FOURIER AND MELLIN INVERSION

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Abstract. We collect some classical, more or less well known, results on inversion of Fourier and Mellin transforms of functions and measures, concentrating on sufficient conditions for pointwise inversion.

Proofs are usually not given.

1. Introduction

The purpose of these notes is to collect some classical, more or less well known, results on inversion of Fourier transforms of functions and measures on $\mathbb{R}$, and of and Mellin transforms of functions on $(0, \infty)$. We concentrate on sufficient conditions for pointwise inversion formulas. We try to give the most general statements for each type of condition, without going too far inte technicalities, and the goal is to present results in forms convenient for future references.

We give only a few proofs, and otherwise refer to standard references for proofs.

Remark 1.1. There is also a parallel (and even more classical) theory for *Fourier series* of functions on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which can be identified with $[0, 1)$ or $[0, 2\pi)$. The problem of inversion for Fourier transforms corresponds to the problem of convergence of Fourier series, see e.g. [8, Chapter II and VI]. We will not consider Fourier series in detail here, but we note that many results are essentially the same for Fourier series and Fourier integrals; for questions on convergence, this follows e.g. by [8, Theorem XVI.(1.3) and XVI.(1.10)–(1.14)], which allows transfer of results in both directions. △

1.1. Some notation.

When we say that a limit exists, this tacitly includes that the limit is finite.

For a function f defined in a neighbourhood of $x \in \mathbb{R}$, we define, when the limits exist,

$$f(x+) := \lim_{y \downarrow x} f(y), \qquad f(x-) := \lim_{y \uparrow x} f(y). \tag{1.1}$$

For a function f defined in a (finite or infinite) interval I, we define its *modulus of continuity* by

$$\omega(\delta; f, I) := \sup\{|f(x) - f(y)| : x, y \in I \text{ with } |x - y| \leqslant \delta\}, \qquad \delta > 0. \tag{1.2}$$

The function f is *Hölder*(α) *on* I, where $0 < \alpha \leqslant 1$, if

$$\omega(\delta; f, I) = O\big(\delta^{\alpha}\big) \tag{1.3}$$

as $\delta \to 0$; equivalently, if (1.3) holds for all $\delta \leqslant 1$, say.

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The function f is *locally Hölder*(α) *on* I, where $0 < \alpha \leqslant 1$, if it is Hölder(α) on a neighbourhood J_x of every $x \in I$; this is equivalent to f being Hölder(α) on every compact interval $J \subseteq I$. (The implicit constant in (1.3) may thus depend on J.)

$L^p(\mathbb{R})$ (where $p > 0$ is a real number) is the space of complex-valued (Lebesgue) measurable functions f on $\mathbb{R}$ such that $\int_{\mathbb{R}} |f|^p < \infty$. We will mainly be concerned with the case $p = 1$: $L^1(\mathbb{R})$ is the space of integrable functions on $\mathbb{R}$.

$M(\mathbb{R})$ is the space of complex (i.e., complex-valued) Borel measures on $\mathbb{R}$. (Recall that by definition, complex measures are finite.) A function $f \in L^1(\mathbb{R})$ can be identified with the complex measure $f(x)\,\mathrm{d}x$; this identification embeds $L^1(\mathbb{R})$ isometrically as a subspace of $M(\mathbb{R})$. Complex measures of the form $f(x)\,\mathrm{d}x$ are called *absolutely continuous*; the function $f(x)$ is called the *density* of the measure.

$BV(\mathbb{R})$ is the space of complex functions of bounded variation on $\mathbb{R}$. Every $F \in BV(\mathbb{R})$ defines a complex measure $\mathrm{d}F \in M(\mathbb{R})$. This map $F \mapsto \mathrm{d}F$ is onto $M(\mathbb{R})$, i.e, every complex measure $\mu \in M(\mathbb{R})$ equals $\mathrm{d}F$ for some $F \in BV(\mathbb{R})$. Moreover, the map is not injective, but if $NBV(\mathbb{R})$ is the subspace of functions $F \in BV(\mathbb{R})$ that are right-continuous and are normalized by $\lim_{x\to-\infty} F(x) = 0$, then $F \mapsto \mathrm{d}F$ is a bijection $NBV(\mathbb{R}) \to M(\mathbb{R})$.

A function F on $\mathbb{R}$ is *absolutely continuous* if F is continuous and of bounded variation, and the measure $\mathrm{d}F$ is absolutely continuous. (Then the density of $\mathrm{d}F$ is the derivative $F'(x)$, which exists a.e.)

2. The Fourier transform

Let f be an integrable complex-valued function on $\mathbb{R}$. (In other words, using standard notation, $f \in L^1(\mathbb{R})$.) We define its *Fourier transform* by

$$\widehat{f}(t) := \int_{-\infty}^{\infty} f(x) e^{\mathrm{i}tx}\,\mathrm{d}x, \qquad t \in \mathbb{R}. \tag{2.1}$$

It is well-known that $\widehat{f}$ is a continuous, bounded function on $\mathbb{R}$, and $\widehat{f}(t) \to 0$ as $t \to \pm\infty$ (the Riemann–Lebesgue lemma).

Remark 2.1. This is one of the common definitions, but there are several other versions, differing by simple changes of variable and sometimes constant factors; some other common versions are $\int_{-\infty}^{\infty} f(x)e^{-\mathrm{i}tx}\,\mathrm{d}x$, $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-\mathrm{i}tx}\,\mathrm{d}x$, $\int_{-\infty}^{\infty} f(x)e^{2\pi\mathrm{i}tx}\,\mathrm{d}x$, $\int_{-\infty}^{\infty} f(x)e^{-2\pi\mathrm{i}tx}\,\mathrm{d}x$. The results below trivially transfer to all such versions, *mutatis mutandis.* △

Similarly, if μ is a complex Borel measure on $\mathbb{R}$ (i.e., $\mu \in M(\mathbb{R})$), we define its Fourier transform by

$$\widehat{\mu}(t) := \int_{-\infty}^{\infty} e^{\mathrm{i}tx}\,\mathrm{d}\mu(x), \qquad t \in \mathbb{R}. \tag{2.2}$$

It is well-known that $\widehat{\mu}$ is a continuous, bounded function on $\mathbb{R}$. However, in general the Riemann–Lebesgue lemma does not hold for measures.

Note that if we identify an integrable function f with the measure $f(x)\,\mathrm{d}x$, then the definitions (2.1) and (2.2) are consistent and yield the same Fourier transform.

Remark 2.2. When μ is a probability measure, the Fourier transform (2.2) is known (in probability theory) as the *characteristic function* of μ. △

Remark 2.3. We focus in these notes on pointwise formulas, and we consider except in a few comments only integrable functions f, i.e. $f \in L^1(\mathbb{R})$; in this case (and only in this case), the integral (2.1) is absolutely convergent for every $t \in \mathbb{R}$, and thus $\widehat{f}(t)$ is well defined.

However, it should be noted that there are important extensions that we do not discuss in detail here. In particular, $\widehat{f}$ can be defined also for $f \in L^2(\mathbb{R})$ (in a somewhat different way since the integral (2.1) in general is not absolutely convergent, and not even conditionally convergent for all t); then $\widehat{f} \in L^2(\mathbb{R})$ (Plancherel's theorem); see e.g. [8, Theorem XVI.(2.17)].

An elegant, and much wider, extension is obtain by going beyond functions and considering *tempered distributions*, which formally are defined as elements of the dual space $\mathcal{S}'(\mathbb{R})$ of the space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing test functions; this space includes $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, the space $C_{\mathsf{b}}(\mathbb{R})$ of continuous bounded functions on $\mathbb{R}$, the space $M(\mathbb{R})$ of complex measures, and much more. The Fourier transform has a natural definition as a linear operator $\mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ which includes as special cases the Fourier transforms on $L^1(\mathbb{R})$, $M(\mathbb{R})$, and $L^2(\mathbb{R})$ defined above. See e.g. [5, 1.16(v)(vii)] for a quick summary, and [2, Section 2.3] or [6, Section I.3] for a more detailed treatment; deeper functional analytic properties are treated in [7].

In the settings of $L^2(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$, the Fourier transform is a bijection and topological isomorphism $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$, and there are "perfect" inversion formulas. However, these formulas are not of the pointwise type considered here. △

3. Fourier-Stieltjes transforms

Let $F : \mathbb{R} \to \mathbb{C}$ be a function of bounded variation. As mentioned above, F defines a complex measure $\mathrm{d}F$ on $\mathbb{R}$. The Fourier transform (2.2) of $\mathrm{d}F$ is known as the *Fourier–Stieltjes transform* of F; it is thus given by

$$\widetilde{F}(t) := \widehat{\mathrm{d}F}(t) = \int_{-\infty}^{\infty} e^{\mathrm{i}tx} \,\mathrm{d}F(x). \tag{3.1}$$

Remark 3.1. In the setting of distributions, see Remark 2.3, every $F \in BV(\mathbb{R})$ can be regarded as a tempered distribution, and its derivative in distribution sense is the complex measure $\mathrm{d}F$. It follows that

$$\widetilde{F}(t) := \widehat{\mathrm{d}F}(t) = -\mathrm{i}t\widehat{F}(t) \tag{3.2}$$

in distribution sense. In the special case of a function $F \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$, (3.2) thus holds in the usual sense, with all appearing functions continuous on $\mathbb{R}$; this also follows by integration by parts. (Note that in this case, $\lim_{x\to-\infty} F(x) = \lim_{x\to\infty} F(x) = 0$.)

△

4. Fourier Inversion

4.1. **Absolute convergence.** The Fourier transform is injective on $L^1(\mathbb{R})$, and more generally on $M(\mathbb{R})$ (and in fact on $\mathcal{S}'(\mathbb{R})$), and thus a function or measure is determined by its Fourier transform. There exist several types of inversion formulas giving explicit formulas for f or μ given its Fourier transform, under suitable conditions. The simplest inversion formula is the following.

Theorem 4.1. (i) *Let $f \in L^1(\mathbb{R})$ and suppose that $\widehat{f} \in L^1(\mathbb{R})$. Then f is a.e. equal to a continuous function, and if f is continuous (which we thus may assume), then*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}xt} \widehat{f}(t) \,\mathrm{d}t, \qquad x \in \mathbb{R}. \tag{4.1}$$

In other words, in this case

$$\widehat{\widehat{f}}(x) = 2\pi f(-x). \tag{4.2}$$

(ii) *Let $\mu \in M(\mathbb{R})$ and suppose that $\widehat{\mu} \in L^1(\mathbb{R})$. Then μ is absolutely continuous with a continuous density f, and* (4.1) *holds; equivalently,*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}xt} \widehat{\mu}(t) \,\mathrm{d}t, \qquad x \in \mathbb{R}. \tag{4.3}$$

Sketch of proof. A short proof uses the theory of distributions mentioned in Remark 2.3. For tempered distributions, the Fourier transform is a bijection, with the inversion formula (4.2) (in distribution sense). In (ii), apply this to μ, regarded as a distribution. Since $\widehat{\mu} \in L^1(\mathbb{R})$, its Fourier transform in distribution sense equals its Fourier transform as an integrable function, which means that $\widehat{\widehat{\mu}}(-x)$ is the continuous function given by the integral in (4.3); the result follows. Finally, (i) is a special case of (ii).

For (i), see also [6, Corollary 1.21] and [8, Section XVI.2, p. 247]. ☐

We note also the following result [6, Corollary I.1.26].

Theorem 4.2. *Let $f \in L^1(\mathbb{R})$ and suppose that f is continuous at* 0 *and that $\widehat{f}(t) \geqslant 0$ for all $t \in \mathbb{R}$. Then $\widehat{f} \in L^1$ and thus Theorem 4.1*(i) *applies; hence* (4.1) *holds a.e., and in particular at every continuity point of f.*

4.2. **Symmetric limits.** However, even if f is integrable, $\widehat{f}$ does not have to be integrable; such cases are more complicated since then the integral (4.1) is not absolutely convergent for any x. In many (but not all) cases, (4.1) still holds with the integral conditionally convergent in the following sense.

$$f(x) = \frac{1}{2\pi} \lim_{A\to\infty} \int_{-A}^{A} e^{-\mathrm{i}xt} \widehat{f}(t) \,\mathrm{d}t. \tag{4.4}$$

Three classical sufficient conditions for (4.4) are given in the following theorems.

Remark 4.3. The main conditions are local, i.e., for some small neighbourhood of x, and thus a function f may satisfy one of them at some point but not at at others. In fact, the inversion formula (4.4) is itself a local property: if it holds for some integrable function f, then it holds for any integrable functions g that equals f in a neighbourhood of x. ([8, Theorem II.(6.6)] for Fourier series; this transfers by Remark 1.1. Alternatively, apply Theorem 4.5 below to the difference $f - g$, which is 0 in an interval around x.) △

Remark 4.4. An interchange of the order of integration shows that, for $f \in L^1(\mathbb{R})$ and any $A > 0$,

$$\begin{aligned} \frac{1}{2\pi} \int_{-A}^{A} e^{-\mathrm{i}xt} \widehat{f}(t) \,\mathrm{d}t &= \frac{1}{2\pi} \int_{-A}^{A} \int_{-\infty}^{\infty} e^{-\mathrm{i}xt+\mathrm{i}yt} f(y) \,\mathrm{d}y \,\mathrm{d}t \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(A(x-y))}{x-y} f(y) \,\mathrm{d}y. \end{aligned} \tag{4.5}$$

The kernel $\frac{\sin(Ax)}{\pi x}$ in (4.5) is known as the *Dirichlet kernel* on $\mathbb{R}$. △

Theorem 4.5 (Dini test)**.** *Suppose that $f \in L^1(\mathbb{R})$ and that $x \in \mathbb{R}$ is such that*

$$\int_0^\delta \frac{|f(x+y)+f(x-y)-2f(x)|}{y}\,\mathrm{d}y < \infty \tag{4.6}$$

for some (and thus every) $\delta > 0$. Then (4.4) holds.

In particular, (4.4) holds if

$$\int_{-\delta}^\delta \frac{|f(x+y)-f(x)|}{|y|}\,\mathrm{d}y < \infty. \tag{4.7}$$

Proof. See [8, Theorem II.(6.1)]. (For Fourier series; the result transfers by Remark 1.1.) □

Corollary 4.6. *Suppose that $f \in L^1(\mathbb{R})$ and that f is locally Hölder(α) for some $\alpha > 0$. Then (4.4) holds for every $x \in \mathbb{R}$.*

If we consider an interval instead of just a single point x, then Theorem 4.5 (at least partly) and Corollary 4.6 can be improved.

Theorem 4.7 (Dini–Lipschitz test)**.** *Suppose that $f \in L^1(\mathbb{R})$ is continuous, and that*

$$\omega(\delta; f, I) = o\bigl(|\log\delta|^{-1}\bigr) \qquad \text{as } \delta \to 0 \tag{4.8}$$

for some interval I. Then (4.4) holds for every x in the interior I°. Moreover, (4.4) holds uniformly on every compact subinterval of I°.

In particular, if (4.8) holds locally on $\mathbb{R}$, i.e., for every compact interval $I \subset \mathbb{R}$ (with implicit constant that may depend on I), then (4.4) holds for every $x \in \mathbb{R}$, uniformly on every compact interval.

Proof. See [8, Theorems II.(10.3) and II.(10.5)]. (For Fourier series; the result transfers by Remark 1.1.) □

The third theorem does not use modulus of continuity or related continuity properties; instead it assumes (locally) bounded variation.

Theorem 4.8 (Dirichlet–Jordan test)**.** *Suppose that $f \in L^1(\mathbb{R})$ and that $x \in \mathbb{R}$ is such that f is of bounded variation on some interval $(x-\delta, x+\delta)$ with $\delta > 0$. Then*

$$\frac{f(x+)+f(x-)}{2} = \frac{1}{2\pi}\lim_{A\to\infty}\int_{-A}^{A} e^{-\mathrm{i}xt}\widehat{f}(t)\,\mathrm{d}t. \tag{4.9}$$

(The limits $f(x \pm 0)$ exist since f is of bounded variation near x.)

In particular, if furthermore f is continuous at x, then (4.4) holds.

Proof. See [8, Theorem II.(8.14)]. (For Fourier series; the result transfers by Remark 1.1.) □

Corollary 4.9. *Suppose that $f \in L^1(\mathbb{R})$ and that f is continuous and locally of bounded variation, i.e., that f is of bounded variation in any finite interval. Then (4.4) holds for every $x \in \mathbb{R}$.*

Finally, we give a simple condition that is sufficient for many applications.

Corollary 4.10. *Suppose that $f \in L^1(\mathbb{R})$ and that f is continuously differentiable. Then (4.4) holds for every $x \in \mathbb{R}$.*

Proof. This is a corollary of any of the above results Theorem 4.5, Corollary 4.6, Theorem 4.7, Theorem 4.8, Corollary 4.9. □

Remark 4.11. Theorems 4.5 and 4.7 are essentially the best possible of this type; see [8, Chapter VIII, in particular Theorems VIII.(2.4), VIII.(2.1)]. Note also that Theorem 4.7 requires the estimate (4.8) in an interval. It is *not* true that the corresponding estimate at a single point x (i.e., keeping x fixed in (1.2)) implies convergence at x, see [8, Section VIII.2, p. 303]. △

Remark 4.12. None of the conditions in the theorems above imply that the integral (4.1) is absolutely convergent. For Fourier series this is shown in [8, Theorem VI.(3.1) and the example VI.(3.7)] (not even Hölder($\frac{1}{2}$) is enough; bounded variation and continuity, and even absolute continuity, is not enough). This transfers to Fourier transforms by extending the examples periodically to $\mathbb{R}$ and then multiplying by a suitable Fejér kernel; we omit the details. See also Example A.3 below, which shows that neither (4.8) nor absolute continuity (and thus also not bounded variation and continuity) is enough. △

Remark 4.13. A famous (and difficult) result by Carleson [1] says that if $f \in L^2(\mathbb{R})$ (or equivalently, $\widehat{f} \in L^2(\mathbb{R})$), then (4.4) holds for *almost every* $x \in \mathbb{R}$ (but not necessarily for *every* x). This extends to $f \in L^p(\mathbb{R})$ for any $p > 1$ [4], but *not* to arbitrary $f \in L^1(\mathbb{R})$; in fact, there exist $f \in L^1(\mathbb{R})$ such that (4.4) does not hold for *any* x [8, Theorem VIII.(4.1)]. (These references show the corresponding results for convergence of Fourier series, but the results transfer to Fourier integrals by Remark 1.1.) △

4.3. **Asymmetric limits.** In the inversion formulas above with (4.4), it is in general important that the symmetric limit is used in (4.4). We may ask whether, more strongly,

$$f(x) = \frac{1}{2\pi} \lim_{A,B\to\infty} \int_{-A}^{B} e^{-\mathrm{i}xt} \widehat{f}(t)\,\mathrm{d}t, \tag{4.10}$$

where the upper and lower limits of integration tend to $\pm\infty$ independently. (This is clearly true when the integral (4.1) is absolutely convergent.)

Obviously, the limit in (4.4) is a special case of the limit in (4.10), so if the general limit in (4.10) exists, then so does the symmetric limit in (4.4) and they are equal. Consequently, (4.10) implies (4.4). Examples A.1–A.3 below show that the converse does not hold. We make a simple observation.

Lemma 4.14. *Suppose that $f \in L^1(\mathbb{R})$, and $x \in \mathbb{R}$. Then the following are eqivalent.*

(i) (4.10) *holds.*

(ii)
$$f(x) = \frac{1}{2\pi} \left(\lim_{A\to\infty} \int_{-A}^{0} e^{-\mathrm{i}xt} \widehat{f}(t)\,\mathrm{d}t + \lim_{B\to\infty} \int_{0}^{B} e^{-\mathrm{i}xt} \widehat{f}(t)\,\mathrm{d}t \right) \tag{4.11}$$
with both limits existing.

(iii) (4.4) *holds and the limit*
$$\lim_{A\to\infty} \int_{-A}^{A} e^{-\mathrm{i}xt} \widehat{f}(t) \operatorname{sgn}(t)\,\mathrm{d}t \tag{4.12}$$
exists.

Proof. (i) $\iff$ (ii): It is easily seen using the Cauchy criterion that if the limit in (4.10) exists, then so do the two limits in (4.11). The converse is obvious. Moreover, when all limits exist, the right-hand sides of (4.10) and (4.11) are equal.

(ii) $\implies$ (iii): We have seen that (4.10) holds, and thus also (4.4). Furthermore,

$$\int_{-A}^{A} e^{-\mathrm{i}xt}\widehat{f}(t)\operatorname{sgn}(t)\,\mathrm{d}t = -\int_{-A}^{0} e^{-\mathrm{i}xt}\widehat{f}(t)\,\mathrm{d}t + \int_{0}^{A} e^{-\mathrm{i}xt}\widehat{f}(t)\,\mathrm{d}t, \tag{4.13}$$

where both integrals on the right-hand side have limits as $A \to \infty$ by (ii).

(iii) $\implies$ (i): We have

$$\int_{0}^{B} e^{-\mathrm{i}xt}\widehat{f}(t)\,\mathrm{d}t = \frac{1}{2}\int_{-B}^{B} e^{-\mathrm{i}xt}\widehat{f}(t)\,\mathrm{d}t + \frac{1}{2}\int_{-B}^{B} e^{-\mathrm{i}xt}\widehat{f}(t)\operatorname{sgn}(t)\,\mathrm{d}t, \tag{4.14}$$

where (iii) implies that both integrals on the right-hand side have limits as $B \to \infty$, and thus the second limit in (4.11) exists. Similarly, the first limit exists. This implies that the limit in (4.10) exists. As noted above, the limit then equals the limit in (4.4); since we now assume that (4.4) holds, (4.10) follows. □

Remark 4.15. The limit in (4.12) (if it exists) is, up to an imaginary numerical factor irrelevant to us, known as the *conjugate function* $\tilde{f}(x)$. △

The Dini test in Theorem 4.5 (in the one-sided version) actually yields also (4.10).

Theorem 4.16. *Suppose that $f \in L^1(\mathbb{R})$ and that $x \in \mathbb{R}$ is such that* (4.7) *holds. Then* (4.10) *holds.*

Proof. This too follows from [8, Theorem II.(6.1)] (for Fourier series); now together with Lemma 4.14 and an application of [8, Theorem XVI.(1.3) with XVI.(1.2)] to show the existence of the conjugate function (4.12). □

Thus we can strengthen Corollary 4.6.

Corollary 4.17. *Suppose that $f \in L^1(\mathbb{R})$ and that f is locally Hölder(α) for some $\alpha > 0$. Then* (4.10) *holds for every $x \in \mathbb{R}$.*

However, Theorems 4.7 and 4.8 (and thus Corollary 4.9) cannot be strengthened in this way; their assumptions do not imply (4.10); see Example A.1–A.3.

4.4. **The generalized integral in the inversion formula is known to exist.**

In the results above, the existence of the limit in (4.4) or (4.10) is part of the conclusion. If we know in advance that the limit exists, then the inversion formula holds assuming only continuity of f.

Theorem 4.18. *Suppose that $f \in L^1(\mathbb{R})$ and that f is continuous at x. If the limit in* (4.4) *exists, then* (4.4) *holds. Similarly, if the limit in* (4.10) *exists, then* (4.10) *holds.*

Proof. The second claim follows from the first, since if the limit in (4.10) holds, then so does the limit in (4.4), and the limits are equal.

The first claim follows from Theorem 4.20 below. □

4.5. **Summability.** As noted above, the inversion formula (4.4) does not always hold, even for continuous $f \in L^1(\mathbb{R})$. An important substitute is a weaker property called *(Cesàro) summability*, where we take the average of the right-hand side in (4.4) over all A in an interval $[0, B]$, and let $B \to \infty$: we thus ask whether

$$f(x) = \frac{1}{2\pi} \lim_{B\to\infty} \frac{1}{B} \int_0^B \mathrm{d}A \int_{-A}^{A} e^{-\mathrm{i}xt} \widehat{f}(t) \,\mathrm{d}t. \tag{4.15}$$

Note that (4.15) holds whenever (4.4) holds. The following theorem shows that (4.15) holds under very general conditions.

Remark 4.19. It follows from (4.5) and an interchange of the order of integration that

$$\begin{aligned}\frac{1}{2\pi B} \int_0^B \mathrm{d}A \int_{-A}^{A} e^{-\mathrm{i}xt} \widehat{f}(t) \,\mathrm{d}t &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-\cos(B(y-x))}{B(y-x)^2} f(y) \,\mathrm{d}y \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-\cos(By)}{By^2} f(x+y) \,\mathrm{d}y. \end{aligned} \tag{4.16}$$

The kernel $\frac{1-\cos(By)}{\pi By^2} = \frac{B}{2\pi}\bigl(\frac{\sin(By/2)}{By/2}\bigr)^2$ in (4.18) is known as the *Fejér kernel* on $\mathbb{R}$. △

Theorem 4.20. *Suppose that $f \in L^1(\mathbb{R})$, and that the limits $f(x+)$ and $f(x-)$ exist for some $x \in \mathbb{R}$. Then*

$$\frac{f(x+)+f(x-)}{2} = \frac{1}{2\pi} \lim_{B\to\infty} \frac{1}{B} \int_0^B \mathrm{d}A \int_{-A}^{A} e^{-\mathrm{i}xt} \widehat{f}(t) \,\mathrm{d}t. \tag{4.17}$$

In particular, if f is continuous at x, then (4.15) *holds.*

Moreover, (4.15) *holds if x is a Lebesgue point of f; hence,* (4.15) *holds for a.e. $x \in \mathbb{R}$.*

Proof. The corresponding results for Fourier series are [8, Theorems III.(3.4) and III.(3.9)], and again the results transfer to $\mathbb{R}$ by the references in Remark 1.1. We can also argue directly as follows, see [8, Remark in Section XVI.2, p. 247]. It follows from (4.16) that, for any $a > 0$,

$$\begin{aligned}&\frac{1}{2\pi B} \int_0^B \mathrm{d}A \int_{-A}^{A} e^{-\mathrm{i}xt} \widehat{f}(t) \,\mathrm{d}t \\ &\qquad = \frac{1}{\pi} \int_{-Ba}^{Ba} \frac{1-\cos(y)}{y^2} f(x+y/B) \,\mathrm{d}y + \frac{1}{\pi} \int_{|y|>a} \frac{1-\cos(By)}{By^2} f(x+y) \,\mathrm{d}y, \end{aligned} \tag{4.18}$$

and if $f(x\pm)$ exist, then the result (4.17) follows by dominated convergence. The result (4.15) when x is a Lebesgue point follows also easily from (4.16), see [6, Theorem I.1.25]. □

Remark 4.21. The a.e. summability for any $f \in L^1(\mathbb{R})$ given by Theorem 4.20 is in contrast to convergence, which does not have to hold anywhere, as said in Remark 4.13. △

Remark 4.22. It is obvious that we need some kind of regularity of f at x for (4.15) to hold (since otherwise we can change the value of f at x without changing $\widehat{f}(t)$ and thus without changing the right-hand side of (4.15)). However, the theorem says that only a very weak condition is needed, for example that x is a Lebesgue point of f. △

5. Fourier-Stieltjes inversion

For the Fourier–Stieltjes transform in Section 3, there is a general inversion formula:

Theorem 5.1. *Let $F \in BV(\mathbb{R})$, and let $F^\circ(x) := \big(F(x+) - F(x-)\big)/2$. (Thus, in particular, $F^\circ(x) = F(x)$ when F is continuous at x.) Then, for any $x, y \in \mathbb{R}$,*

$$F^\circ(y) - F^\circ(x) = \frac{1}{2\pi} \lim_{A\to\infty} \int_{-A}^{A} \frac{e^{-\mathrm{i}ty} - e^{-\mathrm{i}tx}}{-\mathrm{i}t} \widetilde{F}(t)\,\mathrm{d}t. \tag{5.1}$$

Remark 5.2. The integrand in (5.1) is continuous, also at $t = 0$, and thus the integral exists (as a Lebesgue integral or Riemann integral) for every $A < \infty$. The integral $\int_{-\infty}^{\infty}$ does not always exist as an (absolutely convergent) Lebesgue integral; moreover, the symmetric limit in (5.1) in general cannot be replaced by the asymmetric limit $\lim_{A,B\to\infty} \int_{-A}^{B}$, see Examples A.4–A.5. △

Proof. When $\mathrm{d}F$ is a probability measure, this is known as the inversion formula for characteristic functions, see e.g. [3, Theorem 4.1.3]. (Usually stated for right-continuous F, but that does not matter since neither $\widetilde{F}$ nor F° is changed if F is replaced by the function $F(x+)$.) The general case follows by linearity.

Alternatively, this is equivalent to [8, Theorem XVI.(4.5)]. □

6. Mellin transform

If f is a function defined on $(0,\infty)$, its *Mellin transform* is defined by

$$f^*(s) := \int_0^\infty f(x) x^{s-1}\,\mathrm{d}x \tag{6.1}$$

for all complex s such that the integral converges absolutely. It is well known that the domain of such s is a strip $\mathcal{D} = \{s : \operatorname{Re} s \in J\}$ for some interval $J_f \subseteq \mathbb{R}$ (possibly empty or degenerate), and that $f^*(s)$ is analytic in the interior $\mathcal{D}^\circ$ of $\mathcal{D}$ (provided $\mathcal{D}^\circ$ is non-empty, i.e., J_f is neither empty nor degenerate).

We write $s = \sigma + \mathrm{i}\tau$, with $\sigma, \tau \in \mathbb{R}$. The change of variables $x = e^y$ gives, for $\sigma \in J_f$,

$$f^*(\sigma + \mathrm{i}\tau) = \int_{-\infty}^{\infty} f(e^y) e^{(\sigma+\mathrm{i}\tau)y}\,\mathrm{d}y = \int_{-\infty}^{\infty} f(e^y) e^{\sigma y} e^{\mathrm{i}\tau y}\,\mathrm{d}y = \widehat{g}_\sigma(\tau), \tag{6.2}$$

where

$$g_\sigma(y) := e^{\sigma y} f(e^y), \qquad y \in \mathbb{R}. \tag{6.3}$$

Note that $g \in L^1(\mathbb{R})$ when $\sigma \in J_f$.

Thus, the Mellin transform restricted to a vertical line can be regarded as a Fourier transform. Hence, the results in Section 4 can be applied, and yield in particular the following, for a (locally integrable) function f defined on $(0,\infty)$.

Theorem 6.1. *Let $\sigma \in J_f$, and suppose that $f^*(\sigma + \mathrm{i}\tau) \in L^1(\mathrm{d}\tau)$. Then f is a.e. equal to a continuous function, and if f is continuous (which we thus may assume), then*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^{-(\sigma+\mathrm{i}\tau)} f^*(\sigma + \mathrm{i}\tau)\,\mathrm{d}\tau = \frac{1}{2\pi\mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} x^{-s} f^*(s)\,\mathrm{d}s, \qquad x > 0. \tag{6.4}$$

Proof. Theorem 4.1 and (6.2)–(6.3) yield, for $y \in \mathbb{R}$,

$$f(e^y) = e^{-\sigma y} g_\sigma(y) = e^{-\sigma y} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}y\tau} \widehat{g}_\sigma(\tau) \,\mathrm{d}\tau$$
$$= e^{-\sigma y} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}y\tau} f^*(\sigma + \mathrm{i}\tau) \,\mathrm{d}\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(\sigma+\mathrm{i}\tau)y} f^*(\sigma + \mathrm{i}\tau) \,\mathrm{d}\tau, \qquad (6.5)$$

which yields (6.4). ☐

Theorem 6.2 (Dini test)**.** *Let $\sigma \in J_f$, and suppose that $x > 0$ is such that*

$$\int_{-\delta}^{\delta} \frac{|f(x+y) - f(x)|}{|y|} \,\mathrm{d}y < \infty \qquad (6.6)$$

for some $\delta > 0$. Then

$$f(x) = \frac{1}{2\pi\mathrm{i}} \lim_{A\to\infty} \int_{\sigma-\mathrm{i}A}^{\sigma+\mathrm{i}A} x^{-s} f^*(s) \,\mathrm{d}s. \qquad (6.7)$$

Proof. By Theorem 4.5 and (6.2)–(6.3). Note that

$$|g_\sigma(x+y) - g_\sigma(x)| \leqslant |e^{\sigma y} - 1| e^{\sigma x} |f(x)| + e^{\sigma(x+y)} \bigl| f(e^{x+y}) - f(e^x) \bigr|, \qquad (6.8)$$

and it follows easily that (6.6) implies that g_σ satisfies (6.6) (which equals (4.7)) at $\log x$, with some new δ. ☐

Corollary 6.3. *Let $\sigma \in J_f$, and suppose that f is locally Hölder(α) on $(0,\infty)$ for some $\alpha > 0$. Then* (6.7) *holds for every $x \in (0,\infty)$.*

The following results follow similary from Theorems 4.7, 4.8, 4.18, and their corollaries; we omit the details.

Theorem 6.4 (Dini–Lipschitz test)**.** *Let $\sigma \in J_f$, and suppose that f is continuous on $(0,\infty)$, and that*

$$\omega(\delta; f, I) = o\bigl(|\log\delta|^{-1}\bigr) \qquad \text{as } \delta \to 0 \qquad (6.9)$$

holds locally, i.e., for every compact interval $I \subset (0,\infty)$ (with implicit constant that may depend on I). Then (6.7) *holds for every $x \in (0,\infty)$, uniformly on every compact interval.*

Theorem 6.5 (Dirichlet–Jordan test)**.** *Let $\sigma \in J_f$, and suppose that $x \in (0,\infty)$ is such that f is of bounded variation on some interval $(x-\delta, x+\delta)$ with $\delta > 0$. Then*

$$\frac{f(x+) + f(x-)}{2} = \frac{1}{2\pi\mathrm{i}} \lim_{A\to\infty} \int_{\sigma-\mathrm{i}A}^{\sigma+\mathrm{i}A} x^{-s} f^*(s) \,\mathrm{d}s. \qquad (6.10)$$

In particular, if furthermore f is continuous at x, then (6.7) *holds.*

Corollary 6.6. *Let $\sigma \in J_f$, and suppose that f is continuous and locally of bounded variation, i.e., that f is of bounded variation in any compact interval interval $I \subset (0,\infty)$. Then* (6.7) *holds for every $x \in (0,\infty)$.*

Corollary 6.7. *Let $\sigma \in J_f$, and suppose that f is continuously differentiable. Then* (6.7) *holds for every $x \in (0,\infty)$.*

Theorem 6.8. *Let $\sigma \in J_f$, and suppose that f is continuous at $x \in (0,\infty)$. If the limit in* (6.7) *exists, then* (6.7) *holds.*

Further results, for example on asymmetric limits as in Theorem 4.16, are left to the reader.

Appendix A. Some counter examples

We give a few simple counter examples, showing that the symmetric limits in some results above cannot be replaced by general asymmetric limits.

In the first three examples, the symmetric limit (4.4) exists but not the general limit (4.10) in Section 4.3; these examples show that the conditions of Theorems 4.7 and 4.8 do not imply (4.10).

Example A.1. Let

$$f(x) := \begin{cases} e^{-x}, & x \geqslant 0, \\ 0, & x < 0. \end{cases} \tag{A.1}$$

Then

$$\widehat{f}(t) = \int_0^\infty e^{\mathrm{i}tx - x}\,\mathrm{d}x = \int_0^\infty e^{-(1-\mathrm{i}t)x}\,\mathrm{d}x = \frac{1}{1-\mathrm{i}t}, \qquad t \in \mathbb{R}. \tag{A.2}$$

f is smooth at all $x \neq 0$, and thus (4.4) and (4.10) hold there by Theorem 4.16. At $x = 0$, f is not continuous, but f is of bounded variation and Theorem 4.8 applies and yields

$$\frac{f(0+) + f(0-)}{2} = \frac{1}{2} = \frac{1}{2\pi} \lim_{A\to\infty} \int_{-A}^{A} \frac{1}{1-\mathrm{i}t}\,\mathrm{d}t \tag{A.3}$$

as is easily seen directly.

However, we have, for $A, B \geqslant 0$,

$$\int_{-A}^{B} \widehat{f}(t)\,\mathrm{d}t = \int_{-A}^{B} \frac{1}{1-\mathrm{i}t}\,\mathrm{d}t = \mathrm{i}\big(\log(1-\mathrm{i}B) - \log(1+\mathrm{i}A)\big) \tag{A.4}$$

which does not converge as $A, B \to \infty$; in fact, the imaginary part of (A.4) is, for $A, B \geqslant 1$,

$$\log|1-\mathrm{i}B| - \log|1+\mathrm{i}A| = \log B - \log A + O(1). \tag{A.5}$$

Hence (4.10) does not hold. Consequently, Theorem 4.8 cannot be strengthened to yield (4.10) without further assumptions.

In Lemma 4.14, we similarly see that, for $x = 0$, the imaginary parts of the integrals in (4.11) tend to $-\infty$ and $+\infty$ as $A \to \infty$ and $B \to \infty$; furthermore the imaginary part of the integral in (4.12) tends to $+\infty$. $\triangle$

Example A.2. The preceding example concerned a discontinuous function f; however, it may be modified to a continuous function with similar behaviour. Let

$$f(x) := g(x)e^{-x}, \qquad x \in \mathbb{R}, \tag{A.6}$$

where g is a continuous increasing function on $\mathbb{R}$ with $g(x) = 0$ for $x \leqslant 0$ and and $g(x) = 1$ for $x \geqslant 1$, which furthermore is continuously differentiable on $(0,1)$ with $g'(x)$ weakly decreasing. Obviously, $f \in L^1(\mathbb{R})$. Note that both $g(x)$ and e^{-x} are monotone and bounded on $[0,\infty)$, and thus of bounded variation there; consequently their product $f(x)$ is of bounded variation on $[0,\infty)$, and thus on $\mathbb{R}$. Moreover, since f is continuous, and continuously differentiable except at 0 and (possibly) 1, the measure $\mathrm{d}f$ is absolutely continuous, and thus $f(x)$ is an absolutely continuous function on $\mathbb{R}$.

We have, using integration by parts,

$$\widehat{f}(t)=\int_0^\infty g(x)e^{-(1-\mathrm{i}t)x}\,\mathrm{d}x=\frac{1}{1-\mathrm{i}t}\int_0^\infty g'(x)e^{-(1-\mathrm{i}t)x}\,\mathrm{d}x. \tag{A.7}$$

Let $I_0:=[0,\pi/(2t)]$ and $I_n:=[(n-\frac12)\pi/t,(n+\frac12)\pi/t]$ for $n\geqslant 1$. We have

$$\operatorname{Re}\int_0^\infty g'(x)e^{-(1-\mathrm{i}t)x}\,\mathrm{d}x=\int_0^\infty g'(x)e^{-x}\cos(tx)\,\mathrm{d}x=\sum_{n=0}^\infty\int_{I_n} g'(x)e^{-x}\cos(tx)\,\mathrm{d}x. \tag{A.8}$$

Let $a_n:=\int_{I_n} g'(x)e^{-x}\cos(tx)\,\mathrm{d}x$. Since $g'(x)\geqslant 0$, we have, for $t\geqslant 1$ and a constant $c:=e^{-1}\cos(1)>0$,

$$a_0\geqslant\int_0^{1/t} g'(x)e^{-x}\cos(tx)\,\mathrm{d}x\geqslant c\int_0^{1/t} g'(x)\,\mathrm{d}x=cg(1/t). \tag{A.9}$$

Furthermore, since $g'(x)$ is (weakly) decreasing,

$$|a_1|\leqslant\int_{I_1} g'(x)e^{-x}\,\mathrm{d}x\leqslant|I_1|g'(1/t)=\frac{\pi}{t}g'(1/t) \tag{A.10}$$

and, for $k\geqslant 1$,

$$a_{2k}+a_{2k+1}=\int_{I_{2k}}\Bigl(g'(x)e^{-x}-g'(x+\pi/t)e^{-x-\pi/t}\Bigr)\cos(tx)\,\mathrm{d}x\geqslant 0. \tag{A.11}$$

Consequently, (A.8) yields

$$\operatorname{Re}\int_0^\infty g'(x)e^{-(1-\mathrm{i}t)x}\,\mathrm{d}x=\sum_{n=0}^\infty a_n\geqslant a_0-|a_1|\geqslant cg(1/t)-\frac{\pi}{t}g'(1/t). \tag{A.12}$$

Furthermore, crudely,

$$\left|\operatorname{Im}\int_0^\infty g'(x)e^{-(1-\mathrm{i}t)x}\,\mathrm{d}x\right|\leqslant\left|\int_0^\infty g'(x)e^{-(1-\mathrm{i}t)x}\,\mathrm{d}x\right|\leqslant\int_0^\infty g'(x)\,\mathrm{d}x=g(1)=1. \tag{A.13}$$

We obtain by (A.7) and (A.12)–(A.13), with $Q(t):=\int_0^\infty g'(x)e^{-(1-\mathrm{i}t)x}\,\mathrm{d}x$, for large t,

$$\begin{aligned}\operatorname{Im}\widehat{f}(t)&=\operatorname{Im}\frac{1+\mathrm{i}t}{1+t^2}Q(t)=\frac{t}{1+t^2}\operatorname{Re}Q(t)+\frac{1}{1+t^2}\operatorname{Im}Q(t)\\&\geqslant\frac{c}{2t}g(1/t)-\frac{\pi}{t^2}g'(1/t)-\frac{1}{t^2}.\end{aligned} \tag{A.14}$$

For example, if we take

$$g(x)=\frac{1}{|\log x|}=\frac{1}{\log(1/x)} \tag{A.15}$$

for small $x>0$, then (for such x)

$$g'(x)=\frac{1}{x\log^2 x} \tag{A.16}$$

and thus (A.14) shows that for large $t>0$ and some (new) $c>0$,

$$\operatorname{Im}\widehat{f}(t)\geqslant\frac{c}{t\log t}. \tag{A.17}$$

Consequently,

$$\operatorname{Im}\int_0^B \widehat{f}(t)\,\mathrm{d}t \to +\infty \tag{A.18}$$

as $B \to \infty$, and thus (4.10) does not hold. Hence, this example shows that Theorem 4.8 cannot be strengthened to yield (4.10), even if we further assume that $f(x)$ is an absolutely continuous function on $\mathbb{R}$. $\triangle$

Example A.3. For another, slightly smoother, example, take again $f(x)$ as in (A.6), now with

$$g(x) = \frac{1}{\log(1/x)\log\log(1/x)} \tag{A.19}$$

for small $x > 0$, which entails

$$g'(x) \sim \frac{1}{x\log^2(1/x)\log\log(1/x)} \tag{A.20}$$

as $x \downarrow 0$. Thus (A.14) shows that as $t \to +\infty$, for some $c > 0$,

$$\operatorname{Im}\widehat{f}(t) \geqslant \frac{c + o(1)}{t\log t\log\log t}. \tag{A.21}$$

This function too is not integrable on any interval (a, ∞). Consequently,

$$\operatorname{Im}\int_0^B \widehat{f}(t)\,\mathrm{d}t \to +\infty \tag{A.22}$$

as $B \to \infty$, and thus (4.10) does not hold.

As Example A.2, this example shows that Theorem 4.8 cannot be strengthened to yield (4.10), even if we further assume that $f(x)$ is an absolutely continuous function on $\mathbb{R}$. Moreover, it is easily seen that the function f in this example has modulus of continuity

$$\omega(\delta; f, I) \sim g(\delta) = \frac{1}{\log(1/\delta)\log\log(1/\delta)} = o\Bigl(\frac{1}{\log(1/\delta)}\Bigr) \tag{A.23}$$

as $\delta \to 0$; hence, (4.8) holds. Thus, this example shows that also Theorem 4.7 cannot be strengthened to yield (4.10) without further assumptions. $\triangle$

The next two examples show that the symmetric limit in (5.1) in general cannot be replaced by a general asymmetric limit, and in particular that the corresponding integral $\int_{-\infty}^{\infty}$ is not always absolutely convergent.

Example A.4. Let $F(x) := \mathbf{1}\{x \geqslant 0\}$ (the Heaviside function), so that $\mathrm{d}F = \delta_0$, the (Dirac) point mass at 0. Then $\widetilde{F}(t) := \widehat{\mathrm{d}F}(t) = 1$ for all $t \in \mathbb{R}$. Take $x = -1$ and $y = 0$ in (5.1); then (5.1) says

$$\frac{1}{2} - 0 = \frac{1}{2\pi}\lim_{A\to\infty}\int_{-A}^{A} \mathrm{i}\frac{1 - e^{\mathrm{i}t}}{t}\,\mathrm{d}t, \tag{A.24}$$

where it is obvious that the integral $\int_{-\infty}^{\infty}$ is not absolutely convergent. Moreover,

$$\operatorname{Im}\int_{-A}^{B} \mathrm{i}\frac{1 - e^{\mathrm{i}t}}{t}\,\mathrm{d}t = \int_{-A}^{B} \frac{1 - \cos t}{t}\,\mathrm{d}t = \int_{A}^{B} \frac{1 - \cos t}{t}\,\mathrm{d}t = \log B - \log A + o(1) \tag{A.25}$$

as $A, B \to +\infty$. Hence, as asserted in Remark 5.2, the symmetric limit in (5.1) cannot be replaced by the asymmetric limit $\lim_{A,B\to\infty}\int_{-A}^{B}$. △

Example A.5. Let $f(x)$ be as in Example A.2 or A.3. Then f is integrable and absolutely continuous on $\mathbb{R}$, and thus $\widetilde{f}(t) = -\mathrm{i}t\widehat{f}(t)$ by (3.2) and the comments after it. Hence, the inversion formula (5.1) becomes

$$f(y) - f(x) = \frac{1}{2\pi}\lim_{A\to\infty}\int_{-A}^{A}\bigl(e^{-\mathrm{i}ty} - e^{-\mathrm{i}tx}\bigr)\widehat{f}(t)\,\mathrm{d}t. \tag{A.26}$$

We observe that this formula also can be obtained by taking the difference between (4.4) for the two points x and y; note that (4.4) holds by Theorem 4.8. Similarly, by formally taking the difference between (4.10) for x and y, we see that if they hold, then

$$\begin{aligned} f(y) - f(x) &= \frac{1}{2\pi}\lim_{A,B\to\infty}\int_{-A}^{B}\bigl(e^{-\mathrm{i}ty} - e^{-\mathrm{i}tx}\bigr)\widehat{f}(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\lim_{A,B\to\infty}\int_{-A}^{B}\frac{e^{-\mathrm{i}ty} - e^{-\mathrm{i}tx}}{-\mathrm{i}t}\widetilde{F}(t)\,\mathrm{d}t. \end{aligned} \tag{A.27}$$

Conversely, if we let $y := -1$, so f vanishes in a neighbourhood of y and thus (4.10) holds at y by Theorem 4.16, then (A.27) holds if and only if (4.10) holds. In particular, by Examples A.2 and A.3, (A.27) does not hold for $x = 0$ and $y = -1$. △

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