varphi(\{a,b\}) = \{z,w\}. \tag{4.5}$$

Hence, if $v \in (z,w)$, then $v \notin \psi([0,1])$. Thus, (T2i) does not hold.

(T2f) $\Longrightarrow$ (T2d): Let $x,y,z \in T$, and let $w := \gamma(x,y,z)$. By Lemmas 3.2 and 3.5,

$$[y,w] \cap [w,z] = [w,y] \cap [w,z] \subseteq [x,y] \cap [x,z] = [x,w] \tag{4.6}$$

and thus, since $w \in [x,y]$,

$$[y,w] \cap [w,z] \subseteq [x,w] \cap [w,y] = \{w\}. \tag{4.7}$$

Define $\psi : [0, d_{y,w} + d_{w,z}] \to T$ as the concatenation $\psi := \varphi_{y,w} * \varphi_{w,z}$, see (1.1). Then ψ is continuous, and it follows from (4.7) that ψ is injective. Consequently, (T2f) implies, after a change of variables,

$$w = \psi(d_{y,w}) \in [\psi(0), \psi(d_{y,w} + d_{w,z})] = [y,z]. \tag{4.8}$$

(T2f) $\Longleftrightarrow$ (T2e): An immediate consequence of Lemma 3.3.

(T2f) $\Longrightarrow$ (T2h): Suppose that $\psi : [0,1] \to T$ is an injective continuous map. Let $x := \psi(0)$ and $y := \psi(1)$. By (T2f), $\psi : [0,1] \to [x,y]$. Since $\varphi_{x,y}$ is a homeomorphism, $h := \varphi_{x,y}^{-1} \circ \psi : [0,1] \to [0, d_{x,y}]$ is an injective continuous map with $h(0) = 0$ and $h(1) = d_{x,y}$. In particular, the image $h([0,1])$ is connected, and it follows that $h([0,1]) = [0, d_{x,y}]$. Hence h is a continuous bijection, and thus a homeomorphism $[0,1] \to [0, d_{x,y}]$; furthermore, h has to be stricly increasing. Finally, the definition of h yields $\psi = \varphi_{x,y} \circ h$.

(T2h) $\Longrightarrow$ (T2g): Trivial.

(T2g) $\Longrightarrow$ (T2f): Trivial.

This completes the proof of Theorem 2.1. □

5. Subtrees

We have, as a simple consequence of the definition and Theorem 2.1 a simple result for subsets of a real tree.

Theorem 5.1. *Let T be a real tree, and let $S \subseteq T$ be a nonempty subset of T, regarded as a metric space with the induced metric. Then the following are equivalent.*

(i) *S is a real tree.*
(ii) *S is connected.*
(iii) *S is pathwise connected.*
(iv) *If $x, y \in S$, then $[x,y] \subseteq S$. (Here $[x,y]$ is taken in the real tree T.)*

Proof. (i) $\Longrightarrow$ (iii), (iv) $\Longrightarrow$ (iii), and (iii) $\Longrightarrow$ (ii) are trivial.

(iii) $\Longrightarrow$ (i): Suppose that S is pathwise connected. Then, if $x, y \in S$, then there exists a continuous map $\psi : [0,1] \to S$ with $\psi(0) = x$ and $\psi(1) = y$. By (T2j) (for the real tree T), $\psi([0,1]) \supseteq [x,y] := \varphi_{x,y}\big([0, d_{x,y}]\big)$, where $\varphi_{x,y}$ is the mapping in (T1) for the real tree T. Hence, $\varphi_{x,y} : [0, d_{x,y}] \to S$, and thus (T1) holds for S too; uniqueness follows because $\varphi_{x,y}$ obviously is unique in S if it is unique in T. Finally, (T2e) holds in S since it holds in T. (In fact, we could here argue with any of (T2a)–(T2j).) Hence, S is a real tree.

(ii) $\Longrightarrow$ (iv): Suppose that S is connected. Let $x, y \in S$, and consider $[x,y]$ (in the real tree T). Let $z \in (x,y)$, and suppose that $z \notin S$. By (T2a) and Lemma 3.1, the components of $T \setminus z$ are disjoint open sets, with x and y in different components. Let U be the component containing x, and V the union of all other components; then $T \setminus z = U \cup V$ where U and V are open disjoint subsets with $x \in U$ and $y \in V$. Consequently, $S = (S \cap U) \cup (S \cap V)$, where $S \cap U$ and $S \cap V$ are two nonempty disjoint open subsets of S. This contradicts the assumption that S is connected, and this contradiction shows that $(x,y) \subseteq S$. Hence, $[x,y] \subseteq S$. □

A *subtree* of a real tree is thus a connected nonempty subset.

Theorem 5.2. *The intersection of any family $\{T_\alpha\}$ of subtrees of a real tree T is a subtree of T, provided is it nonempty.*

Proof. This is an immediate consequence of Theorem 5.1. Let $S := \bigcap_\alpha T_\alpha$. If $x, y \in S$, then Theorem 5.1(iv) shows that $[x,y] \subseteq T_\alpha$ for every α, and thus $[x,y] \subseteq S$. Hence, another application of Theorem 5.1 shows that S is a subtree. □

In particular, it follows that if T is a real tree, then for any nonempty set $U \subseteq T$, there exists a smallest subtree $S \subseteq T$ with $U \subseteq S$; we say that S is the subtree *spanned by* U. This subtree can be described as follows.

Theorem 5.3. *Let T be a real tree and let S be the subtree generated by a nonempty set $U \subseteq T$. Then*

$$S = \bigcup_{x,y \in U} [x, y]. \tag{5.1}$$

Furthermore, for every $x \in U$, we also have

$$S = \bigcup_{y \in U} [x, y]. \tag{5.2}$$

Proof. Denote the unions in (5.1) and (5.2) by S' and S''_x, respectively. Then Theorem 5.1(iv) shows that $S''_x \subseteq S' \subseteq S$.

On the other hand, S''_x is pathwise connected, since every interval $[x, y]$ is and they contain a common point x. Thus Theorem 5.1 shows that S''_x is a subtree. Since $S''_x \supseteq U$, it follows that $S''_x \supseteq S$, and the result follows. □

6. The four-point inequality

A different type of characterization of real trees is given by the following theorem, see e.g. [11, Theorem 3.40] or [7] and the references there. (This characterization is less intuitive, but technically very useful.) The condition (6.1) is called the *four-point inequality* or *four-point condition*; an equivalent condition is 0-*hyperbolicity*, see Definition A.1 and Lemma 6.7.

Theorem 6.1. *A metric space T is a real tree if and only if T is connected and for any four points $x, y, z, w \in X$*

$$d(x, y) + d(z, w) \leqslant \bigl(d(x, z) + d(y, w)\bigr) \vee \bigl(d(x, w) + d(y, z)\bigr). \tag{6.1}$$

Remark 6.2. It is easily verified that (6.1) is trivial if two or more of x, y, z, w coincide; hence it does not matter whether we require x, y, z, w to be distinct or not. □

Remark 6.3. By considering all permutations of x, y, z, w, it follows that (6.1) is equivalent to the condition that (for any x, y, z, w), among the three sums

$$d(x, y) + d(z, w), \qquad d(x, z) + d(y, w), \qquad d(x, w) + d(y, z), \tag{6.2}$$

two are equal and the third is equal or less than the other two. □

Theorem 6.1 is a simple corollary of the following more general result together with Theorem 5.1. For proofs see e.g. Dress [7, Theorem 8], Gromov [14, §6.1], or the references mentioned in [7]; see also [8].

Theorem 6.4. *Let X be a metric space. Then X can be isometrically embedded into a real tree if and only if the four-point inequality* (6.1) *holds for any four points $x, y, z, w \in X$.* □

Proof of Theorem 6.1 from Theorem 6.4. Suppose that T is connected and that (6.1) holds. Then, by Theorem 6.4, $T \subseteq \widehat{T}$ for some real tree $\widehat{T}$. Since T is connected, T is a real tree by Theorem 5.1. □

Among the consequences we mention the following.

Theorem 6.5. *If T is a real tree, then so is its completion $\tilde{T}$.*

Proof. By continuity, if the four-point inequality (6.1) holds in the dense subset T of $\tilde{T}$, then it holds in $\tilde{T}$. Furthermore, since T is connected, so is $\tilde{T}$. hence, $\tilde{T}$ is a real tree by Theorem 6.1. ☐

Remark 6.6. By Theorem 6.5, we may in many situations assume without loss of generality that real trees are complete, since we can replace an arbitrary real tree by its completion. ☐

The four-point inequality (6.1) can be rewritten in several ways. Define, for three points x, y, z in a general metric space (X, d), the *Gromov product*

$$(x, y)_z := \tfrac{1}{2}\big(d(z, x) + d(z, y) - d(x, y)\big). \tag{6.3}$$

Note that $(x, y)_z \geqslant 0$ by the triangle inequality, and that (6.3) meaures how far the triangle inequality is from being an equality. Note also that in a real tree, Lemma 3.6 shows that $(x, y)_z = \Delta(z, x, y)$, which equals the distance from z to $[x, y]$.

Lemma 6.7. *The four-point inequality* (6.1) *is equivalent to*

$$(x, y)_w \geqslant (x, z)_w \wedge (y, z)_w. \tag{6.4}$$

Proof. By the definition (6.3), the inequality (6.4) holds if and only if at least one of the following holds:

$$d_{x,w} + d_{y,w} - d_{x,y} \geqslant d_{x,w} + d_{z,w} - d_{x,z}, \tag{6.5}$$

$$d_{x,w} + d_{y,w} - d_{x,y} \geqslant d_{y,w} + d_{z,w} - d_{y,z}. \tag{6.6}$$

These are equivalent to, respectively,

$$d_{y,w} + d_{x,z} \geqslant d_{z,w} + d_{x,y}, \tag{6.7}$$

$$d_{x,w} + d_{y,z} \geqslant d_{z,w} + d_{x,y}, \tag{6.8}$$

and thus at least one of them holds if and only iff (6.1) holds. ☐

We note also that, in fact, it suffices to verify the four-point inequality for a fixed choice of one of the four points.

Lemma 6.8. *Let T be a metric space and let $o \in T$ be fixed. If the four-point inequality* (6.1) *holds for $w = o$ and all $x, y, z \in T$, then it holds in general, i.e., for all $x, y, z, w \in T$.*

Proof. By Lemma 6.7, this is the special case $\delta = 0$ of Lemma A.5. ☐

7. Rooted real trees

Definition 7.1. A *rooted real tree* (T, ρ) is a real tree T with a distinguished point $\rho \in T$, called the *root*.

In a rooted real tree (T, ρ), we may define a partial order by

$$y \leqslant x \iff y \in [\rho, x], \qquad x, y \in T. \tag{7.1}$$

Theorem 7.2. *Let (T, ρ) be a rooted real tree. Then* (7.1) *defines a partial order in T, with ρ as the minimum element. Moreover, any two points $x, y \in T$ have a greatest common lower bound, which we denote by $x \wedge y$. Recalling the notation of* (2.3) *and Lemma 3.6, we have*

$$x \wedge y = \gamma(\rho, x, y). \tag{7.2}$$

Proof. It is easily seen, using Lemmas 3.2 and 3.4, that (7.1) defines a partial order. It is obvious from (7.1) that $\rho \leqslant x$ for every $x \in T$.

For any $x, y \in T$, by the definition (7.1) and (3.10),

$$\begin{aligned}\{z : z \leqslant x \text{ and } z \leqslant y\} &= \{z : z \leqslant x\} \cap \{z : z \leqslant y\} = [\rho, x] \cap [\rho, y] \\ &= [\rho, \gamma(\rho, x, y)] = \{z : z \leqslant \gamma(\rho, x, y)\},\end{aligned} \tag{7.3}$$

which shows that $\gamma(\rho, x, y)$ is a greatest lower bound $x \wedge y$. □

For any $x, y \in T$, the path $[x, y]$ from x to y is a combination of the paths $[x, x \wedge y]$ and $[x \wedge y, y]$ (where one or both parts may reduce to a single point). Hence, we have

$$[x, y] = [x, x \wedge y] \cup [x \wedge y, y]. \tag{7.4}$$

$$d(x, y) = d(x, x \wedge y) + d(x \wedge y, y). \tag{7.5}$$

We note also that for any subset $\{x_\alpha\}_{\alpha \in \mathcal{A}} \subseteq T$, it follows from Theorem 5.3 that the subtree spanned by these points and the root ρ is $\bigcup_\alpha [\rho, x_\alpha]$; see further Examples 8.5 and 10.3.

8. Leaves and branch points

Recall from Lemma 3.1 that the components of $T \setminus \{z\}$ are also the path components of $T \setminus \{z\}$, and that these are open and are called the *branches* of T at z.

Definition 8.1. Let T be a real tree. The *degree* $\delta(z) = \delta_T(z)$ of a point $z \in T$ is the number of branches at z, i.e., the number of components of $T \setminus \{z\}$. Thus $1 \leqslant \delta(z) \leqslant \infty$ unless T consists of a single point.

Definition 8.2. Let T be a real tree. We say that a point $z \in T$ is a *leaf* if $\delta(z) = 1$, and a *branch point* if $\delta(z) \geqslant 3$.

We denote the set of leaves by $T^{\mathsf{L}} := \{z : \delta(z) \leqslant 1\}$. The *skeleton* of T is the set $T^o := T \setminus T^{\mathsf{L}} = \{z \in T : \delta(z) \geqslant 2\}$, i.e., the set of all non-leaves of T.

We ignore here the trivial case when T consists of a single point. (In this case, the point is defined to be a leaf, by a modification of the definition above, and $T^o = \emptyset$.)

We note that

$$T^o = \bigcup_{x, y \in T} (x, y). \tag{8.1}$$

Remark 8.3. In a rooted real tree, the root is often not regarded as a leaf, even if its degree is 1. □

We note also that the branches at a point can be characterized as follows.

Lemma 8.4. *Let T be a real tree, and let $z \in T$. Then the following are equivalent, for any $x, y \in T \setminus \{z\}$:*

(i) *x and y belong to different branches of T at z.*
(ii) $z \in [x, y]$.
(iii) $[z, x] \cap [z, y] = \{z\}$.

Proof. (i) $\iff$ (ii): x and y belong to the same path component of $T \setminus \{z\}$ if and only if $[x, y] \subseteq T \setminus \{z\}$, i.e., if and only if $z \notin [x, y]$. (Cf. the condition (T2a).)

(ii) $\iff$ (iii): Immediate from Lemmas 3.5 and 3.7. □

Example 8.5. Let (T, ρ) be a rooted real tree and let $\{x_1, \dots, x_n\}$ be a finite set of points in T. By Theorem 5.3, the subtree T_1 spanned by $\{x_1, \dots, x_n\}$ and the root ρ is $\bigcup_{i=1}^{n}[\rho, x_i]$. It is easily seen that the leaves of T_1 are ρ (but see Remark 8.3) and the set of maximal elements of $\{x_1, \dots, x_n\}$, i.e., $\{x_i : x_i \notin [\rho, x_j] \text{ for every } j \neq i\}$. Furthermore the branch points of T_1 are a subset of $\{x_i \wedge x_j : i \neq j\}$. The sets of leaves and branch points are thus finite. □

9. A metric space of compact real trees

Consider the set $\mathbb{T}$ of all compact real trees, or rather the set of all equivalence classes under isometry of compact real trees (so that two isometric real trees are regarded as the same). (The set theoretic difficulties with "all compact real trees" are handled in the standard way: since a compact real tree, as any compact metric space, has cardinality at most c, it suffices to consider real trees that as sets are subsets of, for example, $\mathbb{R}$.)

The set $\mathbb{T}$ can be equipped with a metric, the *Gromov–Hausdorff distance*, which makes $\mathbb{T}$ a complete separable metric space. Similarly, the set $\mathbb{T}_1$ of rooted compact real trees is a complete separable metric space, equipped with (a rooted version of) the Gromov–Hausdorff distance. See [12] for definitions and proofs; see also [6, Section 7.3] for the Gromov–Hausdorff distance for general metric spaces.

The fact that $\mathbb{T}$ and $\mathbb{T}_1$ thus are complete separable metric spaces (and thus Polish topological spaces) makes it possible to define random compact real trees as random elements of one of these spaces, and a lot of standard machinery then is available.

For noncompact real trees, one can similarly use the version of Gromov–Hausdorff convergence in [6, Section 8.1].

10. Some examples

Example 10.1. A *combinatorial tree* is a non-empty set V of vertices (finite or infinite) together with a set E of unorded pairs $\{v, w\}$ of vertices, such that (V, E) is a tree in the usual combinatorial sense. (An edge $\{vw\}$ is often denoted vw for simplicity.)

We may regard a combinatorial tree as a real tree T, by regarding each edge as a copy of $[0, 1]$, with the endpoints identified with the corresponding vertices in V. Equivalently, we may define T as the disjoint union of V and one copy of $(0, 1)$ for each edge in E, with a suitably defined metric. (We omit the details, and the verification that T is a real tree.) In any case, we regard V as a subset of T.

Note that for $v, w \in V$, the distance $d(v, w)$ equals the usual distance in a graph, i.e., the number of edges in a shortest path from v to w.

The degree $\delta_T(z)$ of a vertex $z \in V$ equals the degree of z in the graph (V, E); the degree of any vertex in $T \setminus V$ is 2. In particular, the leaves of T are precisely the leaves of the tree (V, E) (i.e., the vertices in V adjacent to a single edge in E), and the branch points are the vertices in V that have degree $\geqslant 3$.

It is easy to see that T always is complete, that T is separable if and only if V (and thus also E) is countable, and that T is compact if and only if V (and thus also E) is finite. □

Example 10.2. More generally, suppose as in Example 10.1 that (V, E) is a combinatorial tree, and assume also that for every edge $e \in E$, we are given a real number ℓ_e, called the *length* e. We may construct a real tree T as in Example 10.1, but now for each edge e taking an interval of length ℓ_e. (In particular, $\ell_e = 1$ for all e gives back the real tree in Example 10.1.)

We see again that T is separable if and only if V is countable. (In one direction, note that if D is a countable dense subset of T, then every edge contains, in its interior, an element of D; hence E is countable.)

Moreover, T is compact if V is finite, but the converse does not hold. One counterexample is an infinite star which is compact for some (but not all) choices of edge lengths: let $V = \{0, 1, \dots\}$ and $E = \{0i : i \geqslant 1\}$, with length $\ell_{0i} = 2^{-i}$. □

Example 10.3. As in Example 8.5, let T_1 be the subtree of a rooted real tree that is spanned by a finite set of points $\{x_1, \dots, x_n\}$ and the root. It follows from Example 8.5 and (7.4)–(7.5) that the real tree T_1 can be constructed as in Example 10.2 from a finite combinatorial tree (V, E) where $V = \{x_i\} \cup \{x_i \wedge x_j : 1 \leqslant i < j \leqslant n\} \cup \{\rho\}$, and a suitable set of edges E with suitable lengths ℓ_e; we omit the details. □

Example 10.4. The *infinite binary tree* is a combinatorial tree with $V := \bigcup_{n=0}^{\infty} \{0, 1\}^n$, the set of all finite strings from $\{0, 1\}$ (including the empty string $\emptyset$); the edges are all pairs of the type $\{v, v0\}$ and $\{v, v1\}$ for strings $v \in V$. Let $(\ell_n)_1^\infty$ be a sequence of positive real numbers, and let T be the real tree constructed in Example 10.2 with edge lengths defined by $\ell_{\{v,vj\}} := \ell_{|v|+1}$, where $|v|$ is the length of the string v.

No vertex in V is a leaf, and thus, see Example 10.1, the real tree T has no leaf, so $T^{\mathsf{L}} = \emptyset$ and $T^o = T$.

The real tree T is always separable, and never compact. It is easy to see that T is complete if $\sum_n \ell_n = \infty$, but not if $\sum_n \ell_n < \infty$, since in the latter case, the sequence $(0^n)_0^\infty = \emptyset, 0, 00, 000, \dots$ is a Cauchy sequence without a limit. See further the next example. □

Example 10.5. Let T_0 be the infinite binary tree in Example 10.4 and assume that $L := \sum_n \ell_n < \infty$. Note that $d(\emptyset, z) < L$ for every $z \in T_0$. For every $s < L$, the set $\{z \in T_0 : d(\emptyset, z) \leqslant s\}$ is closed and contained in a finite number of edges, and thus it is compact.

Let $(z_n)_1^\infty$ be a Cauchy sequence in T_0. Then the sequence $d(\emptyset, z_n)$ is a Cauchy sequence, so it converges to some limit $d_\infty \leqslant L$. If the limit $d_\infty < L$, then we see that the Cauchy sequence (z_n) belongs to a compact subset of T_0, and thus it converges. On the other hand, if $d(\emptyset, z_n) \to L$, then the Cauchy sequence cannot converge, since a limit z would have to satisfy $d(\emptyset, z) = \lim_{n\to\infty} d(\emptyset, z_n) = L$, but no such z exists in T_0.

Consider now the completion T of T_0; T is a real tree by Theorem 6.5, and we call T a *complete infinite binary tree.* We claim that $T \setminus T_0$ may be identified with the set $\{0,1\}^\infty$ of infinite strings from $\{0,1\}$. In fact, if $v = \xi_1\xi_2\cdots \in \{0,1\}^\infty$, then let $v_n := \xi_1\cdots\xi_n \in V$ for each $n \geqslant 0$; we have $d(v_n, v_m) = \sum_{n<i\leqslant m} \ell_i$ when $n \leqslant m$, and thus (v_n) is a Cauchy sequence in T_0 so it has a limit in T which we represent by v. We have $d(\emptyset, v) = L$, and thus $v \in T \setminus T_0$.

Furthermore, if $v = \xi_1\xi_2\cdots$ and $v' = \xi'_1\xi'_2\cdots$ are elements of $\{0,1\}^\infty$, let $D(v, v') := \inf\{i : \xi_i \neq \xi'_i\}$; thus $1 \leqslant D(v, v') < \infty$ if $v \neq v'$, but $D(v, v) = \infty$. Further, let $L^+_n := \sum_{i\geqslant n} \ell_i$; thus $L^+_1 = L$, and $L^+_n \searrow 0$ as $n \to \infty$. It is then easy to see that for any $v, v' \in \{0,1\}^\infty$, regarded as elements of T, we have

$$d(v, v') = 2L^+_{D(v,v')}. \tag{10.1}$$

In particular, this shows that two different strings in $\{0,1\}^\infty$ represent different points in T, so we may regard $\{0,1\}^\infty$ as a subset of T. Note also that, since $L^+_n \to 0$ as $n \to \infty$, the metric (10.1) induces the product topology on $\{0,1\}^\infty$; thus $\{0,1\}^\infty$ is a compact subset of T, homeomorphic to the Cantor set.

Finally, if (z_n) is any Cauchy sequence in T_0 without limit in T_0, we have seen that $d(\emptyset, z_n) \to L$, and since $\ell_k \to 0$ as $k \to \infty$, it follows easily that we may approximate each z_n by $z'_n \in V$ such that $d(z_n, z'_n) \to 0$ as $n \to \infty$. Then z'_n is a finite string; we extend it (arbitrarily) to an infinite string $z''_n \in \{0,1\}^\infty$ and note that

$$d(z'_n, z''_n) = L^+_{|z'_n|+1} \to 0. \tag{10.2}$$

Hence $d(z_n, z''_n) \to 0$, and thus also (z''_n) is a Cauchy sequence; furthermore (z''_n) lies in the compact metric space $\{0,1\}^\infty$. Consequently $z''_n \to z$ for some $z \in \{0,1\}^\infty$, and thus also $z_n \to z$. This shows that every Cauchy sequence in T_0 has a limit either in T_0 or in $\{0,1\}^\infty$, and thus $T = T_0 \cup \{0,1\}^\infty$ as claimed above.

The complete infinite binary tree T is compact; this follows either by using a modification of the argument above to show that an arbitrary sequence (z_n) in T has a subsequence that converges, or by noting that for every $\varepsilon > 0$, there is a finite ε-net in T, since $\{0,1\}^\infty$ is compact, and so is the set $\{z \in T : d(z, \{0,1\}^\infty) \geqslant \varepsilon\}$; we omit the details.

Note that the complete infinite binary tree $T = T_0 \cup \{0,1\}^\infty$ regarded as a set is the same for every sequence $(\ell_n)_1^\infty$ satisfying the assumptions $\ell_n > 0$ and $\sum_n \ell_n < \infty$; furthermore, it is easily seen that the topology of T is the same for all such $(\ell_n)_1^\infty$. However, the metric on T depends on $(\ell_n)_1^\infty$, as is seen e.g. by (10.1).

It is easily seen that the set of leaves $T^{\mathsf{L}} = \{0,1\}^\infty$, and thus the skeleton $T^o = T_0$. □

Example 10.6. Let $\ell > 0$ and let $g : [0, \ell] \to [0, \infty)$ be a non-negative continuous function defined on $[0, \ell]$ with $g(0) = g(1) = 0$. We define a semimetric $d = d_g$ on $[0,\ell]$ by

$$d(s,t) := g(s) + g(t) - 2 \min_{u\in[s,t]} g(u), \tag{10.3}$$

when $s,t \in [0,\ell]$ with $s \leqslant t$ (and, of course, $d(s,t) := d(t,s)$ when $s > t$). It is easily verified that this is a semimetric; thus, if we define an equivalence relation on $[0,\ell]$ by $s \equiv t$ if $d(s,t) = 0$, then the quotient space $T_g := [0,\ell]/\equiv$ is a metric space; moreover, it is not difficult to show that T_g is connected and satisfies the 4-point inequality; thus T_g is a real tree by Theorem 6.1. The quotient map $[0,\ell] \to T_g$ is continuous, and thus T_g is a compact real tree.

Note that if $s \leqslant t$, then $s \equiv t \iff g(u) \geqslant g(s) = g(t)$ for every $u \in [s,t]$. (Informally, we may think of obtaining T_g by putting glue on the downside of the graph of g, and then compressing the x-axis.)

As a simple example, a finite combinatorial tree as in Example 10.1 or 10.2 can be constructed in this way by taking $g(t)$ to be the *contour* function of the tree, defined as the height (distance to the root) of a particle that moves with unit speed along the "outside" of the tree, starting and ending at the root.

In fact, every compact rooted real tree may be constructed in this way (up to isometry) using a suitable function $g : [0,1] \to [0,\infty)$ [17, Remark 3.2].

In applications, as in the following two examples, $g(t)$ is usually a random function, and then T_g is a random real tree. If we for simplicity let ℓ be fixed (for example, $\ell = 1$), then the map $g \mapsto T_g$ is a continuous map from $C[0,\ell]$ to the set $\mathbb{T}_1$ of rooted compact real trees with the Gromov–Hausdorff metric in Section 9, see [10]. In particular, this map is (Borel) measurable, so if g is a random element of $C[0,\ell]$, then T_g is a well-defined random element of the Polish space $\mathbb{T}_1$ of rooted compact real trees. □

Example 10.7. The *Brownian continuum random tree*, originally constructed (in several different ways) by Aldous [1, 2, 3], is the random real tree $T_{\mathbf{e}}$ obtained by the construction in Example 10.6 letting $g(t)$ be a random (normalized) Brownian excursion $\mathbf{e} : [0,1] \to [0,\infty)$; see [3, Corollary 22]. (Actually, Aldous defined the Brownian continuum random tree to be $T_{2\mathbf{e}}$ in our notation, but the convention has later changed to $T_{\mathbf{e}}$; of course, the results differ only by a scaling.) See e.g. [1; 2; 3], [11] and [16] for properties of this random real tree. In particular, $T_{\mathbf{e}}$ has almost surely a countably infinite number of branch points, all of degree 3, and an uncountable number of leaves. □

Example 10.8. More generally, a *Lévy tree* is a random real tree constructed as in Example 10.6 letting g be a random continuous fuction known as the *height process* of a Lévy process (with certain conditions), see [9; 10]. In the special case when the Lévy process is Brownian motion, this height process is a Brownian excursion and we obtain the Brownian continuum random tree as in Example 10.7.

Other special cases are the *stable trees*, see [16]. □

Example 10.9. Let T be a partially ordered set such that

(i) Any two elements $x,y \in T$ have a greatest lower bound $x \wedge y$.

(ii) For every $x \in T$, the set $L_x := \{y \in T : y \leqslant x\}$ is linearly ordered.
(iii) There is a *height function* $h : T \to \mathbb{R}$ such that for every $x \in T$, the restriction $h : L_x \to \mathbb{R}$ is an order-preserving bijection onto an interval $(a, h(x)]$ or $[(a, h(x)]$ for some $a \in \mathbb{R} \cup \{-\infty\}$ (where thus the interval is $(-\infty, h(x)]$ if $a = -\infty$).

It is easily seen from (i) and (ii) that a in (iii) cannot depend on x. Moreover, either $h(L_x) = [a, h(x)]$ for all x, and then T has a smallest element o with $h(o) = a$, or $h(L_x) = (a, h(x)]$ for every x, and then T has no minimum (or minimal) element.

Define

$$d(x, y) := h(x) + h(y) - 2h(x \wedge y), \qquad x, y \in T. \tag{10.4}$$

It is easily seen that d is a metric on T, which makes T a real tree. The path $[x, y]$ between two points $x, y \in T$ consist of the two parts $[x, x \wedge y]$ and $[x \wedge y, y]$, which are subsets of L_x and L_y, respectively.

If T is has a minimum element o, we choose o as a root, and then the partial order defined in (7.1) is the original order. Moreover, $h(x) = d(x, o)+ a$ with $a := h(o)$.

Conversely, if (T, ρ) is a rooted real tree, the partial order defined in (7.1) satisfies (i)–(iii) above with the height function $h(x) := d(x, \rho)$, and the construction above returns the original metric on T.

It is easily verified that the trees constructed in Example 10.6 are of this type, with height function g (after identifying equivalent points). $\square$

Example 10.10. Let T be the collection of all bounded non-empty subsets of R that contain their supremum. Let $h(A) := \sup A$ for $A \in T$, and define a partial order by letting $A \leqslant B$ if $A = (B \cap (-\infty, t]) \cup \{t\}$ for some $t \in \mathbb{R}$ with $t \leqslant h(B)$ (necessarily $t = h(A)$).

It is easily verified that $\leqslant$ is a partial order, and that it satisfies (i)–(iii) in Example 10.9 with the height function h (with $a = -\infty$). Hence, (10.4) defines a metric that makes T inte a real tree.

Note that this is a very large tree. Its cardinality is $2^{\mathfrak{c}}$, and every point in T has uncountable degree (more precisely, also of cardinality $2^{\mathfrak{c}}$). In particular, T is not separable.

See [11, Examples 3.18 and 3.45] for further properties of this real tree. $\square$

11. The length measure

Every real tree has a natural measure on it, defined as follows. (See e.g. [13, 2.10] for the definition and properties of Hausdorff measures.) We let $\mathcal{B}(T)$ denote the collection of Borel subsets of T.

Definition 11.1. Let $T = (T, d)$ be a real tree. The *length measure* λ on T is the 1-dimensional Hausdorff measure $\mathcal{H}^1$ on the skeleton T^o, regarded as a Borel measure on T. In other words, for a Borel set $A \in \mathcal{B}(T)$,

$$\lambda(A) := \mathcal{H}^1(A \cap T^o), \tag{11.1}$$

where $\mathcal{H}^1$ is the Hausdorff measure on T^o.

Note that, by definition, $\lambda(T^{\mathsf{L}}) = 0$.

In the definition (11.1), if A is a Borel set in T, then $A \cap T^o$ is a Borel subset of T^o, and thus $\lambda(A)$ is well defined. If T^o is a Borel subset of T, or more generally a $\mathcal{H}^1$-measurable subset of T, then we can also define the length measure by (11.1) interpreting $\mathcal{H}^1$ as the Hausdorff measure on T. In general, T^o is *not* measurable (see Example 11.6 below), but it is in most cases of interest (and in particular for all compact T) by Theorem 11.4 below. Alternatively, we can always (even if T^o is not measurable) define λ by (11.1) interpreting $\mathcal{H}^1$ as the *outer* Hausdorff meaure on T.

Remark 11.2. We have here defined λ as a Borel measure. Alternatively, we may more generally define it by (11.1) for every $A \subseteq T$ such that $A \cap T^o$ is a $\mathcal{H}^1$-measurable subset of T^o. □

We note some elementary properties of λ, which justify the name length measure. Note that for every $x, y \in T$, the set $[x, y]$ is isometric to the interval $[0, d(x, y)] \subset \mathbb{R}$, and is thus compact. Hence, $[x, y]$ and $(x, y) = [x, y] \setminus \{x, y\}$ are Borel sets in T.

Theorem 11.3. *The length measure λ is a continuous measure on T, i.e., if $x \in T$, then $\lambda\{x\} = 0$. Moreover, if $x, y \in T$, then*

$$\lambda([x, y]) = \lambda((x, y)) = d(x, y). \tag{11.2}$$

Proof. For every $x \in T$, we have $\lambda\{x\} = 0$ by (11.1).

Hence, for every $x, y \in T$, $\lambda([x, y]) = \lambda((x, y))$. Furthermore, (x, y) is a subset of T^o isometric to the interval $(0, d(x, y)) \subset \mathbb{R}$, and thus $\mathcal{H}^1((x, y)) = \mathcal{H}^1((0, d(x, y)) = d(x, y)$, since the Hausdorff measure $\mathcal{H}^1$ on $\mathbb{R}$ equals the Lebesgue measure. □

When T is separable, we can say more.

Theorem 11.4. *Suppose that T is a separable real tree.*

(i) *Then T^{L} and T^o are Borel subsets of T.*
(ii) *The length measure λ is σ-finite.*
(iii) *The length measure is the unique Borel measure λ on T with*

$$\lambda([x, y]) = d(x, y), \qquad x, y \in T, \tag{11.3}$$

and $\lambda(T^{\mathsf{L}}) = 0$.

Proof. (i): Let D be a countable dense subset of T. It is easy to see that then, cf. (8.1),

$$T^o = \bigcup_{x,y \in D} (x, y). \tag{11.4}$$

Furthermore, as noted above, $(x, y) = [x, y] \setminus \{x, y\}$ is a Borel set for any x and y. Hence, T^o is Borel, and thus so is its complement T^{L}.

(ii): This follows from (11.4), since λ is concentrated on T^o by (11.1), and $\lambda((x, y)) < \infty$ for each (x, y) by (11.3).

(iii): We have already shown (11.3) in Theorem 11.3, and $\lambda(T^{\mathsf{L}}) = 0$ follows directly from (11.1).

To show uniqueness, suppose that λ' is another Borel measure on T with $\lambda'([x, y]) = d(x, y)$ for all $x, y \in T$, and $\lambda'(T^{\mathsf{L}}) = 0$. Note first that then

$\lambda'\{x\} = 0$ for every $x \in T$, and thus for all $x, y \in T$, we have, by the assumption and (11.3),

$$\lambda'((x,y)) = \lambda'((x,y]) = \lambda'([x,y]) = \lambda((x,y)) = \lambda((x,y]) = \lambda([x,y]) < \infty. \tag{11.5}$$

We may assume that $T^o \neq \emptyset$. Let $x_0, x_1, \dots$ be a dense subset of T^o, and let, for $n \geqslant 1$,

$$T_n := \bigcup_{i=1}^{n} [x_0, x_i] \subset T^o. \tag{11.6}$$

T_n is a pathwise connected subset of T, and thus T_n is a real tree by Theorem 5.1. (In fact, by Theorem 5.3, T_n is the subtree spanned by $x_0, \dots, x_n$.) We consider the restrictions of λ and λ' to the compact (and thus Borel) subset T_n.

First, for every $n \geqslant 2$ we have $[x_0, x_n] \cap T_{n-1} = [x_0, y_n]$ for some $y_n \in T_{n-1}$, and then $T_n = T_{n-1} \cup (y_n, x_n]$ is a partition into two disjoint Borel subsets; hence, induction and (11.5) yield

$$\lambda'(T_n) = \lambda(T_n) < \infty, \qquad n \geqslant 1. \tag{11.7}$$

It now follows by the monotone class theorem (see e.g. [15, Theorem 1.2.3]) that

$$\lambda'(A) = \lambda(A), \qquad A \in \mathcal{B}(T_n), \tag{11.8}$$

because it follows from (11.7) that the collection $\mathcal{D}_n := \{A \in \mathcal{B}(T_n) : \lambda'(A) = \lambda(A)\}$ is a Dynkin system, the collection $\mathcal{A}_n := \{[x,y] : x, y \in T_n\} \cup \emptyset$ is a π-system (i.e., closed under finite intersections) that generates $\mathcal{B}(T_n)$, and $\mathcal{A}_n \subseteq \mathcal{D}_n$ by (11.5).

Now let $n \to \infty$; then $T_n \nearrow \bigcup_1^\infty T_n = T^o$. Hence it follows from (11.8) that for every $A \in \mathcal{B}(T)$,

$$.\lambda'(A \cap T^o) = \lim_{n\to\infty} \lambda'(A \cap T_n) = \lim_{n\to\infty} \lambda(A \cap T_n) = \lambda(A \cap T^o). \tag{11.9}$$

Finally, by assumption and definition $\lambda'(A \cap T^{\mathsf{L}}) = \lambda(A \cap T^{\mathsf{L}}) = 0$, and thus (11.9) yields

$$.\lambda'(A) = \lambda'(A \cap T^o) = \lambda(A \cap T^o) = \lambda(A), \qquad A \in \mathcal{B}(T). \tag{11.10}$$

□

Example 11.5. Let T be the complete infinite binary tree in Example 10.5, with $\ell_n := 2^{-\gamma n}$ for some $\gamma > 0$. It follows from (10.1) that for every $k \geqslant 0$, $T^{\mathsf{L}} = \{0,1\}^\infty$ can be partitioned into 2^k disjoint balls of radius $c2^{-\gamma k}$, where $c = 2(1-2^{-\gamma})^{1/2}$. It follows by standard arguments (see e.g. [4, Section 1.2]) that the $(1/\gamma)$-dimensional Hausdorff measure $\mathcal{H}^{1/\gamma}(\{0,1\}^\infty)$ is finite and positive, and thus $\{0,1\}^\infty$ has Hausdorff dimension $1/\gamma$. (The Minkowski dimension [4, Section 1.1] of $\{0,1\}^\infty$ is the same.)

The Hausdorff dimension of $T^{\mathsf{L}} = \{0,1\}^\infty$ can thus be any number in $(0,\infty)$. We see also that $\mathcal{H}^1(T^{\mathsf{L}}) = \mathcal{H}^1(\{0,1\}^\infty) > 0$ when $\gamma \leqslant 1$ (and ∞ when $\gamma < 1$); this shows that in general, the length measure is not equal to the Hausdorff measure $\mathcal{H}^1$ on T.

We see also that the total length measure $\lambda(T) = \sum_1^\infty 2^{n-\gamma n}$ is finite for $\gamma > 1$ but infinite for $\gamma \leqslant 1$. □

Example 11.6. Consider the complete infinite binary tree T in Example 10.5, for some sequence $(\ell_n)_1^\infty$, and let $A \subseteq T^{\mathsf{L}}$ be an arbitrary subset of $T^{\mathsf{L}} = \{0,1\}^\infty$. Define the real tree T_A by attaching an interval $[x, x']$ of length 1 to every $x \in A$ in the obvious way.

Then T_A is a real tree with

$$\begin{aligned} T_A^{\mathsf{L}} &= \{x \in \{0,1\}^\infty : x \notin A\} \cup \{x' : x \in A\} \\ &= \bigl(\{0,1\}^\infty \setminus A\bigr) \cup \{x' : x \in A\}. \end{aligned} \tag{11.11}$$

Suppose that T_A^{L} is a Borel set in T_A. Then so is

$$T_A^{\mathsf{L}} \cap \{x \in T_A : d(\emptyset, x) = L\} = \{0,1\}^\infty \setminus A. \tag{11.12}$$

Hence, A then is a Borel set in $\{0,1\}^\infty$. Consequently, if we choose A as a subset of $\{0,1\}^\infty$ that is not a Borel set, then T_A^{L} and its complement T_A^o are not Borel sets in T_A.

If we specialize to the case $\ell_n = 2^{-n}$, then (see Example 11.5) $\{0,1\}^\infty$ has finite and positive Hausdorff measure $h := \mathcal{H}^1(\{0,1\}^\infty)$. Since $\{0,1\}^\infty$ is a Polish space, and $\mathcal{H}^1$ is continuous and with full support, it follows that the measure space $(\{0,1\}^\infty, \mathcal{H}^1)$ is isomorphic to $([0,1], h\nu)$, where ν denotes the Lebesgue measure; this holds both for the Borel σ-fields and their completions, which are the $\mathcal{H}^1$-measurable sets in $\{0,1\}^\infty$ and the Lebesgue measurable sets in $[0,1]$, respectively. Hence, there exists $A \subset \{0,1\}^\infty$ such that A is not measurable for $\mathcal{H}^1$, and it follows that T_A^{L} and T_A^o are not measurable for the Hausdorff measure $\mathcal{H}^1$. □

12. Leaf measure

In some applications, a real tree is equipped with a different (Borel) measure, which, in contrast to the length measure in Section 11, is supported on the set of leaves T^{L}. For simplicity, suppose that T is a separable real tree, so that T^{L} is a Borel set by Theorem 11.4. We then may call any Borel measure supported on T^{L} a *leaf measure*. Note that a leaf measure thus has to be specified, and is not automatically determined by T, unlike the length measure in Section 11.

Usually, one considers leaf measures that are probability measures; they thus give a meaning to "a random leaf".

Aldous [3] defines a *continuum tree* as a rooted real tree (with some extra conditions) equipped with a nonatomic probability measure that is supported on the set of leaves (and thus a leaf measure in our sense), and furthermore has full support in the sense that for any x in the skeleton of the tree, the set $\{y : y > x\}$ (recall (7.1)) has positive measure.

Example 12.1. If T is constructed from a finite combinatorial tree as in Example 10.1 or 10.2, then T has a finite number of leaves, and a natural leaf measure is given by the uniform distribution on T^{L}. □

Example 12.2. If T is constructed from a continuous function $g : [0,\ell] \to [0,\infty)$ as in Example 10.6, then the natural (quotient) mapping $[0,\ell] \to T$ is continuous, and thus measurable, so it maps the Lebesgue measure on $[0,\ell]$ to a measure μ on T. This is in general not a leaf measure (one counterexample is when g is the contour function of a finite combinatorial tree as in

Example 10.1 or 10.2; then the measure μ is 2 times the length measure on T, as is easily seen). However, μ is a leaf measure under some conditions [3, Theorem 13]; in particular, μ is almost surely a leaf measure when, as in Example 10.7, $g = \mathbf{e}$, the (random) standard Brownian excursion; see [3, Corollary 22]. □

Appendix A. Gromov hyperbolic spaces

This appendix is a long remark on an approximate version of the four-point inequality, which is important in other contexts; the appendix can be skipped by those interested in trees only. $X = (X, d)$ is a metric space.

Gromov [14] defined *δ-hyperbolic metric spaces* as metric spaces where the four-point inequality holds up to an error δ. More precisely, he used the version (6.4), and the definition is as follows, for a given $\delta \geqslant 0$:

Definition A.1 (Gromov)**.** A metric space $X = (X, d)$ is δ-hyperbolic if, for all $x, y, z, w \in X$,

$$(x, y)_w \geqslant (x, z)_w \wedge (y, z)_w - \delta. \tag{A.1}$$

By Lemma 6.7, Theorem 6.4 can be formulated as follows.

Theorem A.2. *A metric space is* 0*-hyperbolic if and only if it can be isometrically embedded in a real tree.*

However, the main interest of Definition A.1 is for $\delta > 0$. The value of δ is often not important, and one says that a metric space X is *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta < \infty$.

Remark A.3. Intuitively, at large distances, the δ in (A.1) is insignificant, and thus the large scale geometry of a Gromov hyperbolic space is "tree-like". □

There are several alternatives to Definition A.1, with conditions that are equivalent in the sense that if one holds, then so do the others, but with δ replaced by $C\delta$ for some (small) constant C. (The conditions are in general not equivalent with the same δ.) Some of these alternative conditions are given (or implicit) in the following lemmas. See further e.g. [14], [6] and [18].

Lemma A.4. *The inequality* (A.1) *holds if and only if*

$$d(x, y) + d(z, w) \leqslant \bigl(d(x, z) + d(y, w)\bigr) \vee \bigl(d(x, w) + d(y, z)\bigr) + 2\delta. \tag{A.2}$$

Proof. As the proof of Lemma 6.7, adding 2δ to the left-hand sides of (6.5)–(6.8). □

Lemma A.5 ([14])**.** *Let X be a metric space and let $o \in X$ be fixed. Let $\delta \geqslant 0$. If* (A.1) *holds for $w = o$ and all $x, y, z \in X$, then it holds for all $x, y, z, w \in X$ with δ replaced by 2δ.*

Proof. We first show that the assumption implies that, for any x, y, z, w,

$$(x, y)_o + (z, w)_o \geqslant \bigl((x, z)_o + (y, w)_o\bigr) \wedge \bigl((x, w)_o + (y, z)_o\bigr) - 2\delta. \tag{A.3}$$

To see this, we first note that both sides are symmetric in x and y, and also in z and w; hence, by interchanging (x, y) and/or (z, w) if necessary, we may assume that $(x,z)_o$ is the largest of the four numbers $(x,z)_o, (x,w)_o, (y,z)_o, (y,w)_o$. In this case, the assumption implies

$$(x,y)_o \geqslant (x,z)_o \wedge (y,z)_o - \delta = (y,z)_o - \delta, \tag{A.4}$$

$$(z,w)_o \geqslant (x,z)_o \wedge (x,w)_o - \delta = (x,w)_o - \delta, \tag{A.5}$$

and thus

$$(x,y)_o + (z,w)_o \geqslant (y,z)_o + (x,w)_o - 2\delta, \tag{A.6}$$

verifying (A.3).

Next, by exanding all Gromov products in (A.3) according to the definition (6.3), we see that (A.3) does not depend on the choice of o; hence (A.3) holds for every $o,x,y,z,w \in X$.

In particular, we can choose $o = w$ in (A.3). Since $(w,v)_w = 0$ for every v by (6.3), we obtain

$$(x,y)_w \geqslant (x,z)_w \wedge (y,z)_w - 2\delta, \tag{A.7}$$

as asserted. □

A *geodesic* in a metric space X is an isometric curve, i.e., an isometric mapping of an interval $I \subseteq \mathbb{R}$ into X. If a geodesic φ is defined on an interval $I = [a,b]$ that is closed and finite, we say that the geodesic has the endpoints $\varphi(a)$ and $\varphi(b)$, and that it *joins* $\varphi(a)$ and $\varphi(b)$. (Then, the geodesic necessarily has length $d_{x,y}$, and we may choose $I = [0, d_{x,y}]$.)

Note that condition (T1) says that for every $x,y \in X$, there is a unique geodesic from x to y (provided we normalize $I = [0, d_{x,y}]$).

More generally, a metric space $X = (X,d)$ is *geodesic*, if every pair $x,y \in X$ is joined by a geodesic. In other words, we assume the existence part of (T1), but uniqueness is not assumed. In particular, every real tree is geodesic.

Gromov hyperbolicity is often studied for geodesic metric spaces. This can be done without real loss of generality by the following result by Bonk and Schramm [5], to which we refer for a proof.

Theorem A.6 (Bonk and Schramm [5, Theorem 4.1])**.** *Let $\delta \geqslant 0$. A metric space is δ-hyperbolic if and only if it can be isometrically embedded into a δ-hyperbolic complete geodesic metric space.* □

In particular, for $\delta = 0$, we recover Theorem 6.4 (together with Theorem 6.5).

For geodetic metric spaces, there are further conditions equivalent to Gromov hyperbolicity. We note first an extension of Lemma 3.3 to general geodesic spaces.

Lemma A.7. *Let (X,d) be a geodesic metric space, and let $x,y,z \in X$. Then $(x,y)_z = 0$ if and only if there exists a geodesic from x to y that contains z.*

Proof. As for Lemma 3.3. □

In a geodesic metric space, it is enough to assume the hyperbolicity condition (A.1) when the left-hand side is 0, i.e., by Lemma A.7, when w is on a geodesic joining x and y.

Lemma A.8. *Let (X, d) be a geodesic metric space. Suppose that, for every $x, y, z \in X$ and every w that lies on a geodesic from x to y, the inequality* (A.1) *holds, or equivalently*

$$(x, z)_w \wedge (y, z)_w \leqslant \delta. \tag{A.8}$$

Then X is 3δ-hyperbolic.

Proof. See [6, p. 286 and Exercise 8.4.5] or [18, Proof of Theorem 2.34]. □

Remark A.9. In particular, using Theorem A.2, a geodesic metric space is a real tree if and only if (A.8) holds with $\delta = 0$. In fact, it is easily seen that this condition implies that geodesics are unique, so (T1) holds. (Given two geodesics from x to y, let z and w be points on them with $d(z, x) = d(w, x)$, and conclude $z = w$ from (A.8).) Then, by Lemma A.7 again, (A.8) with $\delta = 0$ when $w \in [x, y]$ is equivalent to $[x, y] \subseteq [x, z] \cup [y, z]$, which is (T2b). □

The condition in Lemma A.8 can be regarded as a condition on the geometry of the triangle xyz. One can pursue this point of view further.

Definition A.10. Let (X, d) be a geodesic metric space. A *triangle xyz* in X is a set of three *vertices* $x, y, z \in X$ together with three *sides* xy, xz and yz that are geodesics between the pairs of vertices.

Note that in general, the sides of a triangle are not uniquely determined by the vertices.

Definition A.11. Let (X, d) be a geodesic metric space. A triangle is δ-*slim* if each side is contained in the closed δ-neighbourhood of the union of the two other sides, i.e., if every $w \in xy$ has distance $\leqslant \delta$ to $xz \cup yz$, and similarly for the two other sides.

Gromov [14] used a definition that is similar, but somewhat more technical. Note that in a triangle with vertices x, y, z, we have, by (6.3), $(y, z)_x + (x, z)_y = d_{x,y}$, and thus the side xy can be divided into two parts of lengths $(y, z)_x$ and $(x, z)_y$, containing x and y respectively; we call these, the *x-part* and *y-part* of xy. (In a tree, these are the parts of $[x, y]$ before and after $\gamma(x, y, z)$. The definition in [14] is stated in terms of isometric mappings of the three sides into a real tree with three endpoints having the same distances between each other as x, y, z.)

Definition A.12. Let (X, d) be a geodesic metric space. A triangle xyz is δ-*thin* if for each w in the x-part of the side xy, the point $v \in xz$ with $d(v, x) = d(w, x)$ satisfies $d(v, w) \leqslant \delta$, and similarly for the y-part and for the two other sides.

It is immediate that a δ-thin triangle is δ-slim.

Lemma A.13. *Let X be a geodesic metric space.*

(i) *If X is δ-hyperbolic, then every triangle in X is 2δ-thin.*
(ii) *If every triangle in X is δ-thin, then X is 2δ-hyperbolic.*

Proof. See [14, Proposition 6.3.C]. □

Lemma A.14. *Let X be a geodesic metric space.*

(i) *If X is δ-hyperbolic, then every triangle in X is 2δ-slim.*
(ii) *If every triangle in X is δ-slim, then X is 3δ-hyperbolic.*

Proof. (i) follows from Lemma A.13(i); see also [18, Theorems 2.35] (with 3δ).

(ii) is [18, Theorems 2.34 (with $h = 0$)]. ☐

Remark A.15. If every triangle in X is δ-slim (or δ-thin), then any two geodesics with the same endpoints are within distance 2δ of each other. (I.e., their Hausdorff distance is $\leqslant 2\delta$.) This follows by considering the triangle obtained by subdividing one of the geodesics at an arbitrary interior point. (Or, more bravely, by considering the geodesics as two sides of a degenerate triangle with two vertices coinciding.) ☐

Remark A.16. In particular, using Theorem A.2, a geodesic metric space is a real tree if and only if every triangle is 0-slim (or 0-thin).

In fact, if this holds, then by Remark A.15, geodesics are unique, and thus (T1) holds. Then, a triangle xyz is 0-slim if and only if (T2b) holds for all permutations of x, y, z. Conversely, each triangle in a real tree is 0-thin as a consequence of Lemmas 3.5 and 3.6. ☐

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