A PROOF OF A HYPERGEOMETRIC IDENTITY

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1. Introduction

Johan Kåhrström discovered the following identity experimentally in 2006, checking a large number of cases with computer:

$$\sum_{i=\max(0,2k-l-n)}^{\min(k,m-l)} (-1)^i \frac{(m-i)!\,(n-k+i)!}{i!\,(k-i)!\,(m-l-i)!\,(n+l-2k+i)!} = (-1)^{k+l}, \tag{1}$$

for all integers k, l, m, n with $0 \le l \le k \le \min(m, n)$.

Some reformulations are given in (2)–(4) below. For the background and application to certain bilinear froms on $\mathfrak{sl}_2$-modules, see [2].

Johan Kåhrström (then a graduate student) and his supervisor Volodymyr Mazorchuk proved several special cases and then asked for a general proof. Several different proofs were quickly (and independently) found by Herbert Wilf & Doron Zeilberger, Christian Krattenthaler, Tobias Ekholm, Ganna Kudryavtseva and myself, see [2] for brief descriptions. The purpose of this note is to present my elementary proof.

2. Some reformulations

The identity (1) may be rewritten in several ways; we give some here.

The version first presented to me was

$$\sum_{i=\max(0,l+k-m-n)}^{\min(k,l)} (-1)^i \frac{(n-i)!\,(m+i)!}{(n+m-k-l+i)!\,(k-i)!\,(l-i)!\,i!} = (-1)^{n+k+l}, \tag{2}$$

for all integers k, l, m, n with $m \ge 0$ and $n \ge l \ge n - k \ge 0$.

If we first substitute $l \to n - l$ and $m \to m - k$ in (2) and then interchange m and n, we obtain (1) so the two versions are equivalent. (The conditions on k, l, m, n also translate.)

If we in (2) substitute $k = b+c$, $l = b+d$, $n = b+c+d$, $m = a$, we have $a = m$, $b = k + l - n$, $c = n - l$, $d = n - k$, and the conditions translate to $a, b, c, d \ge 0$. Hence, letting also $i = b + j$, (1) and (2) are also equivalent to the symmetric formula

$$\sum_{j=-\min(a,b)}^{\min(c,d)} (-1)^j \frac{(a+b+j)!\,(c+d-j)!}{(a+j)!\,(b+j)!\,(c-j)!\,(d-j)!} = 1, \qquad a, b, c, d \ge 0. \tag{3}$$

Date: 10 January 2011 (original proof 28 October 2006).

The summation limits in (1)–(3) make all arguments of the factorials non-negative integers. However, we can allow negative arguments in the denominator, with the standard interpretation $1/i! = 0$ for $i < 0$; thus we may change the lower limit in (1) or (2) to, for example, 0, and the upper limit to, for example, k, $m-l$ or m in (1) and k, l or n in (2); the value of the sum will not change since all additional terms are 0.

The ratios of successive summands in (1)–(3) are rational functions of i, and the summands can thus be seen as hypergeometric terms [1, Section 5.7]. If further $2k \le n+l$, we can rewrite (1) as the hypergeometric evaluation [2]

$$ {}_3F_2\left(\begin{matrix} n-k+1,\ -k,\ l-m \\ -m,\ n+l-2k+1 \end{matrix} \,\middle|\, 1 \right) = (-1)^{k+l} \frac{k!\,(m-l)!\,(n+l-2k)!}{m!\,(n-k)!}, \tag{4} $$

for all integers k, l, m, n with $0 \le l \le k \le \min(m, n)$ and $2k \le n+l$, where the hypergeometric function on the left hand side is a polynomial of degree $\min(k, m-l)$.

3. Proof of the identity

We show (2), which is equivalent to (1) and (3) by simple changes of variables, see Section 2.

Let

$$ p(i) = \frac{(n-i)!}{(l-i)!} = \prod_{j=1}^{n-l} (l-i+j) $$

and

$$ q(i) = \frac{(m+i)!}{(n+m-k-l+i)!} = \prod_{j=1}^{k+l-n} (n+m-k-l+i+j); $$

these are polynomials in i of degrees $n-l \ge 0$ and $k+l-n \ge 0$ with leading terms $(-1)^{n-l} i^{n-l}$ and i^{k+l-n}, respectively. The product $p(i)q(i)$ is thus a polynomial in i of degree k with leading term $(-1)^{n-l} i^k$.

Let the sum in (2) be denoted by S. As discussed in Section 2, we can change the summation limits to $\sum_0^k$, and then the sum may be written

$$ S = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} p(i) q(i). \tag{5} $$

Let Δ be the difference operator $\Delta f(x) = f(x+1) - f(x)$, and note that if f is any polynomial of degree k with leading term $a_k x^k$, then $\Delta^k f(x) = a_k k!$ for every

x. Thus $\Delta^k(pq)(x) = (-1)^{n-l}k!$ and, by (5),

$$\begin{aligned}
S &= \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}p(i)q(i) \\
&= \frac{1}{k!}(-\Delta)^k(pq)(0) \\
&= \frac{1}{k!}(-1)^k(-1)^{n-l}k! \\
&= (-1)^{k+l+n}.
\end{aligned}$$

□

References

[1] R.L. Graham, D.E. Knuth & O. Patashnik, *Concrete Mathematics.* 2nd ed., Addison–Wesley, Reading, Mass., 1994.

[2] Johan Kåhrström, Bilinear forms on $\mathfrak{sl}_2$-modules and a hypergeometric inequality. Tech. report 2007:8, Dept. of Mathematics, Uppsala University.

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