THE VOLUME OF A BALL IN $\mathbb{R}^n$

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Abstract. This note gives a simple proof of the standard formula for the volume of the unit ball in $\mathbb{R}^n$. We use a modification of the standard argument that integrates $e^{-|x|^2}$ over $\mathbb{R}^n$ and then uses polar coordinates.

Recall that the Gamma function is defined by

$$\Gamma(s) = \int_0^\infty y^{s-1} e^{-y}\, dy \tag{1}$$

for $s > -1$, and that $\Gamma(s+1) = s\Gamma(s)$, $s > -1$. (This is easily shown by integration by parts.) In particular, when s is a positive integer, induction yields $\Gamma(s) = (s-1)!$. We may thus extend the factorial function by the definition $x! = \Gamma(x+1)$.

Theorem. *Let $n \geq 1$. A ball with radius r in $\mathbb{R}^n$ has volume $v_n r^n$, where*

$$v_n = \frac{\pi^{n/2}}{(n/2)!} = \frac{\pi^{n/2}}{\Gamma(n/2+1)}. \tag{2}$$

Proof. It is clear by homogeneity that a ball $B(x_0, r)$ of radius r has volume $v_n r^n$ where v_n is the volume of the unit ball.

Write the points in $\mathbb{R}^{n+1}$ as (x, y) with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, and let D be the subset of $\mathbb{R}^{n+1}$ given by $D = \{(x, y) : |x|^2 < y\}$. Integrate e^{-y} over D, and use Fubini's theorem. Integrating first over $x \in \mathbb{R}^n$ yields, using (1),

$$\int_{y=0}^\infty \int_{|x^2|<y} e^{-y}\, dx\, dy = \int_0^\infty v_n y^{n/2} e^{-y}\, dy = v_n \Gamma(n/2+1), \tag{3}$$

since the inner integral is over the ball $B(0, y^{1/2})$ of volume $v_n y^{n/2}$.

On the other hand, integrating first over y yields

$$\int_{x\in\mathbb{R}^n} \int_{y=|x|^2}^\infty e^{-y}\, dy\, dx = \int_{x\in\mathbb{R}^n} e^{-|x|^2}\, dx = \prod_{i=1}^n \int_{-\infty}^\infty e^{-x_i^2}\, dx_i = I^n, \tag{4}$$

where we define

$$I = \int_{-\infty}^\infty e^{-x^2}\, dx.$$

Consequently, Fubini's theorem shows that the values in (3) and (4) are equal, i.e.

$$v_n \Gamma(n/2+1) = I^n. \tag{5}$$

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In the special case $n = 2$, we have $v_2 = \pi$. Hence (5) with $n = 2$ yields $\pi = I^2$ and $I = \pi^{1/2}$. Consequently, (2) follows by using (5) again. $\square$

Remark. We have also given a proof of the standard formula

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \pi^{1/2}; \tag{6}$$

again, our argument is a modification of the standard one.

Corollary. *The surface area ω_n of the unit ball is*

$$\omega_n = nv_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{7}$$

Proof. Polar coordinates yields

$$v_n = \int_0^1 \omega_n r^{n-1}\,dr = \frac{\omega_n}{n}.$$

Equivalently, computing the area of a sphere with radius r,

$$\omega r^{n-1} = \frac{d}{dr}\big(v_n r^n\big) = nv_n r^{n-1}.$$

$\square$

Example 1. In the case $n = 1$, the unit ball $B(0,1)$ is the interval $(-1,1)$ of length 2; thus $v_1 = 2$. This is by (2) equivalent to $\Gamma(\frac{3}{2}) = \pi^{1/2}/2$, or, equivalently, to the well-known formula $\Gamma(\frac{1}{2}) = \pi^{1/2}$.

Example 2. For $n = 3$ we have, see Example 1, $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{4}\pi^{1/2}$, and thus the theorem yields $v_3 = 4\pi/3$. For $n = 4$ we have $v_4 = \pi^2/2$.

n	1	2	3	4	5	6
v_n	2	π	$\frac{4\pi}{3}$	$\frac{\pi^2}{2}$	$\frac{8\pi^2}{15}$	$\frac{\pi^3}{6}$
ω_n	2	2π	4π	$2\pi^2$	$\frac{8\pi^2}{3}$	π^3

TABLE 1. Some numerical values

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