Random Networks

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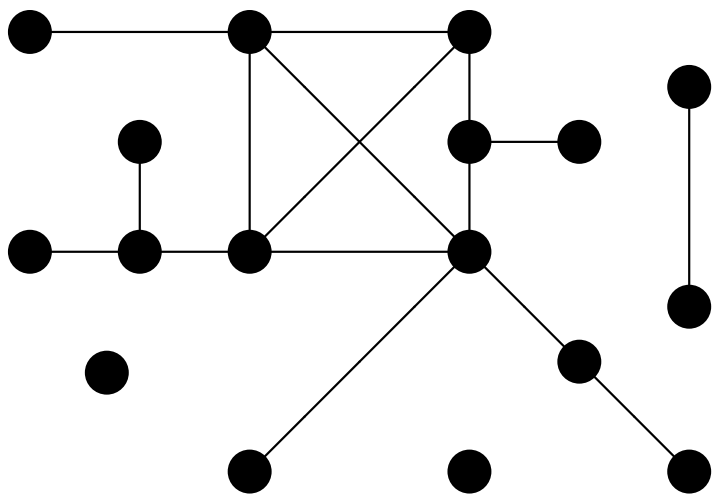
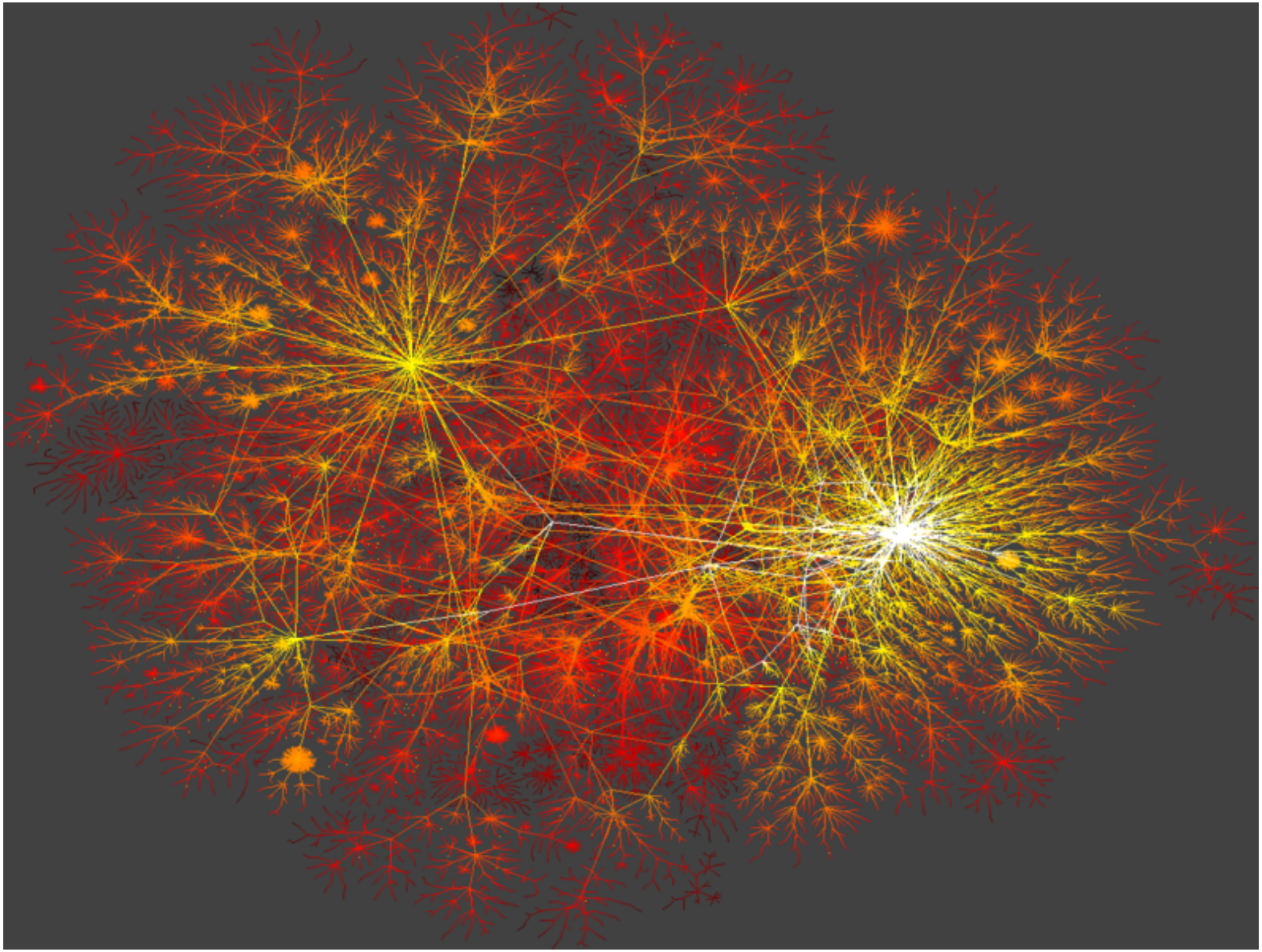
Networks

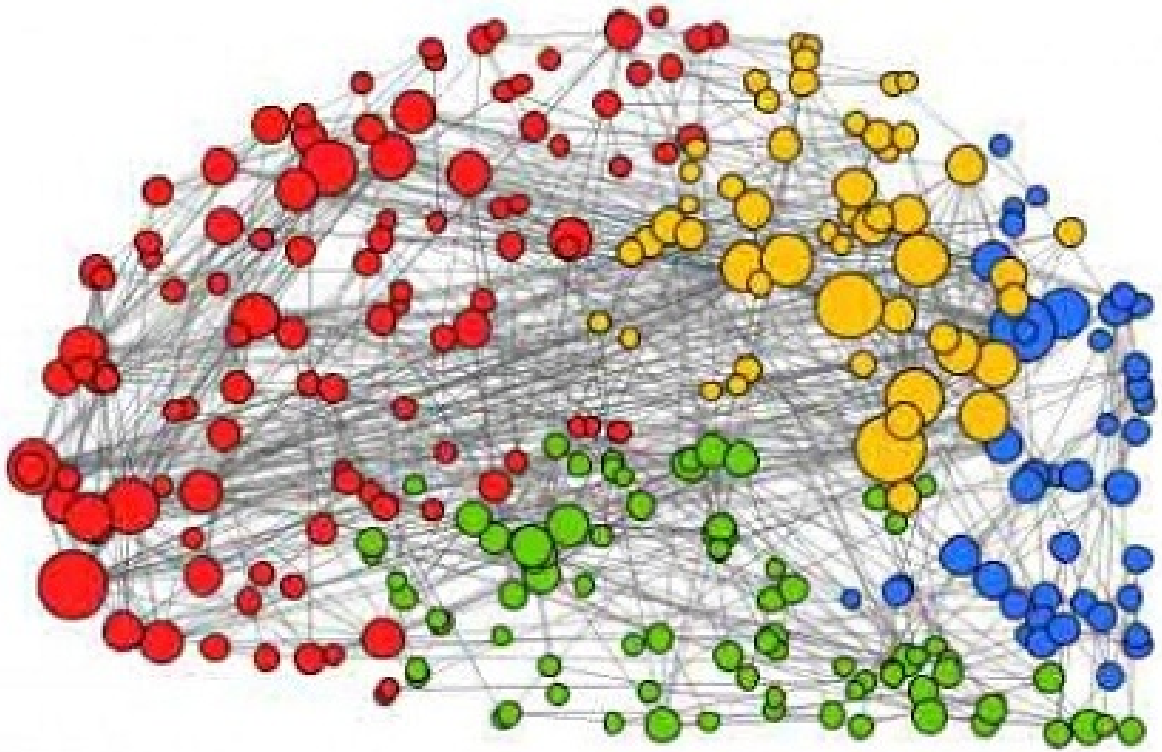
A network (graph) is a set of *nodes* (or *vertices*) together with *edges* (or *links*), where each edge connects two nodes.
Sometimes edges are directed from one node to the other, and sometimes there is additional structure, such as various attributes of nodes or edges (e.g. length, capacity, cost, . . . of an edge; size, age, sex, . . . of a node).



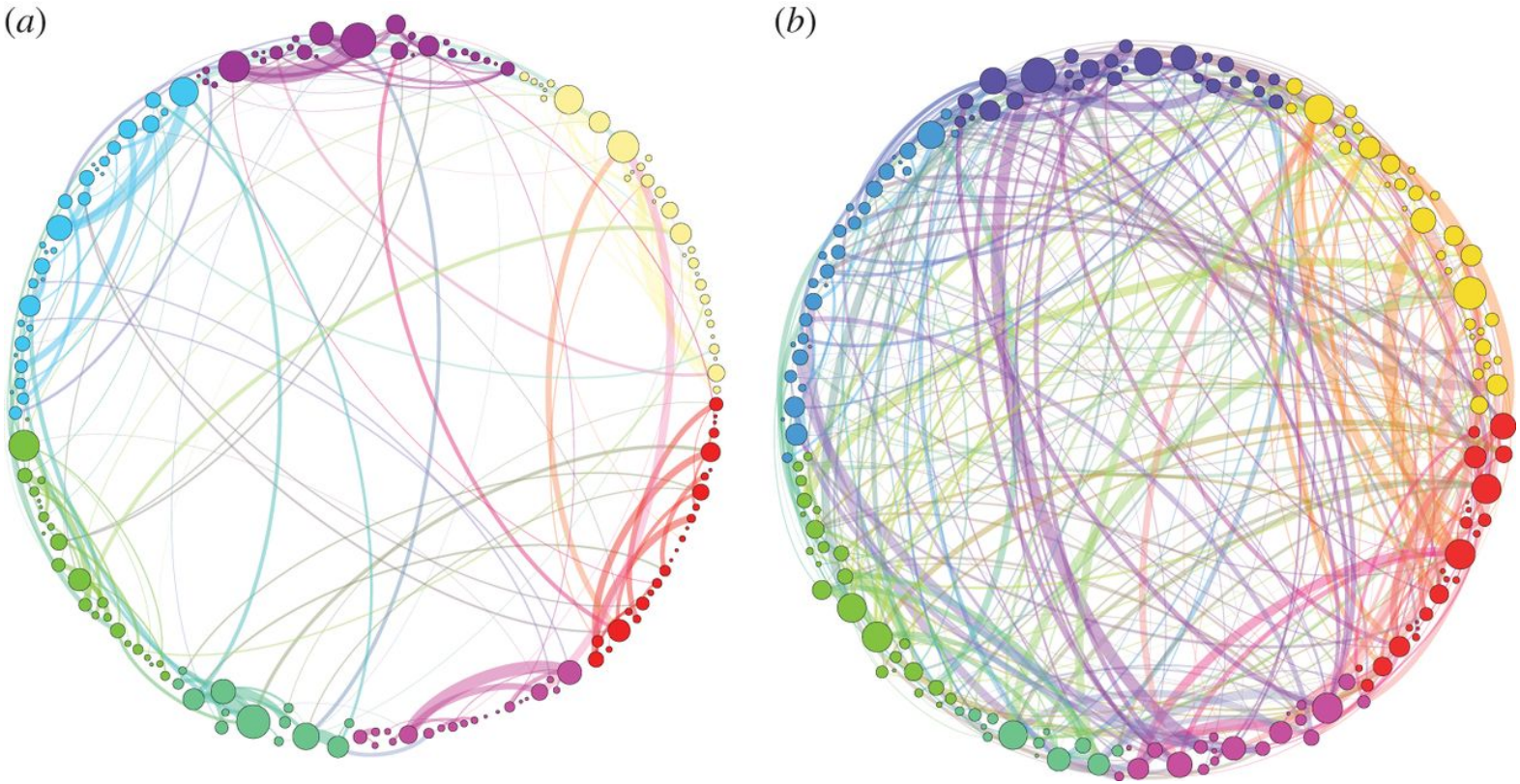
Some examples:



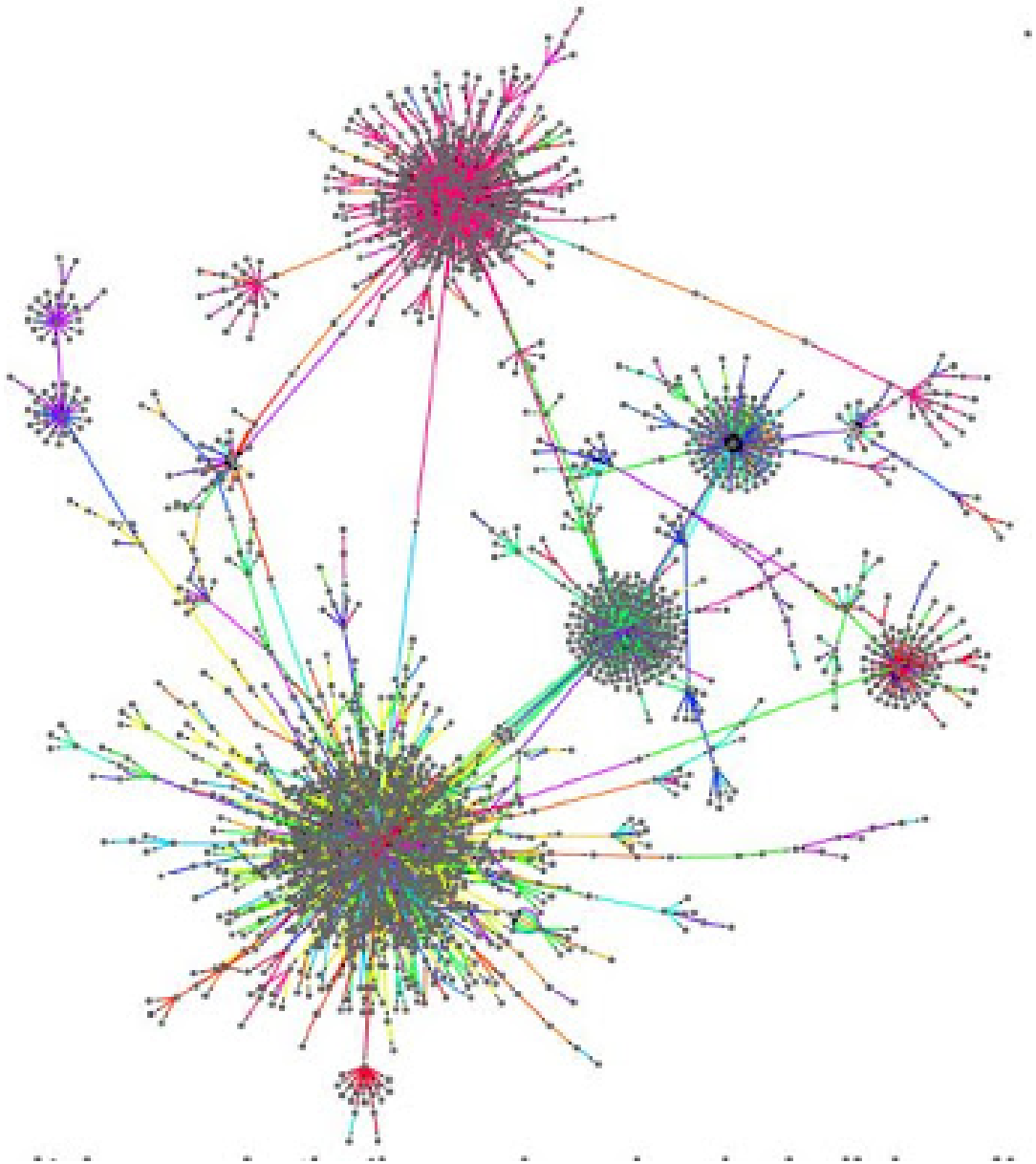
A macroscopic snapshot of Internet connectivity, with selected backbone ISPs (Internet Service Provider) colored separately. By K. C. Claffy (www.caida.org)



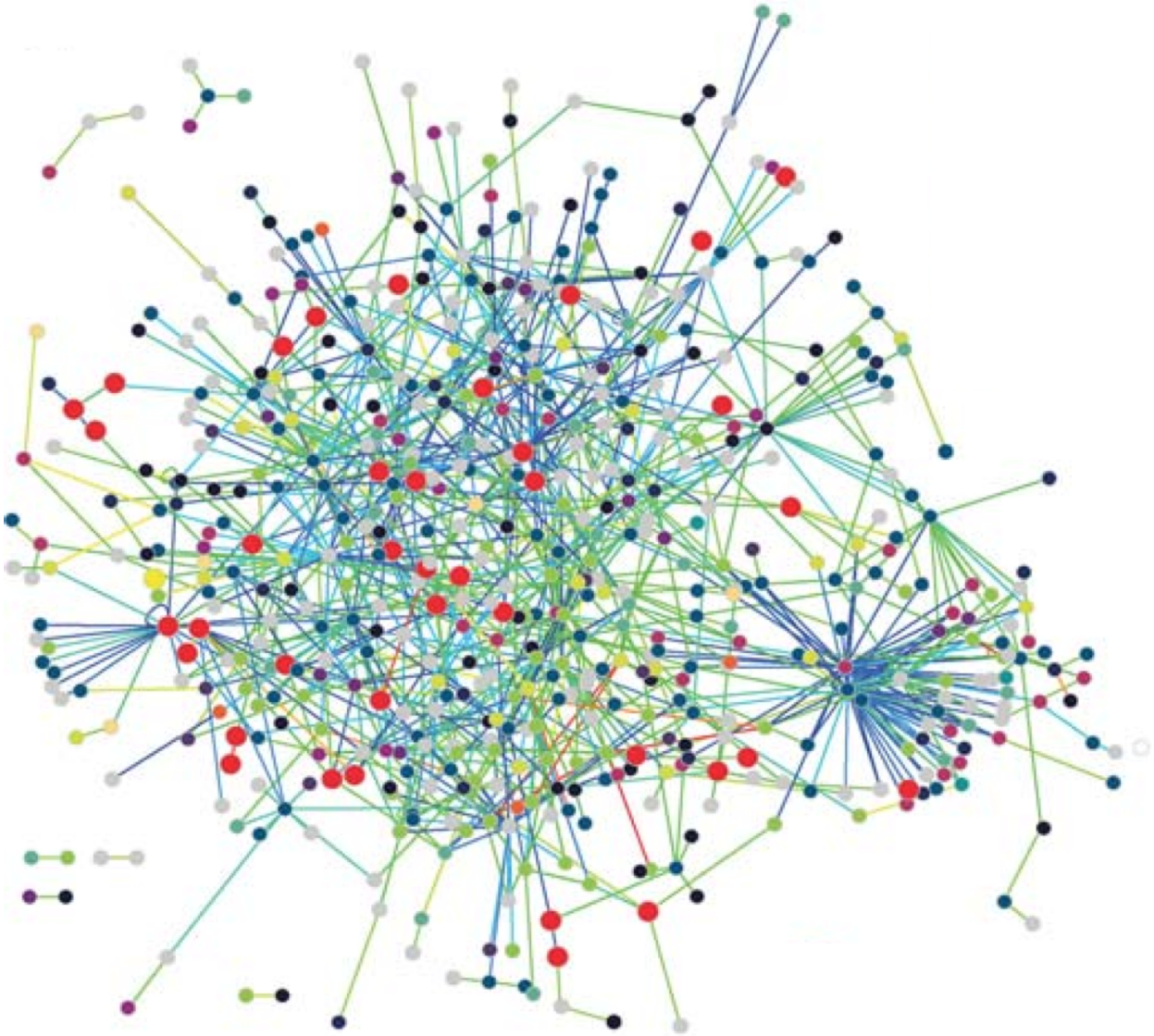
Wiring the brain, University of Cambridge



(a) placebo, (b) psilocybin. Homological scaffolds of brain functional networks, Petri et al., J. Royal Society 2014



ReTweets on Twitter. twiterhero.blogspot.se



The protein interaction network of Treponema pallidum. Peter Uetz

Random networks

A *random network* is a network where nodes or edges or both are created by some random procedure.
First example: (classical random networks studied by Erdős and Rényi and many others from 1959 and until today – often called Erdős–Rényi graphs)

Fix two (large) numbers n (number of nodes) and m (number of edges). Number the nodes $1, \ldots, n$. Draw a pair of nodes at random and join them by an edge. Repeat m times (without replacement). Denoted $G(n, m)$.

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A variant: Fix n (number of nodes) and a probability p. For each pair of nodes, make a random choice and connect the nodes by an edge with probability p. (Toss the same biased coin for each pair of nodes.) Denoted $G(n, p)$.

Why?

Some reasons why a statistician or applied mathematician might want to study random networks:

- Networks are used to describe a lot of things; besides the examples shown above, e.g.: interactions between people (in different ways, for example collaborations beteen scientists), animals (e.g. food webs), microbes, proteins, ... Typically, the network is not known exactly (and even if it is, it may be different tomorrow). A suitable random network may be used as a model.

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- For example, networks are used to describe possible infection routes for an infectious disease.
  A suitable random model may be useful. We can study mathematically how an infection spreads on the random network. If we believe in our model, we can use it to predict e.g. the spread of a real epidemic.

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- If a simple random model reproduces some interesting properties of a network, that is a strong warning that we should not give too much significance to those features and waste time inventing complicated explanations for them.
- Conversely, if a random network model fails to reproduce some properties, this shows that there are other reasons behind these properties, and it is meaningful to study them further.

Why?

Some reasons why a pure mathematician (like me) might want to study random networks:

- Random networks provide many interesting problems in probabilistic combinatorics.
  There are many nice theorems with results that are easy to state, and sometimes surprising. Some of them are easy to prove, while others are quite difficult. Many different methods from probability theory or combinatorics are used, and there is plenty of opportunity to develop new methods.

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- There are also applications inside mathematics, for example in extremal combinatorics.
  For many problems in graph theory, an extremal or almost extremal case can be obtained by a suitable random graph; this includes many cases where no explicit construction is known.

Example - Networks with high girth and chromatic number

The *chromatic number* of a network is the smallest number of colours that can be used to colour the nodes such that adjacent nodes always have different colours.
The *girth* of a network is the length of the smallest cycle.
It is easy to construct networks with high chromatic number if we allow small girth (e.g. 3) – just put in many edges.

Theorem (Erdős, 1959)

For any numbers k and m, there exists a graph with girth $> k$ and chromatic number $> m$.

Proof.

Take a random graph $G(n,p)$ with $p = A/n$, where A is a large constant (depending on m). Then, as $n \to \infty$,
$P(G(n,p)$ has chromatic number $\leq m) \to 0$,
$P(G(n,p)$ has girth $> k) \to q > 0$. □

Properties of random networks

Many properties of random networks have been studied mathematically.
Here are a few important examples.

Remark

Mathematicians are usually interested in asymptotic results as $n \to \infty$.

Node degrees

The *degree* of a node is the number of links connecting the node to other nodes.
The classical Erdős–Rényi random graphs have node degrees that are random, but with a rather small random dispersion and very small probability of having a degree that is much larger than the average. In fact, the distribution is Binomial or Hypergeometric, and asymptotically Poisson.

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This is one reason for considering other random networks.

Small subgraphs

Let H be a fixed graph, for example K_3. Let X_n be the number of copies of H as subgraphs of $G(n,p)$.
Let H have v_H vertices and e_H edges, and let $d_H = e_H/v_H$.

$$\mathbb{E}\, X_n \sim c n^{v_H} p^{e_H}$$

Thus $\mathbb{E}\, X_n \to \infty \iff p \gg n^{-v_H/e_H} = n^{-1/d_H}$.

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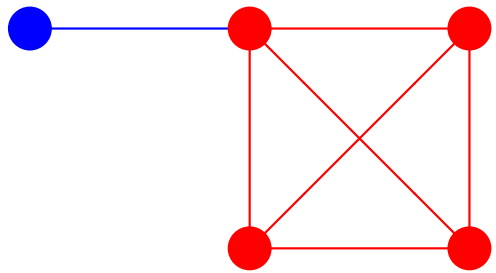
Thus $\mathbb{E}\, X_n \to \infty \iff p \gg n^{-v_H/e_H} = n^{-1/d_H}$.
If $d_F \leq d_H$ for all subgraphs $F \subset H$ (H is balanced), this is also necessary and sufficient for $\mathbb{P}(X_n > 0) \to 1$. ($X_n > 0$ w.h.p.)

Example:



An unbalanced graph. $d_H = 7/5 = 1.4$, $d_F = 6/4 = 1.5$.

Theorem

(A typical threshold result.)

Let $m_H = \max\{d_F : F \subseteq H\}$. *Then*

$X_H > 0$ *w.h.p.* $\iff p \gg n^{-1/m_H}$.

$X_H = 0$ *w.h.p.* $\iff p \ll n^{-1/m_H}$.

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Theorem

If $d_H > d_F$ *for all* $F \subset H$ *(*H *is strictly balanced), and* $p \sim cn^{-1/d_H}$, *then* $X_H \xrightarrow{\mathrm{d}} \mathrm{Po}(\lambda)$ *for some* $\lambda > 0$.

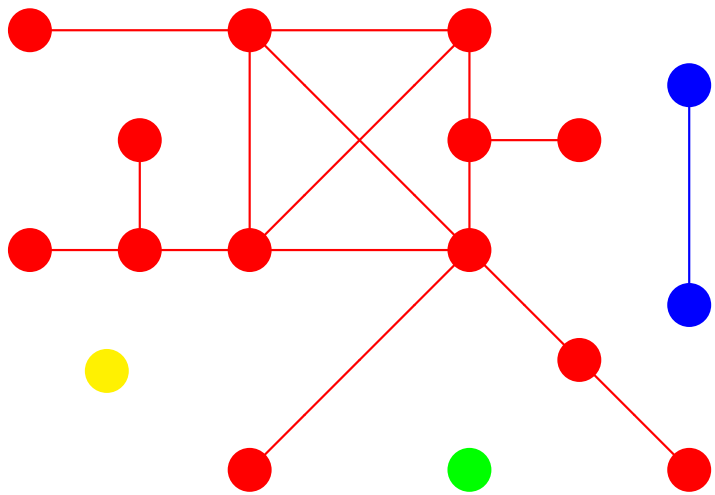
Proof.

$$X_H = \sum_{\alpha} I_\alpha$$

where α are the copies of H in the complete graph K_n.
The indicators are weakly dependent.
Etc. □

Components

A *component* in a network is a connected part of the network.
Example:



This network has 4 components.
The network is *connected* if there is only one component.

A typical case is that there is a giant component containing a large part of all nodes, possibly together with many very small components with only one or a few nodes each.

Another case, typical for very sparse networks, is that there are many small components but no really big one.

This is often interesting in applications. For example, in epidemiology, a component may represent the set of people that will become ill, if one vertex is infected from the outside. Hence a network with a large component is likely to give large epidemics, but a network with only small components is safer.

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On the other hand, if the network represents Internet after a computer virus attack, then we want a large component.

Theorem (Erdős and Rényi)

A classical Erdős–Rényi random graph $G(n, m)$ with n nodes and m edges has a giant component if $m > n/2$ but not otherwise.

More formally: If $n \to \infty$ and $m \sim cn$ for some constant c, and C_1 is the largest component of the random network, then

$$\frac{|C_1|}{n} \xrightarrow{\text{p}} \begin{cases} 0 & \text{if } c \leq 1/2, \\ \rho(2c) > 0 & \text{if } c > 1/2. \end{cases}$$

If $c < 1/2$, then $|C_1| = O_p(\log n)$.
The same holds for $G(n, p)$ with $p \sim c'/n$, with $c' = 2c$ so the threshold is $c' = 1$, i.e. $p = 1/n$.

Theorem (Erdős and Rényi)

A classical Erdős–Rényi random graph $G(n, m)$ with n nodes and m edges is connected if $m > n \log n/2$ but not otherwise.

More precisely: If $n \to \infty$ and $m = n \log n/2 + cn/2 + o(n)$ for some constant c, then

$$P(\text{connected}) \to e^{-e^{-c}}.$$

The number of isolated nodes is asymptotically $\text{Po}(e^{-c})$.

The same holds for $G(n, p)$ with $p = (\log n + c + o(1))/n$.

Branching processes

One useful technique to study Erdős–Rényi networks and many other random network models is to start at some node, find first its neighbours, then their neighbours, and so on, until an entire component has been explored.
For some models (for example the Erdős–Rényi networks), the number of new nodes found each time are (asymptotically) independent and with the same distribution, so this yields a *branching process*, more precisely a *Galton–Watson process*, where each individual gets a random number of children, and these numbers are i.i.d.

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Theorem

A Galton–Watson process has a positive probability of surviving for ever if and only if the expected number of children is > 1.

Branching processes

For an Erdős–Rényi network $G(n, p)$ the expected number of neigbours is np, so the branching process approximation yields the condition $c = np > 1$.

It follows also that $\rho(c)$ is the survival probability in a Galton–Watson process with offspring distribution $\mathrm{Po}(c)$, which is given by

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In some other random network models, we obtain instead a *multi-type branching process* where individuals may be of different types, with different offspring distributions. (Typically an infinite number of types are needed.)

Susceptibility

The *susceptibility* or *mean cluster size* $\chi(G)$ is the expected size of the component containing a random node. Equivalently, it is n times the probability that two random nodes lie in the same component (and thus may be connected by a path in the network). If the components are $C_1, C_2, \ldots$, then

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Theorem

For $G(n,p)$, as $n \to \infty$:

$$\chi(G(n,p)) \sim_p \begin{cases} \frac{1}{1-np}, & 1-np \gg n^{-1/3} \\ n\rho(np)^2, & np-1 \gg n^{-1/3}. \end{cases}$$

Distances and diameter

Given that two nodes are in the same component, we may ask for the *distance* between them, i.e., the shortest path between them in the network. The maximum distance is the *diameter*. The *average distance* between two random nodes is often at least as interesting.

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In many networks (random and "real-world"), the diameter and average distance are of the order $\log n$, and thus quite small even when the number n of nodes is large. This phenomenon is often called *Small Worlds*.

It says, essentially, that the number of vertices within distance x grows exponentially.

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Example: The giant component of $G(n, p)$ with $p = c/n$, $c > 1$.
Counter example: A lattice, and other geometric networks

Other random networks

The classical Erdős–Rényi random networks have node degrees that are random, but with a rather small random dispersion and very small probability of having a degree that is much larger than the average. (The distribution is Binomial or Hypergeometric, and asymptotically Poisson, with an exponentially decreasing tail..)

Power laws

Many networks from "reality" seem to have node degrees that are distributed according to a *power law*, i.e., there are constants γ and C_1 such that

$$\text{number of nodes with degree } k \approx C_1 k^{-\gamma}$$

or, which is roughly equivalent, with another constant C_2,

$$\text{number of nodes with degree at least } k \approx C_2 k^{-(\gamma-1)}.$$

(Since the networks are finite (although large), this can of course hold only in some (large) range and not for all k.)
Networks with a power law are often called *scale-free*.

The degree of a random node has (asymptotically) finite mean $\iff \gamma > 2$.

The degree of a random node has (asymptotically) finite variance $\iff \gamma > 3$.

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As always when something becomes popular, it is easy to overinterpret data, and see power laws also when the evidence really is weak. (*Seek and ye shall find.*)

I'm not an expert on any example, but I think that one should be suspicious.

Random network models

A classical (Erdős–Rényi) random network is thus too homogeneous to be a good model in many applications. (In others it is excellent.)

Many other random network models have been proposed and studied, especially the last 10 years, often with power laws for the node degrees.
This has given new input and new life to the theory of random network, and has stimulated the mathematical development.
(Independently of whether the models are good models for anything or not.)

Some proposed random network models

1. (Inequality depends on other properties.)
1a. Fix n (number of nodes). Give each node i a number a_i which we call *activity*. Let $A = \sum_{i=1}^{n} a_i$ be the sum of all activities. For each pair i, j of nodes, make a random choice and join them by an edge ij with probability $p_{ij} = \frac{a_i a_j}{A}$.
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- If we choose all activities a_i equal, say c, we get a classical random network $G(n, p)$ with $p = c/n$.
- If we choose a_i according to a power law, we get (asymptotically) the same power law for the degrees.
- In general: the degrees of most nodes are close to their activities, so by choosing the activities, we can obtain almost any desired distribution of the node degrees.

Example. Let a_i be proportional to $\sqrt{n}/\sqrt{i}$; this yields $p_{ij} \approx c/\sqrt{ij}$ for some constant $c > 0$. In this case, the node degrees follow a power law with exponent $\gamma = 3$.

Theorem

With high probability, there exist a giant component if $\frac{1}{A}\sum_{i=1}^{n} a_i^2 > 1$, *but not otherwise.*

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Example. Power law with $\gamma \leq 3$ (as in the example above): since the harmonic series diverges, there is always a giant component even if many edges are removed. (Good for communication network. Bad for epidemics.)

Proof.

Branching process approximation, with the links to the neighbours as individuals in the branching process.

The expected number of children of a link is

$$\sum_i \frac{a_i}{\sum_j a_j} a_i = \frac{\sum_i a_i^2}{\sum_i a_i}.$$

□

Variants

Variants

1b Determine the probabilities p_{ij} by some other formula. Example 2b below is the case $p_{ij} = c/\max(i,j)$. Bollobás, Janson and Riordan (2006) have studied the case $p_{ij} = \kappa(i/n, j/n)/n$ for a given function κ on $[0,1]^2$ (and more generally). ("*Inhomogeneous random graph*")

Theorem (Bollobás, Janson and Riordan)

This random network has a giant component if and only if the integral operator with kernel κ has norm on $L^2[0,1]$ (or, equivalently, spectral radius) greater than 1.

1c Fix a (finite or infinite) space S of "types", and a probability measure μ on S, and a symmetric function $W : S \times S \to [0, 1]$. Give each vertex i a random type $X_i \in S$, i.i.d. with distribution μ. Conditioned on these types X_i, add an edge ij with probability $W(X_i, X_j)$, independently of all other edges.

Stochastic Block Model (S finite); Graphon model.

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- Again, constant $W = p$ gives $G(n, p)$.
- $S = [0, 1]$ is popular in mathematical studies, but not necessary.
- These random networks are exchangeable. Conversely, every infinite exchangeable network can be constructed in this way. (By an extension of De Finetti's theorem to random arrays by Aldous and Hoover.)

1d Fix the node degrees as a given sequence $d_1, \ldots, d_n$, and take a random network, uniformly among all possible networks with these degrees.
("*Configuration model*". Bender and Canfield (1978), Bollobás (1985), Molloy and Reed (1995, 1998), and others.)

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Theorem (Molloy and Reed)

The random network with given node degrees $d_1, d_2, \ldots, d_n$ has a giant component if and only if $\sum_i d_i(d_i - 2) > 0$.

Proof.

Branching process approximation.

The expected number of children of a link is

$$\sum_i \frac{d_i}{\sum_j d_j}(d_i - 1) = \frac{\sum_i d_i(d_i - 1)}{\sum_i d_i}$$

and

$$\frac{\sum_i d_i(d_i - 1)}{\sum_i d_i} > 1 \iff \sum_i d_i(d_i - 2) > 0.$$

□

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Theorem

In this case, there will be a giant component if $c > 1/4$, but not otherwise.

2c Add new node one by one. Each time, add with probability c also a new edge, with two endpoints chosen at random among all nodes (old and new). (c must be less than 1.) Known as the *CHKNS model*, after Callaway, Hopcroft, Kleinberg, Newman and Strogatz (2001).

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Theorem

For the CHKNS model, there is a giant component if $c > 1/8$, but not otherwise.

More precisely, if $c = \frac{1}{8} + \epsilon$, the proportion of the nodes belonging to the giant component is (asymptotically)

$$\exp\left(-\frac{\pi}{2\sqrt{2}}\epsilon^{-1/2} + O(\log \epsilon)\right).$$

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These random networks do not have power laws for the degrees.

More examples: (Inequality depends on chance, and *"For whosoever hath, to him shall be given"*.)

3a Add nodes one by one. Join every node by an edge to one old node; this node is chosen with probability proportional to the number of edges that node already has.

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3b (More generally.) Let c be a positive integer. Add nodes one by one. Join every new node by c edges to c old nodes; these nodes are chosen with probabilities proportional to the number of edges that the nodes already have.

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3b (More generally.) Let c be a positive integer. Add nodes one by one. Join every new node by c edges to c old nodes; these nodes are chosen with probabilities proportional to the number of edges that the nodes already have.

This model is called *preferential attachment*. (Barabási and Albert, 1999).

Theorem

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Theorem

Let $a > -1$ be fixed and let the probability of attaching a new edge to an old node of degree k be proportional to $k + a$. Then the node degrees have a power law distribution with exponent $3 + a$.

Theorem (Oliveira and Spencer)

Let $\alpha > 1$ be fixed and let the probability of attaching a new edge to an old node of degree k be proportional to k^{α}. Then only a finite number of nodes will ever have degrees larger than $1/(\alpha - 1)$, so the proportion of such vertices tends to 0.
If $\alpha > 2$, then moreover, after some random time, all new nodes will attach to the same node.

Stochastic processes

Another useful tool is to introduce a time parameter and study suitable stochastic processes that describe, for example, the exploration of the successive neighbourhoods of a vertex.

Standard theorems about convergence of a stochastic process to a deterministic function, or to a Gaussian process, then can be applied.

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The existence and size of a giant component in a random network with given vertex degrees.

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Example

The existence and size of a giant component in a random network with given vertex degrees.

Example

Constuct a random network with given vertex degrees. Consider an epidemic, where an infected vertex infects each neighbour with some fixed rate β, and recovers and becomes immune with rate ρ. Study the total number of infected, and also the time evolution.

A random network