Simply generated trees and conditioned Galton–Watson trees

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The trees that we consider are rooted and ordered (= plane); thus each node v has a number of children, ordered in a sequence $v_1, \dots, v_d$, where $d = d(v) \geq 0$ is the *outdegree* of v. (See [1] for more information on these and other types of trees; the trees we consider are there called *planted plane trees.*) We let $\mathfrak{T}_n$ denote the set of all ordered rooted trees with n nodes (including the root) and let $\mathfrak{T}_{\mathrm{f}} := \bigcup_{n=0}^{\infty} \mathfrak{T}_n$ be the set of all finite ordered rooted trees.

Let $(w_k)_{k\geq 0}$ be a fixed *weight sequence* of non-negative real numbers. We then define the *weight* of a tree $T \in \mathfrak{T}_{\mathrm{f}}$ by

$$w(T) := \prod_{v \in T} w_{d(v)},$$

taking the product over all nodes v in T. Trees with such weights are called *simply generated trees* and were introduced by Meir and Moon [5]. To avoid trivialities, we assume that $w_0 > 0$ and that there exists some $k \geq 2$ with $w_k > 0$.

We let $\mathcal{T}_n$ be the random tree obtained by picking an element of $\mathfrak{T}_n$ at random with probability proportional to its weight, i.e.,

$$\mathbb{P}(\mathcal{T}_n = T) = \frac{w(T)}{Z_n}, \qquad T \in \mathfrak{T}_n,$$

where the normalizing factor Z_n, known as the *partition function*, is given by

$$Z_n := \sum_{T \in \mathfrak{T}_n} w(T).$$

We consider only n such that $Z_n > 0$.

One particularly important case is when $\sum_{k=0}^{\infty} w_k = 1$, so the weight sequence (w_k) is a probability distribution on $\mathbb{Z}_{\geq 0}$. In this case, the random tree $\mathcal{T}_n$ is the same as the random Galton–Watson tree $\mathcal{T}$ with offspring distribution (w_k) conditioned on $|\mathcal{T}| = n$. In this case the random tree $\mathcal{T}_n$ is thus called a *conditioned Galton–Watson tree.*

The distribution of the tree $\mathcal{T}_n$ does not change if w_k is replaced by $\widetilde{w}_k := ab^k w_k$ for some $a, b > 0$. Using this, we can always reduce to one of the three following cases, where $\rho \in [0, \infty]$ is the radius of convergence of the generating function $\Phi(x) := \sum_{k=0}^{\infty} w_k x^k$ and $\mu := \sum_{k=0}^{\infty} k w_k = \Phi'(1)$:

(i) Critical Galton–Watson: (w_k) a probability distribution with mean $\mu = 1$. (In this case $\rho \geq 1$.)

(ii) Subcritical Galton–Watson: (w_k) a probability distribution with mean $\mu < 1$ and $\rho = 1$.

(iii) Not Galton–Watson: $\rho = 0$.

Case (i) is the standard case, and most work has been done for this case only (often with additional conditions like $\operatorname{Var} \xi < \infty$).

Probabilists, including myself, have often dismissed the remaining cases as uninteresting exceptional cases. However, some researchers, including mathematical physicists, have studied such cases and found a condensation, showing that there are interesting phenomena in the exceptional cases as well. The purpose of this talk is to give a unified limit theorem of $\mathcal{T}_n$ as $n \to \infty$ for all simply generated trees, extending the well-known result in the standard case (i), and to encourage further research in the other cases too.

A limit theorem

In cases (i) and (ii), let ξ be an integer-valued random variable with distribution (w_k); thus $0 < \mathbb{E}\,\xi \le 1$. In case (iii), let $\xi = 0$, so $\mathbb{E}\,\xi = 0$. In all cases, let $\widehat{\xi}$ be a random variable with values in $\{0, 1, \dots, \infty\}$ with the distribution

$$\mathbb{P}(\widehat{\xi} = k) := \begin{cases} k\,\mathbb{P}(\xi = k), & k = 0, 1, 2, \dots, \\ 1 - \mathbb{E}\,\xi, & k = \infty. \end{cases}$$

In case (i), this is the usual size-biased transformation of ξ.

We define the modified Galton–Watson tree $\widehat{\mathcal{T}}$ as follows: There are two types of nodes: *normal* and *special*, with the root being special. Normal nodes have offspring (outdegree) according to independent copies of ξ, while special nodes have offspring according to independent copies of $\widehat{\xi}$. Moreover, all children of a normal node are normal; when a special node gets an infinite number of children, all are normal; when a special node gets a finite number of children, one of its children is selected uniformly at random and is special, while all other children are normal.

The special nodes form a path from the root; we call this path the *spine* of $\widehat{\mathcal{T}}$. $\widehat{\mathcal{T}}$ behaves differently in our three different cases:

(i) In the critical Galton–Watson case, the spine is an infinite path. Each outdegree $d(v)$ in $\widehat{\mathcal{T}}$ is finite, so the tree is infinite but locally finite. This is the size-biased Galton–Watson tree defined by Lyons, Pemantle and Peres [4].

(ii) In the subcritical Galton–Watson case, the spine is a.s. finite with a number L of vertices that has a (shifted) Geometric distribution $\mathrm{Ge}(1-\mu)$:

$$\mathbb{P}(L = \ell) = (1-\mu)\mu^{\ell-1}, \qquad \ell = 1, 2, \dots.$$

(iii) In the non-Galton–Watson case, the spine consists of the root only; the root has infinitely many children, and all its children are leaves. $\widehat{\mathcal{T}}$ is thus an infinite star. (This is the limiting case $\mu = 0$ of case (ii).)

In case (i), all vertices have finite degree, while in cases (ii) and (iii), the tree has (a.s.) exactly one node with infinite outdegree, viz. the top of the spine.

Our main theorem extends a result by Lyons, Pemantle and Peres to cases (ii) and (iii) in complete generality. For special cases, see [3] and [2].

Theorem 1. *In all three cases, T_n converges in distribution to $\widehat{\mathcal{T}}$ as $n \to \infty$, in the topology defined by convergence of all finite parts of the tree.*

The topology can, equivalently, be defined as convergence of each outdegree.

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