

ANOTHER NOTE ON THE DROOP QUOTA AND ROUNDING

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1. INTRODUCTION

Consider an election where v votes are cast and there are s seats to be filled. The standard Droop quota, as defined by Henry Droop [1], is $v/(s+1)$ rounded up to the nearest strictly larger integer, i.e., $\lfloor v/(s+1) \rfloor + 1$. This is the quota used in many STV elections, but it has later been realized that there is no compelling reason to use an integer quota except in versions of STV that only transfer whole votes (perhaps in the traditional way by physically moving ballots by hand between different stacks). Thus the exact (unrounded) Droop quota $v/(s+1)$ is also used in some modern versions of STV, for example the ERS rules [3].

The two versions of the Droop quota were compared by Lundell and Hill [2], who concluded that the exact version generally is better. (To avoid some problems, they then also recommended that candidates are elected only when they *exceed* the quota, a rule suggested by Mann [4].) The purpose of the present note is to add two further reasons for using the exact Droop quota whenever possible.

Remark 1. As discussed by Lundell and Hill [2], the terminology is varying, with other names sometimes used for the exact version. (For example “NB quota” in [4].) I agree with them on using “Droop quota” for both versions; I will distinguish the two versions by calling them “exact Droop quota” and “rounded Droop quota”. (Strictly speaking, the “rounded” version is not rounded, since it is increased even when the exact quota happens to be an integer.)

Remark 2. In practice, the calculations are usually done to a fixed number of decimal places, such as 2 for the ERS rules [3]. (Similarly, the Irish senate rules, where each vote is counted as 1000, may be regarded as doing calculations to 3 decimal places.) In such cases, even the exact Droop quota is rounded (usually upwards) to the used precision. I will disregard this and assume that calculations are done exactly, for example by rational arithmetic or by using a sufficiently large precision, preferably with guard digits. (This is discussed further by Lundell and Hill [2].)

Date: 20 March, 2011.

In my opinion, if the result of an election depends on the chosen accuracy of the calculations, only the result that agrees with exact (rational) calculations can be defended.

2. MONOTONICITY

Monotonicity means that increased support cannot harm a candidate. It is well-known that monotonicity can fail with STV: in some situations, a candidate may lose a seat by getting additional support (either by getting additional votes or by moving up on some ballots, perhaps to become the first preference). Woodall [5] discusses several different cases and examples; see also Woodall [6] for a detailed discussion of one example. To use the rounded Droop quota adds further possibilities of violations of monotonicity (of Woodall's type *mono-add-plump* and *mono-add-top* [5]): a candidate or party can lose a seat by attracting a new voter. Example:

(5 seats)	
ABCDE	500
F	99

The rounded Droop quota is $\lfloor 599/6 \rfloor + 1 = 100$, and A, B, C, D, E are elected. Suppose now that the ABCDE party gets a new voter, raising their vote to 501. Then the quota becomes $\lfloor 600/6 \rfloor + 1 = 101$; A, B, C, D are elected but the surplus transferred to E is only 97, so E is eliminated and F is elected to the final seat. (With the exact Droop quota, A, B, C, D, E are elected in both cases, with votes strictly exceeding the quota.)

For simplicity, this example uses a party (coalition) with all its voters voting in the same way (as is approximatively the case in Australian Senate elections), but note that the result remains the same if the ABCDE voters vote for these candidates in different orders of preference (except that someone else than E may be the one losing a seat). In particular, adding a single ballot E to the example above would be just as bad for E.

The reason for this counterintuitive behaviour is that the quota increases in steps of 1 (or not at all); in this example, the exact Droop quota increases (from $599/6$ to $600/6$) by a factor $1/599$, but the rounded Droop quota increases by a factor $1/100$. The new vote increases the votes for party ABCDE by a factor $1/500$. Thus, using the exact Droop quota, the ratio *votes/quota* goes up, as it always does, but with the rounded Droop quota, the ratio goes down in this example because the quota increases by a larger factor than the number of votes for ABCDE. In this example, the effect is that one seat is lost.

Note that although rounding looks innocuous when the number of votes is large, the problem is not limited to small elections. For example, we can modify the example above to 5 000 000 and 999 999 votes with the same result.

3. HOMOGENEITY

Homogeneity means that the result only depends on the *proportion* of ballots of each possible type [5]. STV with the exact Droop quota is obviously homogeneous. Woodall [5] regards STV as homogeneous; he notes that finite precision calculation might give violations but sees this as a minor practical problem. I agree as long as the exact Droop quota is used, but when the rounded Droop quota is used as a matter of principle (perhaps out of tradition), I think that it is justified to regard STV as non-homogeneous. Example:

(9 seats)	
ABCDEF	71
XYZ	30

The rounded Droop quota is $\lfloor 101/10 \rfloor + 1 = 11$, so first A, B, C, D, E, F and X, Y are elected, leaving G with 5 transferred votes and Z with 8; thus the final seat goes to Z and the ABCDEF party gets 6 seats. If all votes are multiplied by 10, then the quota becomes $\lfloor 1010/10 \rfloor + 1 = 102$, and at the end G has 98 votes against Z with 96; thus G takes the final seat, giving the ABCDEF party 7 seats. (This is also the outcome with the exact Droop quota, in both cases. It is further the only outcome consistent with the *Droop proportionality criterion (DPC)* in [5]; for another example of DPC violation with the rounded Droop quota see [2].)

Again, the effect exists also for large elections; we can modify the example to 7 000 001 and 3 000 000 votes.

4. CONCLUSION

Failure of monotonicity is unfortunately an unavoidable problem for STV. To use the rounded Droop quota adds to this problem. While the added cases might be of minor practical importance, it seems better to reduce the problem as much as possible by using the exact Droop quota.

The property of homogeneity is perhaps less important, but it is certainly desirable and again this is an argument for using the exact quota.

Note that in practice, the exact and rounded Droop quotas usually give the same result, especially in large elections. But in the cases where the difference matters, the exact quota seems to be the better choice; see Lundell and Hill [2] for further examples and discussions. I thus agree with the conclusion of Lundell and Hill [2] that the exact quota should be used when possible.

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