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Mini-Workshop: Probability Theory on Trees and Analysis of Algorithms

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Introduction by the Organisers

The Mini-Workshop: Probability Theory on Trees and Analysis of Algorithms workshop, organised by Max Muster (München) and Bill E. Xample (New York) was held January 11-17. This meeting was well attended with over 30 participants with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds ...

Workshop on Mini-Workshop: Probability Theory on Trees and Analysis of Algorithms

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Abstracts

Random records and cuttings in trees
Svante Janson

We consider random cutting down of rooted trees, defined as follows [6]. If T is a rooted tree with number of vertices $|T| \geq 2$, we make a random cut by choosing one edge at random. Delete this edge so that the tree separates into two parts, and keep only the part containing the root. Continue recusively until only the root is left. We let $X(T)$ denote the (random) number of cuts that are performed until the tree is gone.

The same random variable appears when we consider records in a tree. Let each edge e have a random value λ_e attached to it, and assume that these values are i.i.d. with a continuous distribution. Say that a value λ_e is a *record* if it is the largest value in the path from the root to e. Then the number of records is again given by $X(T)$, as is easily seen.

There are also vertex versions of cuttings and records. For cuttings, choose a vertex at random and destroy it together with all its descendants. Continue until the root is chosen and thus the whole tree is destroyed. For records, we assign i.i.d. values λ_v to the vertices, and define a record as above. Again, there is an equivalence between cuttings and records. The edge and vertex versions are closely related, and the results are essentially the same.

These random variables can be studied both for deterministic trees and for random trees.

If the tree is a path, we have the classical record problem studied by Rényi, [9].

Our main results are for the case when the tree T_n itself is random, more precisely a random conditioned Galton–Watson tree (also known as simply generated tree) with n vertices. (It is well-known that examples include random labelled trees and random binary trees.) Since both the records (or cutting) and the tree now are random, $X(T_n)$ can be regarded in (at least) two ways.

First, we can regard $X(T_n)$ as a random variable, obtained by picking a random tree T_n and then a random cutting of it. This point of view has been taken by Meir and Moon [6] (mean and variance for Cayley trees), Chassaing and Marchand [3] (asymptotic distribution for Cayley trees), Panholzer [7, 8] (asymptotic distribution for some special families of simply generated trees, and for non-crossing trees). We extend these results to all conditioned Galton–Watson trees. (All unspecified limits are as $n \to \infty$.)

Theorem 1. *Let T_n be a conditioned Galton–Watson tree of size n, defined by an offspring distribution ξ with mean $\mathbb{E}\,\xi = 1$ and finite variance $\sigma^2 > 0$. Then,*

$$\frac{X(T_n)}{\sigma n^{1/2}} \xrightarrow{\mathrm{d}} Z,$$

where Z has a Rayleigh distribution with density $xe^{-x^2/2}$, $x > 0$. Moreover, if $\mathbb{E}\,\xi^m < \infty$ for every $m > 0$, then all moments converge and thus, for every $r > 0$,

$$\mathbb{E}\,X(T_n)^r \sim \sigma^r n^{r/2}\,\mathbb{E}\,Z^r = 2^{r/2}\sigma^r\Gamma\big(\tfrac{r}{2}+1\big)n^{r/2}.$$

The other point of view is to study $X(T_n)$ as a random variable conditioned on T_n. In other words, we consider the random procedure in two steps: First we choose a random tree $T = T_n$. Then we keep this tree fixed and consider random cuttings of it; this gives a random variable $X(T)$ with a distribution that depends on T. Normalizing as in the theorem above, we consider the distribution of $\sigma^{-1}n^{-1/2}X(T_n)$ given T_n; this is thus a random probability distribution. We then can show that this random probability distribution converges in distribution to a random probability distribution (that does not depend on ξ); this random distribution has moments that can be expressed as functionals of a Brownian excursion.

The proofs are based on Aldous' theory of the continuum random tree [1, 2].

Finally, we study the case when the tree is a (deterministic) complete binary tree of size n. In this case, both the methods and results are different. There is now a periodicity in the result. This is not surprising for complete binary trees, but it is a bit surprising that the periodicity is in the fractional part $\{\lg n - \lg\lg n\}$.

Theorem 2. *Suppose that $n \to \infty$ such that $\{\lg n - \lg\lg n\} \to \gamma \in [0,1]$. Then*

$$\Big(X(T_n) - \frac{n}{\lg n} - \frac{n\lg\lg n}{\lg^2 n}\Big) \Big/ \frac{n}{\lg^2 n} \xrightarrow{\mathrm{d}} -W_\gamma$$

where W_γ has an infinitely divisible distribution with characteristic function

$$\mathbb{E}\,e^{\mathrm{i}tW_\gamma} = \exp\Big(\mathrm{i}f(\gamma)t + \int_0^\infty \big(e^{\mathrm{i}tx} - 1 - \mathrm{i}tx\mathbf{1}[x<1]\big)\,d\nu_\gamma(x)\Big),$$

where $f(\gamma) := 2^\gamma - 1 - \gamma$ and the Lévy measure ν_γ is supported on $(0,\infty)$ and has density

$$\frac{d\nu_\gamma}{dx} = 2^{\{\lg x+\gamma\}}x^{-2}.$$

The strategy of the proof is to approximate $X(T_n)$ by a sum of independent random variables derived from $\{\lambda_e\}$; it turns out that only exceptionally small values at level $\approx \lg\lg n$ have a significant influence on $X(T_n)$. We will then apply a classical limit theorem for triangular arrays.

For details, see [4, 5].

References

[1] D. Aldous, The continuum random tree II: an overview. *Stochastic Analysis (Proc., Durham, 1990)*, 23–70, London Math. Soc. Lecture Note Ser. 167, Cambridge Univ. Press, Cambridge, 1991.

[2] D. Aldous, The continuum random tree III. *Ann. Probab.* **21**, no. 1, 248–289.

[3] P. Chassaing & R. Marchand. In preparation.

[4] S. Janson, Random cutting and records in deterministic and random trees. Preprint, 2003. Available from `http://www.math.uu.se/~svante/papers`

[5] S. Janson, Random records and cuttings in complete binary trees. In *Mathematics and Computer Science III, Algorithms, Trees, Combinatorics and Probabilities (Vienna 2004)*, Eds. M. Drmota, P. Flajolet, D. Gardy, B. Gittenberger, Birkhäuser, Basel, 2004,
[6] A. Meir & J.W. Moon, Cutting down random trees. *J. Australian Math. Soc.* **11** (1970), 313–324.
[7] A. Panholzer, Cutting down very simple trees. Preprint, 2003.
[8] A. Panholzer, Non-crossing trees revisited: cutting down and spanning subtrees. *Proceedings, Discrete Random Walks 2003*, Cyril Banderier and Christian Krattenthaler, Eds., *Discr. Math. Theor. Comput. Sci.* **AC** (2003), 265–276.
[9] A. Rényi, (1962). On the extreme elements of observations. *MTA III, Oszt. Közl.* **12** (1962) 105–121. Reprinted in *Collected Works*, Vol III, pp. 50-66, Akadémiai Kiadó, Budapest, 1976.

Branching random walks on trees and the Brownian snake

Svante Janson
(joint work with Jean-François Marckert)

Consider a rooted ordered finite tree T where each edge is assigned a real number called *value*. We then let, for every vertex v, S_v be the sum of the values of the edges along the paths from the root to v. We will assume that the values of the edges are independent random variables with a common distribution. We let Y denote one of these values.

We study the case when tree itself is random, more precisely a random conditioned Galton–Watson tree (or simply generated tree) with n vertices.

Each S_v is a sum of i.i.d. variables, and the number of terms is the depth of v, which typically is of the order $n^{1/2}$. Hence, by the central limit theorem, S_v is typically of order $n^{1/4}$ if $\mathbb{E}\,Y = 0$ and $\operatorname{Var} Y < \infty$, but S_v is typically of order $n^{1/2}$ if $\mathbb{E}\,Y \neq 0$.

To study the case $\mathbb{E}\,Y = 0$ in more detail, we take the values $n^{-1/4}S_v$ in the order given by the depth first walk on the tree, extend this by linear interpolation to a continuous function, and rescale to obtain a function $r_n(s)$ on $[0, 1]$.

Before proceeding, recall that the Brownian snake [4, 7] is a random function that can be desribed as follows: Let $\zeta(s)$ be a random non-negative function on a given interval I; in our case, ζ is a standard Brownian excursion on $I = [0, 1]$. (Another common version is with ζ reflected Brownian motion on $[0,\infty)$.) Then the corresponding Brownian snake $W(s, t)$ is a random function of two variables (or stochastic field), $s \in I$ and $t \geq 0$, such that conditioned on ζ, $W(s, t)$ is a Gaussian process with mean $\mathbb{E}\,W(s, t) = 0$ and covariance function, if $s_1 \leq s_2$,

$$\operatorname{Cov}\bigl(W(s_1, t_1), W(s_2, t_2)\bigr) = \min\Bigl(t_1,\, t_2,\, \inf_{u\in[s_1,s_2]} \zeta(u)\Bigr).$$

In particular, for fixed s, $t \mapsto W(s, t)$ is a Brownian motion stopped at $t = \zeta(s)$. Two such Brownian motions for s_1 and s_2 are identical for $t \leq \inf_{u\in[s_1,s_2]} \zeta(u)$, and then evolve independently.

Let $r(s) := W(s, \zeta(s))$ (known as the head of the snake).

Assume $\mathbb{E}\,Y = 0$ and $\operatorname{Var} Y = 1$, and let $n \to \infty$. It has been shown by Markert and Mokkadem [8] and Chassaing and Schaeffer [3] in special cases, and by Gittenberger [5] in general, assuming $\mathbb{E}\,|Y|^{8+\varepsilon} < \infty$, that then $r_n \xrightarrow{\mathrm{d}} r$ in $C[0,1]$ (i.e. in the uniform topology).

We want to weaken the moment condition on Y as far as possible. First, it is easy to see that $r_n \to r$ in the sense of finite-dimensional distributions without further assumptions. Weak convergence in $C[0,1]$ is equivalent to the convergence of the finite-dimensional distributions together with tightness. Often, the tightness is a technical nuisance that can be verified with more or less work. Here, that is not the case and we need a stronger condition on Y in order to obtain convergence.

Theorem 3. *Assume $\mathbb{E}\,Y = 0$. Then $r_n \xrightarrow{\mathrm{d}} r$ in $C[0,1]$ if and only if $\mathbb{P}(|Y| \geq y) = o(y^{-4})$.*

In particular, this holds if $\mathbb{E}\,Y^4 < +\infty$, and no weaker moment condition suffices.

When this condition fails, we do not have convergence because the extreme values of Y will cause thin spikes in r_n. These spikes are at random positions, and are therefore not seen by the finite-dimensional distributions. We also have convergence $r_n \xrightarrow{\mathrm{d}} r$ in, for example, $L^2[0,1]$.

If Y have tails that are exactly of the order y^{-4}, then r_n converges in distribution to a "hairy snake", i.e. a Brownian snake with hairs added. The hairs are vertical line segments going up or down from the snake; their positions and lengths are given by a Poisson process, so the number of them is infinite, but there is only a finite number of them with length larger than a given number.

Note that this limiting object, the hairy snake, is *not* a function, and therefore the convergence does not take place in $C[0,1]$. Instead we identify continuous functions on $[0,1]$ with their graphs, and obtain convergence in the space of compact subsets of $\mathbb{R}^2$. This seems to be a novel type of convergence in this context.

If the tails of Y are even larger, the spikes dominate and we may after suitable rescaling obtain convergence to a flat (or dead) hairy snake, with hairs as above added to the line segment from $(0,0)$ to $(1,0)$, and thus without the Brownian part.

The proofs are based on Aldous' theory of the continuum random tree [1, 2]. For details, see [6].

References

[1] D. Aldous, The continuum random tree II: an overview. *Stochastic Analysis (Proc., Durham, 1990)*, 23–70, London Math. Soc. Lecture Note Ser. 167, Cambridge Univ. Press, Cambridge, 1991.

[2] D. Aldous, The continuum random tree III. *Ann. Probab.* **21** (1993), no. 1, 248–289.

[3] P. Chassaing & G. Schaeffer, Random planar lattices and integrated superBrownian excursion. *Prob. Theo. Rel. Fields*, to appear.

[4] T. Duquesne & J.F. Le Gall, Random trees, Lévy processes and spatial branching processes. *Astérisque* **281** (2002).

[5] B. Gittenberger, A note on "State spaces of the snake and its tour – Convergence of the discrete snake" by J.-F. Marckert and A. Mokkadem. *J. Theo. Prob.*, **16** (2003), 1063–1067.
[6] S. Janson & J.-F. Marckert, Convergence of discrete snakes. Preprint, 2003. Available from `http://www.math.uu.se/~svante/papers`
[7] J.F. Le Gall, *Spatial branching processes, random snakes and partial differential equations.* Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 1999.
[8] J.F. Marckert & A. Mokkadem, States spaces of the snake and of its tour – Convergence of the discrete snake. *J. Theo. Prob.*, **16** (2003), 1015–1046.

Asymptotics for tails and moments

Svante Janson
(joint work with Jim Fill and probably others)

This talk describes some preliminary results and work in progress on tail estimates for certain random variables that appear as limits in problems coming from random trees or analysis of algorithms.

First, it is pointed out that a general theorem by Kasahara and Davies, see [3] gives the equivalence, under quite general conditions, between asymptotics for tail probabilities of a random variable, asymptotics for moments, and asymptotics for the moment generating function.

Secondly, we presented a very recent result (found during this workshop after yesterday's talk by Uwe Rösler) giving estimates for the moments (and thus tail estimates) for solutions to fixed-point equations of the Max-recursive type, for example for the limit variable for MAX-FIND.

Thirdly, we discussed a family of limit random variables that arises in the study of Catalan trees; most of them can be expressed as functionals of a Brownian excursion (using the methods of Aldous [1, 2]) and then the asymptotics can be found using the well-known large deviation principle for Brownian motion. This involves finding a certain constant as the solution to a non-linear variational problem; in some case we can solve this exactly, but in others we only have upper and lower bounds for this constant (differing by a few percent).

References

[1] D. Aldous, The continuum random tree II: an overview. *Stochastic Analysis (Proc., Durham, 1990)*, 23–70, London Math. Soc. Lecture Note Ser. 167, Cambridge Univ. Press, Cambridge, 1991.
[2] D. Aldous, The continuum random tree III. *Ann. Probab.* **21** (1993), no. 1, 248–289.
[3] Ph. Chassaing & S. Janson, The center of mass of the ISE and the Wiener index of trees. *Electronic Comm. Probab.*, to appear.

The contraction method in infinite-dimensional spaces

Svante Janson
(joint work with Michael Drmota and Ralph Neininger)

The contraction method has since it was introduced by Rösler [4] found a number of applications, in particular in the analysis of algorithms. It has been extended to random vectors in finite-dimensional spaces [2, 3] and to infinite-dimensional Hilbert spaces [1] (with the ℓ_2-metric).

We discuss the possibility to use the Zolotarev ζ_s metric in a Banach space. The main problem is completeness. We cannot show this in Banach spaces in general, or in natural spaces like $C[0,1]$, but we can do it in Hilbert spaces.

Theorem 4. *Let $s > 0$ and let H be a separable Hilbert space. The Zolotarev metric ζ_s is a complete metric on the set of all H-valued random variables with given k:th moments for $k = 0, 1, \dots, \lceil s \rceil - 1$. (The k:th moment lies in the k:th projective tensor power $H^{\hat{\otimes} k}$.)*

Using this theorem, the proof in [3] works without changes also for Hilbert space valued variables.

In a tentative application, that is still work in progress, we apply this to the profile of random binary trees (and other trees). The relevant random variables can be represented as continuous functions on an interval $[a, b]$. It is desirable to obtain uniform convergence in $C[a, b]$, but this Banach space is not a Hilbert space and we do not know whether the Zolotarev metric is complete for it. Instead, we observe that the random functions in this case may be extended to analytic functions in a domain in the complex plane; we then use a Bergman space of analytic functions, which is a Hilbert space. We can thus use the contraction method there and obtain convergence in the Bergman space, which implies uniform convergence on compact subsets, and in particular on $[a, b]$.

References

[1] R.M. Burton & U. Rösler, An L_2 convergence theorem for random affine mappings. *J. Appl. Probab.* **32** (1995), 183–192.

[2] R. Neininger, On a multivariate contraction method for random recursive structures with applications to Quicksort. *Random Structures Algorithms* **19** (2001), 498–524.

[3] R. Neininger & L. Rüschendorf, A general limit theorem for recursive algorithms and combinatorial structures. *Ann. Appl. Probab.* **14** (2004), 378–418.

[4] U. Rösler, A limit theorem for "Quicksort". *RAIRO Inform. Théor. Appl.*, **25** (1991), 85–100.

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