Szemerédi's regularity lemma and graph limits

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Abel Prize symposium, KVA, 25 May 2012

In honour of

Endre Szemerédi.

Partly based on papers by Szemerédi, Lovász, Szegedy, Borgs, Chayes, Sós, Vesztergombi, Simonovits, Diaconis, Janson and others.

Szemerédi's regularity lemma

Let G be a graph, with vertex set V. If X and Y are subsets of V, let $e_G(X, Y)$ be the number of edges between X and Y. If U and W are disjoint subsets of V and $\varepsilon > 0$, we say that (U, W) is ε-regular if there exists $d \in [0, 1]$ such that

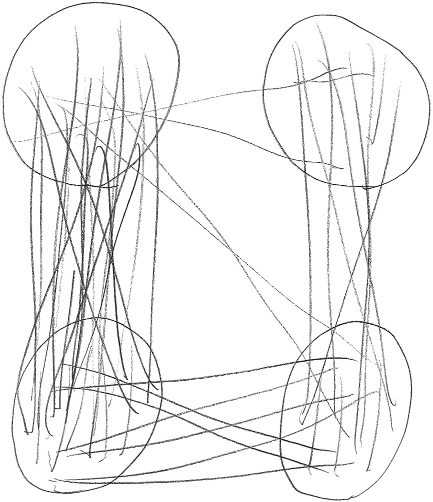
$$\left| \frac{e_G(X, Y)}{|X| \cdot |Y|} - d \right| \leq \varepsilon$$

for all $X \subseteq U$ and $Y \subseteq W$ such that $|X| \geq \varepsilon |U|$ and $|Y| \geq \varepsilon |W|$.
Finally, a partition $V_1, \ldots, V_k$ of V is a equipartition if $\big| |V_i| - |V_j| \big| \leq 1$ for all i, j.

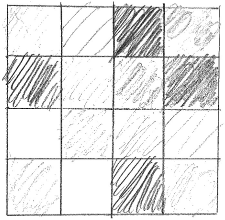
Theorem (Szemerédi's regularity lemma - one version)

Given m and $\varepsilon > 0$, there exists $M = M(m, \varepsilon)$ such that if G is any graph with at least M vertices, then for some ℓ with $m \leq \ell \leq M$, G has an equipartition into ℓ sets $V_1, \ldots, V_\ell$, such that all but $\varepsilon \ell^2$ pairs (V_i, V_j) are ε-regular.

Thus, any graph can be partitioned like this:



Hence, we can approximate a large graph like this:



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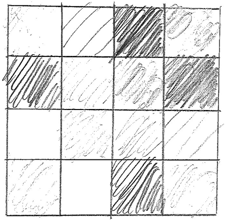


0.4	0.2	1	
0.6	0.3		1
0		0.3	0.2
	0	0.6	0.4

From graphs to functions

Thus large graphs can be approximated by piecewise constant functions on $[0,1]^2$
This suggests that symmetric functions $[0,1]^2 \to [0,1]$ ("graphons") can be seen as limits of large graphs, and conversely.

Definition (in principle)

A sequence $G_1, G_2, \ldots$ of graphs with $|G_n| \to \infty$ converges to a graphon $W : [0,1]^2 \to [0,1]$ if there is a sequence of Szemerédi partitions of G_n such that the corresponding densities, regarded as functions on $[0,1]^2$ converge to W.

Subgraph densities

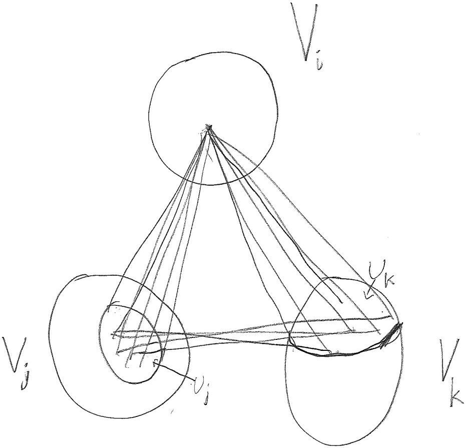
Let G be a graph with a Szemerédi partition $V_1, \dots, V_\ell$, and edge densities d_{ij}. Let $|G| = n$; thus each $|V_i| \approx n/\ell$.

Fix i and j and assume that (V_i, V_j) is ε-regular. Then most (all but $O(\varepsilon|V_i|)$) of the vertices in V_i have at least $(d_{ij} - \varepsilon)|V_j|$ neighbours in V_j.

(Otherwise there would be a large bad set $B \subset V_i$ with $e(B, V_j) < (d_{ij} - \varepsilon)|B|\,|V_j|$, violating the definition of ε-regularity.)

Similarly, most vertices in V_i have at most $(d_{ij} + \varepsilon)|V_j|$ neighbours in V_j.

Triangles



Assume that (V_i, V_j), (V_i, V_k), (V_j, V_k) are ε-regular. Then most vertices in V_i have $(d_{ij} \pm \varepsilon)|V_j|$ neighbours in V_j and $(d_{ik} \pm \varepsilon)|V_k|$ neighbours in V_k; moreover, denoting these sets of neighbours by U_j and U_k, the number of edges between them is

$$e(U_j, U_k) = (d_{jk} \pm \varepsilon)|U_j|\,|U_k| = (d_{ij}d_{ik}d_{jk} + O(\varepsilon))(n/\ell)^2.$$

This holds for most vertices in V_i, and summing over all vertices we find that the number of triangles with one vertex in each of V_i, V_j, V_k is

$$(d_{ij}d_{ik}d_{jk} + O(\varepsilon))(n/\ell)^3.$$

Summing over all i, j, k, including irregular pairs and cases with repetitions, we see that the number of triangles in G is

$$n^3 \left(\ell^{-3} \sum_{i,j,k=1}^{\ell} d_{ij} d_{ik} d_{jk} + O(\varepsilon) + O(\ell^{-1}) \right)$$

Taking the limit as $\ell \to \infty$ and $\varepsilon \to 0$, assuming that d_{ij} corresond to step functions converging to W, we find that the number of triangles is

$$n^3 \iiint W(x,y) W(x,z) W(y,z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z + o(n^3).$$

General subgraph counts

The arguments extend to counts of other subgraphs.

Definition

If F is any fixed graph, with $|F| = f$, let $t(F, G)$ denote the number of (labelled) copies of F in G, divided by $|G|^f$.

Definition

A sequence of graphs G_n with $|G_n| \to \infty$ converges to a graphon W if

$$t(F, G_n) \to t(F, W) := \int_{[0,1]^f} \prod_{ij \in E(F)} W(x_i, x_j)\, \mathrm{d}x_1 \cdots \mathrm{d}x_f.$$

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A sequence $G_1, G_2, \ldots$ of graphs with $|G_n| \to \infty$ is p-quasirandom if $t(F, G_n) \to p^{e(F)}$ for every fixed graph F.

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Definition (with graph limits)

A sequence $G_1, G_2, \ldots$ of graphs with $|G_n| \to \infty$ is p-quasirandom if $G_n \to p$ (the constant graphon $W(x, y) = p$).

Random graphs from Szemerédi partitions

To construct a graph with a given Szemerédi partition $V_1, \ldots, V_\ell$ and given densities d_{ij}:

Take ℓ sets of vertices $V_1, \ldots, V_\ell$ of equal size. For each pair (V_i, V_j), add edges between them at random, with probability d_{ij} for each edge. (Independently. Toss a (biased) coin for each pair of vetices in $V_i \times V_j$ to decide whether to add an edge.)

Random graphs from graphons

Let $W : [0,1]^2 \to [0,1]$ be a graphon. Let $n \geq 1$. Gonstruct a random graph G_n with vertex set $\{1, \ldots, n\}$ as follows:

Construction 1.
Let $x_i = i/n$.
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Theorem
$G_n \to W$ almost surely.

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Theorem

Graph limits may be described by distributions of infinite random graphs H_∞.

Graphons from random graphs

Let H_∞ be the infinite random graph constructed above. Let

$$I_{ij} = \begin{cases} 1, & \text{there is an edge } ij \\ 0, & \text{there is no edge } ij. \end{cases}$$

Then I_{ij} is an *exchangeable* array of indicator random variables. (The distribution is invariant under permutations.)

General representation theorem for exchangeable arrays by Aldous and Hoover $\implies$

$$I_{ij} = f(x_i, x_j, \xi_{ij}), \qquad i < j,$$

for some function f and x_i, ξ_{ij} i.i.d. uniform in $[0, 1]$.

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Theorem

$$G_n \to W.$$

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or by graph limits (involving graphons and analytic arguments).

In any case, they both tell us that there is order in chaos.

Congratulations to the Abel Prize!