Random trees and branching processes

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Part I. Galton–Watson trees

Let ξ be a random variable with $\xi \in \mathbb{N} := \{0, 1, 2, \dots\}$. Let $p_k := \mathbb{P}(\xi = k)$.

A *Galton–Watson process* [Watson and Galton 1875; Bienaymé 1845] starts with a single individual; she gets a number of children, each of these gets a number of children, and so on; the numbers of children of different individuals are independent copies of ξ.
ξ (or its distribution (p_k)) is called the *offspring distribution*.

A Galton–Watson process generates a random rooted tree $\mathcal{T}$ (finite or infinite), with the initial individual as the root. $\mathcal{T}$ is called a *Galton–Watson tree*.

Fundamental theorem: $\mathcal{T}$ is a.s. finite $\iff \mathbb{E}\,\xi \leq 1$.

The case $\mathbb{E}\,\xi = 1$ is called *critical*.
This case turns out to be the most important for our purposes.

A *conditional Galton–Watson tree with n nodes* is the random tree $\mathcal{T}$ conditioned on $|\mathcal{T}| = n$. (Denoted $\mathcal{T}_n$.)

Simply generated trees

Trees are rooted and ordered (the children of each node are ordered). (Unordered trees may be given random orderings.)

$\mathbf{w} = (w_k)_{k\geq 0}$ is a fixed *weight sequence* with $w_k \geq 0$.

The *weight* of a finite tree T is

$$w(T) := \prod_{v \in T} w_{d^+(v)},$$

where $d^+(v)$ is the outdegree (number of children) of v.

Trees with such weights are called *simply generated trees* [Meir and Moon 1978].

We let $\mathcal{T}_n$ be the random simply generated tree obtained by picking a tree with n nodes at random with probability proportional to its weight.

Conditioned Galton–Watson trees again

Let $\mathcal{T}$ be a Galton–Watson tree with offspring distribution ξ. Choose the weight sequence $w_k = \mathbb{P}(\xi = k)$. Then, for every finite tree T,

$$\mathbb{P}(\mathcal{T} = T) = \prod_{v \in T} \mathbb{P}(\xi = d^+(v)) = \prod_{v \in T} w_{d^+(v)} = w(T)$$

Thus the random simply generated tree $\mathcal{T}_n$ is the same as the conditioned Galton–Watson tree $\mathcal{T}_n$.

In other words, the conditioned Galton–Watson trees are the same as the simply generated trees with a weight sequence satisfying $\sum_{k=0}^{\infty} w_k = 1$ (a *probability weight sequence*).

This may seem quite restrictive, but it isn't...

Equivalent weights

Let $a, b > 0$ and change w_k to

$$\widetilde{w}_k := ab^k w_k.$$

Then the distribution of the simply generated tree $\mathcal{T}_n$ is not changed.

In other words, the new weight sequence $(\widetilde{w}_k)$ defines the same simply generated random trees $\mathcal{T}_n$ as (w_k).

We say that the weight sequences (w_k) and $(\widetilde{w}_k)$ are *equivalent*.

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We say that the weight sequences (w_k) and $(\widetilde{w}_k)$ are *equivalent*.

In particular, if there exists $a, b > 0$ such that $\sum_k \widetilde{w}_k = 1$, then the simply generated tree generated by (w_k) is the same as a conditioned Galton–Watson tree.

For many (w_k) there exists an equivalent probability weight sequence; in this case $\mathcal{T}_n$ can thus be seen as a conditioned Galton–Watson tree.
(Not if w_k grows too rapidly, such as $k!$.)

Moreover, in many cases this can be done such that the resulting probability distribution has mean 1. In such cases it thus suffices to consider the case of a probability weight sequence with mean $\mathbb{E}\,\xi = 1$, so $\mathcal{T}_n$ is a critical conditional Galton–Watson tree.
This turns out to be the natural choice of ξ.

Thus, simply generated trees and (critical) conditioned Galton–Watson trees are almost the same.

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– BUT ONLY ALMOST !

Many kinds of random trees occuring in various applications can be seen as simply generated random trees and conditioned Galton–Watson trees. [Aldous, Devroye, ...]

Example $w_k = 1$ yields uniformly random *ordered trees* (*plane trees*).
Also $w_k = 2^{-k-1}$, a *Geometric distribution* $\mathrm{Ge}(1/2)$

Example $w_k = 1/k!$ yields uniformly random *labelled trees*.
Also $w_k = e^{-1}/k!$, a *Poisson distribution* $\mathrm{Po}(1)$.

Example $w_0 = 1$, $w_1 = 2$, $w_2 = 1$, $w_k = 0$ for $k \geq 3$ yields uniformly random *binary trees*.
Also $w_k = \binom{2}{k}\frac{1}{4}$, a *Binomial distribution* $\mathrm{Bi}(2, 1/2)$.

Any other Geometric, Poisson or Binomial $\mathrm{Bi}(2, p)$ distribution will give the same trees. The ones above are the critical distributions, which are the natural choices.

Three types

Three types of simply generated trees:

I. Critical conditioned Galton–Watson tree.
Examples: see above.

II. Subcritical conditioned Galton–Watson tree with the m.g.f. $\mathbb{E}\, e^{t\xi} = \infty$ for $t > 0$; then not equivalent to any critical conditioned Galton–Watson tree.
Example: $\mathbb{P}(\xi = k) = (k+1)^{-3}/\zeta(3)$.

III. Simply generated tree, not equivalent to any conditioned Galton–Watson tree.
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In particular, every conditioned Galton–Watson tree is of type I or II.
I.e., for any offspring distribution ξ, there exists a (unique) equivalent offspring distribution as in I or II.
This is the *natural offspring distribution*.

Critical Galton–Watson trees (Case I) form a nice and natural setting, with many known results (sometimes with extra assumptions, e.g. finite second moment or exponential moment).

Some of these results can be extended to the general case, including cases II and III.

Remark The three cases may be characterized by properties of the generating function $\Phi(z) := \sum_k w_k z^k$. (The p.g.f. for a Galton–Watson tree.)

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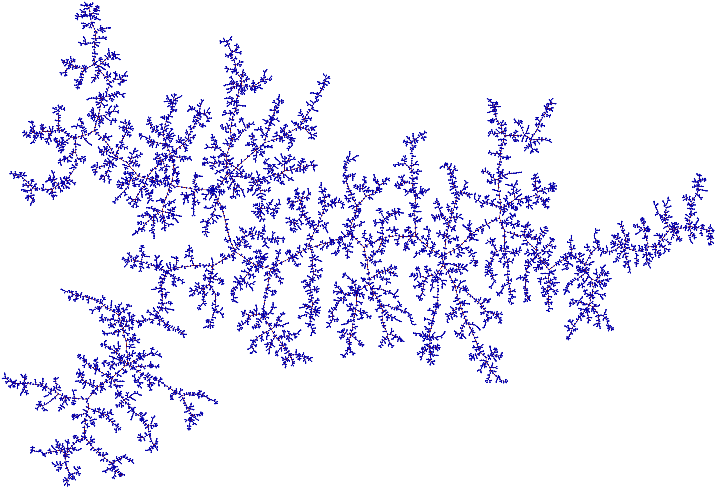
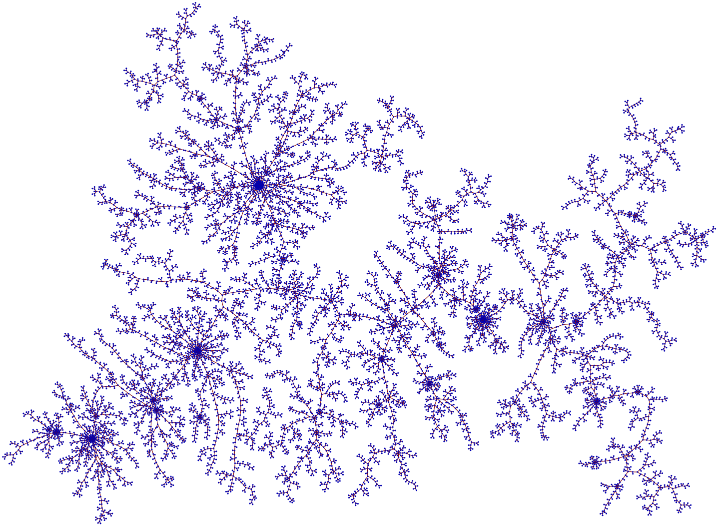
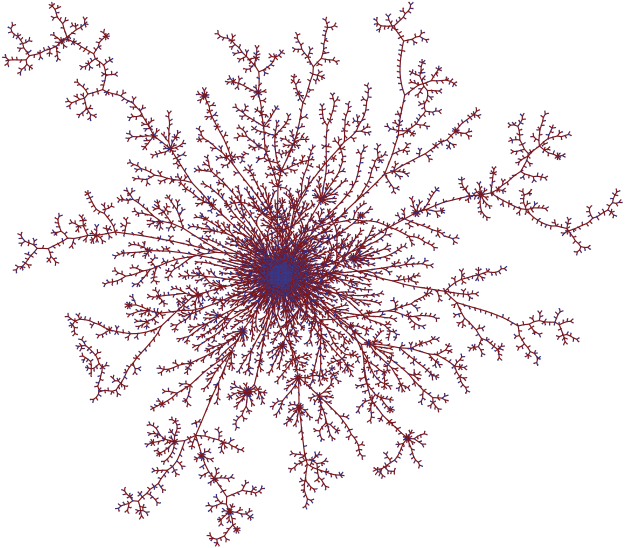
We are interested in limits as $n \to \infty$.



A conditioned Galton–Watson tree. Case I, finite variance.
[Igor Korchemski]



A conditioned Galton–Watson tree. Case I, stable limit.
[Igor Korchemski]



A subcritical conditioned Galton–Watson tree. Case II. [Igor Korchemski]

Node degrees

Theorem (the outdegree of a randomly chosen node)

Let V_n be a uniformly random node in a random simply generated tree $\mathcal{T}_n$.

(i). *As $n \to \infty$, the outdegree $d^+_{\mathcal{T}_n}(V_n)$ of V_n satisfies*

$$d^+_{\mathcal{T}_n}(V_n) \xrightarrow{\mathrm{d}} \xi'$$

for some (finite) random variable ξ'.

(ii). *If $\mathcal{T}_n$ is a conditioned Galton–Watson tree with offspring distribution ξ, then ξ' is equivalent to ξ. Furthermore, ξ' is the unique natural offspring distribution equivalent to ξ. Hence $\xi' = \xi$ iff ξ is natural. (Case I+II)*

(iii). *In particular, if ξ is equivalent to some critical offspring distribution, then ξ' is this critical distribution. (Case I)*

(iv). *In case III (not a conditioned Galton–Watson tree), $\xi' = 0$.*

In the sequel we assume that we have chosen the natural offspring distribution for a conditioned Galton–Watson tree. (In particular, this includes all critical offspring distributions: $\mathbb{E}\,\xi = 1$.)

In the less interesting Case III, we define $\xi := \xi' = 0$.

Thus, in all cases, $\xi' = \xi$.

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Let $\pi_k := \mathbb{P}(\xi = k)$. Then, the theorem says

$$\mathbb{P}(d^+_{\mathcal{T}_n}(V_n) = k) \to \pi_k, \quad k \geq 0.$$

Let also

$$\mu := \mathbb{E}\,\xi = \sum_k k\pi_k.$$

Recall that $\mu = 1$ in Case I, $0 < \mu < 1$ in Case II, and $\mu = 0$ in Case III.

Consider now instead the degree of the root o.

Theorem (the root degree)

As $n \to \infty$,

$$\mathbb{P}(d^+_{\mathcal{T}_n}(o) = k) \to k\pi_k, \quad 0 \le k < \infty.$$

Consequently,

$$d^+_{\mathcal{T}_n}(o) \xrightarrow{\mathrm{d}} \widehat{\xi},$$

where $\widehat{\xi}$ is a random variable in $\{0, 1, \dots, \infty\}$ with the distribution

$$\mathbb{P}(\widehat{\xi} = k) := \begin{cases} k\pi_k, & k = 0, 1, 2, \dots, \\ 1 - \mu, & k = \infty. \end{cases}$$

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Note that $\sum_0^\infty k\pi_k = \mu$ is $= 1$ in Case I, but < 1 in Cases II and III. Hence, we have convergence to a proper finite random variable only in Case I.

Limits close to the root

Theorem

Let $\mathbf{w} = (w_k)_{k\geq 0}$ be any weight sequence with $w_0 > 0$ and $w_k > 0$ for some $k \geq 2$.

Then $\mathcal{T}_n \xrightarrow{\mathrm{d}} \widehat{\mathcal{T}}$ as $n \to \infty$, where $\widehat{\mathcal{T}}$ is an infinite modified Galton–Watson tree (see below).

The limit (in distribution) in the theorem is for a topology where convergence means convergence of outdegree for any fixed node; it thus really means local convergence close to the root.

(It is for this purpose convenient to regard the trees as subtrees of the infinite Ulam–Harris tree.)

Kennedy (1975), Aldous & Pitman (1998), Kolchin (1984), Jonsson & Stefánsson (2011), et al + J

The infinite limit tree

$\widehat{\mathcal{T}}$ can be defined as follows.

There are *normal* and *special* nodes. The root is special.

Normal nodes have offspring (outdegree) as copies of ξ.
Special nodes have offspring as copies of $\widehat{\xi}$, where

$$\mathbb{P}(\widehat{\xi} = k) := \begin{cases} k\pi_k, & k = 0, 1, 2, \dots, \\ 1 - \mu, & k = \infty. \end{cases}$$

When a special node gets a finite number of children, one of its children is selected uniformly at random and is special.
All other children are normal.

(Based on Kesten ($\mu = 1$) + Jonsson & Stefánsson ($\mu < 1$).)

The spine

The special nodes form a path from the root; we call this path the *spine* of $\widehat{\mathcal{T}}$.

There are three cases:

I. $\mu = 1$ (the critical case).

$\widehat{\xi} < \infty$ a.s. Each special node has a special child and the spine is an infinite path. Each outdegree in $\widehat{\mathcal{T}}$ is finite, so the tree is infinite but locally finite.

The distribution of $\widehat{\xi}$ is the *size-biased* distribution of ξ, and $\widehat{\mathcal{T}}$ is the size-biased Galton–Watson tree defined by Kesten (1986).

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Alternative construction: Start with the spine (an infinite path from the root). At each node in the spine attach further branches; the number of branches at each node in the spine is a copy of $\widehat{\xi} - 1$ and each branch is a copy of the Galton–Watson tree $\mathcal{T}$ with offspring distributed as ξ; furthermore, at a node where k new branches are attached, the number of them attached to the left of the spine is uniformly distributed on $\{0, \dots, k\}$.

Since the critical Galton–Watson tree $\mathcal{T}$ is a.s. finite, it follows that $\widehat{\mathcal{T}}$ a.s. has exactly one infinite path from the root, viz. the spine.

II. $0 < \mu < 1$ (the subcritical case).

A special node has with probability $1 - \mu$ no special child. Hence, the spine is a.s. finite and the number L of nodes in the spine has a geometric distribution $\mathrm{Ge}(1 - \mu)$:

$$\mathbb{P}(L = \ell) = (1 - \mu)\mu^{\ell - 1}, \qquad \ell = 1, 2, \dots.$$

The tree $\widehat{\mathcal{T}}$ has exactly one node with infinite outdegree, viz. the top of the spine. $\widehat{\mathcal{T}}$ has no infinite path.

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The tree $\widehat{\mathcal{T}}$ has exactly one node with infinite outdegree, viz. the top of the spine. $\widehat{\mathcal{T}}$ has no infinite path.

Alternative construction: Start with a spine of random length L. Attach further branches that are independent copies of the Galton–Watson tree $\mathcal{T}$; at the top of the spine we attach an infinite number of branches and at all other nodes in the spine the number we attach is a copy of $\xi^* - 1$ where $\xi^* \overset{\mathrm{d}}{=} (\widehat{\xi} \mid \widehat{\xi} < \infty)$ has the size-biased distribution $\mathbb{P}(\xi^* = k) = k\pi_k/\mu$.

The spine thus ends with an explosion producing an infinite number of branches, and this is the only node with an infinite degree.

III. $\mu = 0$ (Not Galton–Watson tree.)

A degenerate special case of II.

A normal node has 0 children. A special node has ∞ children, all normal.

The root is the only special node. The spine has length $L = 1$.

The tree $\widehat{\mathcal{T}}$ is an infinite star. (No randomness.)

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Example

$w_k = k!$.

In the limit, $\mathcal{T}_n$ has $\mathrm{Po}(1)$ branches of length 2; all others have length 1.

Limits in the bulk: Fringe trees

Given a tree T and a node v, let T_v be the fringe tree at v, i.e., the subtree consisting of v and all its descendants.
Let T^* be the fringe tree at a uniformly random node $V \in T$.

Theorem

Let $\mathcal{T}_n^*$ be the random fringe tree of $\mathcal{T}_n$. Then, as $n \to \infty$, $\mathcal{T}_n^*$ converges in distribution to the (unconditioned) Galton–Watson tree $\mathcal{T}$ with offspring distribution π, i.e., for any fixed (finite) tree T,

$$\mathbb{P}(\mathcal{T}_n^* = T) \to \mathbb{P}(\mathcal{T} = T).$$

For $\mu = 1$, i.e., critical Galton–Watson trees, explicit in Aldous (1991), referring to Kolchin (1986).

Extended fringe trees

Even more generally:

Define the extended fringe tree T^{**} by adding also the mother of v, with its descendents, the grandmother, and so on, i.e., by considering T "shifted" with centre at the random node v.

Theorem (Stufler (2018))

The extended fringe tree $\mathcal{T}_n^{**}$ converges to a random tree $\hat{\widehat{\mathcal{T}}}$ constructed as follows:

(i). *If $\mu = 1$ (critical case), add an infinite spine backwards from the root of $\widehat{\mathcal{T}}$; let each node in the spine be special (with a $\widehat{\xi}$ offspring distribution), and add independent forward $\mathcal{T}$ Galton–Watson trees to all their children.*

(ii). *If $\mu < 1$ (subcritical case), assuming a regularity condition (complete condensation), add a spine backwards with special nodes, until the first node with an infinite number of children appears; then continue backwards and add more special nodes until another node with an infinite number of children appears; stop and discard the last node. Then add independent forward Galton–Watson trees as above.*

Quenched version

Let $n_T(\mathcal{T}_n)$ be the number of fringe subtrees of $\mathcal{T}_n$ that are isomorphic to T.

Theorem

Assume $\mu := \mathbb{E}\,\xi = 1$ *and* $\operatorname{Var}\xi < \infty$.

(i). *For any fixed tree* T,

$$\frac{n_T(\mathcal{T}_n)}{n} = \mathbb{P}(\mathcal{T}_n^* = T \mid \mathcal{T}_n) \xrightarrow{\mathrm{p}} \mathbb{P}(\mathcal{T} = T).$$

(ii).

$$\frac{n_T(\mathcal{T}_n) - n\,\mathbb{P}(\mathcal{T} = T)}{\sqrt{n}} \xrightarrow{\mathrm{d}} N(0, \gamma^2)$$

for some $\gamma^2 = \gamma_T^2 < \infty$.

General subtrees

Let $S(T)$ be the number of arbitrary (non-fringe) subtrees of T.

Theorem

Suppose that $0 < \operatorname{Var} \xi < \infty$.

(i). *There exist constants* $\mu, \sigma^2 > 0$ *such that*

$$\frac{\log S(\mathcal{T}_n) - \mu n}{\sqrt{n}} \xrightarrow{\mathrm{d}} N(0, \sigma^2).$$

(ii). *If* ξ *has an exponential moment, i.e.* $\mathbb{E}\, e^{t\xi} < \infty$ *for some* $t > 0$, *then, assuming a technical condition,*

$$\mathbb{E}\, S(\mathcal{T}_n)^m \sim \gamma'_m \tau_m^n$$

for some constants $\gamma_m > 0$ *and* $1 < \tau_1 < \tau_2 < \dots$.

Cai and Janson (2018)

Global limit

Assume $\mathbb{E}\,\xi = 1$ and $\mathbb{E}\,\xi^2 < \infty$. In $\mathcal{T}_n$, the typical distance to the root is of order $\sqrt{n}$, so we scale distances by this factor. The scaled tree converges, as a compact metric space in the Gromov–Hausdorff topology, to a random metric space, the *Brownian continuum random tree*. (Aldous 1990).

One version of this is the following (essentially Aldous 1990). For a tree T with nodes $v_1, \dots, v_n$ in depth-first order, let the height process $h_T : [0, n] \to \mathbb{R}_+$ be defined by $h(i) = d(0, v_i)$, the distance in the tree, (and $h(0) = 0$) and linear interpolation between the integers.

Theorem (Aldous)

Suppose that $\mathbb{E}\,\xi = 1$ *and* $0 < \sigma^2 := \operatorname{Var}\xi < \infty$. *Then*

$$\frac{1}{\sqrt{n}} h_{\mathcal{T}_n}(nt) \xrightarrow{\mathrm{d}} \frac{2}{\sigma} B_{ex}(t), \qquad t \in [0,1],$$

in $C[0,1]$, *where* $B_{ex}(t)$ *is a Brownian excursion.*

For $\mathbb{E}\,\xi = 1$ and $\mathbb{E}\,\xi^2 = \infty$, if ξ is in the domain of a stable law, there are analogues with stable trees and Lévy processes. [Duquesne, Le Gall]

Let, for a tree T,

$$
\begin{aligned}
H(T) &:= \max_{v \in T} d(o,v) && \text{(the } \textit{height}\text{)} \\
W_k(T) &:= |\{v \in T : d(0,v) = k\}| \\
W(T) &:= \max_k W_k(T) && \text{(the } \textit{width}\text{)}
\end{aligned}
$$

Let $M := \max_{0 \le t \le 1} B_{ex}(t)$.

Theorem (Chassaing, Marckert, Yor)

Assume again $\mathbb{E}\,\xi = 1$ *and* $0 < \sigma^2 := \operatorname{Var}\xi < \infty$. *Then*

$$\left(n^{-1/2} H(\mathcal{T}_n), n^{-1/2} W(\mathcal{T}_n)\right) \xrightarrow{\mathrm{d}} \left(2\sigma^{-1} M, \sigma M'\right),$$

with $M' \overset{\mathrm{d}}{=} M$.

Remark $M' \neq M$. In fact, $\operatorname{Corr}(M, M') \doteq -0.6428\ldots$

$H(\mathcal{T}_n) W(\mathcal{T}_n)/n \xrightarrow{\mathrm{d}} 2MM'$ with $\mathbb{E}(2MM') \doteq 3.046$.

Theorem (Addario-Berry, Devroye, J)

Assume $\mathbb{E}\,\xi = 1$ *and* $0 < \sigma^2 := \operatorname{Var}\xi < \infty$. *Then, uniformly in* $x > 0$ *and* $n \geq 1$,

$$\mathbb{P}\big(H(\mathcal{T}_n) > x\sqrt{n}\big) \leq Ce^{-cx^2}$$
$$\mathbb{P}\big(W(\mathcal{T}_n) > x\sqrt{n}\big) \leq Ce^{-cx^2}$$

Corollary

With the same assumptions

$$\mathbb{E}\big(W(\mathcal{T}_n)^r\big)/n^{r/2} \to \sigma^r 2^{-r/2} r(r-1)\Gamma(r/2)\zeta(r),$$
$$\mathbb{E}\big(H(\mathcal{T}_n)^r\big)/n^{r/2} \to \sigma^{-r} 2^{r/2} r(r-1)\Gamma(r/2)\zeta(r).$$

Problem What happens when $\operatorname{Var}\xi = \infty$?
Is still $\mathbb{E}\,H(\mathcal{T}_n) = O(\sqrt{n})$? Is $\mathbb{E}\,H(\mathcal{T}_n) = o(\sqrt{n})$?

Part II – general CMJ branching processes

A Crump–Mode–Jagers process is a branching process in continuous time, where each individual has a random number N of children (with $0 \le N \le \infty$), born at times when the individual itself has ages $\xi_1 \le \xi_2 \dots$; these are also random (and may be dependent in any way). (Technically, best seen as a point process.)

Different individuals have i.i.d. life stories.

Let $\mathcal{T}_\infty$ be the complete family tree of the process, starting with a single individual born at time 0, and let $\mathcal{T}_t$ be the subtree of individuals born up to time t.

We are interested in cases when $\mathcal{T}_\infty$ is infinite but each $\mathcal{T}_t$ a.s. is finite. Thus assume $\mathbb{E}\, N > 1$ (supercritical case) and assume for simplicity $N \ge 1$.

Let $Z_t := |\mathcal{T}_t|$, the number of individuals at time t.

More generally, a *characteristic* of an individual is a random function ϕ of the age $t \geq 0$; we assume $\phi(t) \geq 0$ and $\phi \in D[0, \infty)$. Let, where σ_x is the time individual x is born,

$$Z_t^\phi := \sum_{x:\sigma_x \leq t} \phi_x(t - \sigma_x),$$

the total characteristic of all individuals existing at time t.

Known results (Crump, Mode, Jagers, Nerman, et al):

Assume some technical conditions.

- There exists $\alpha > 0$ (the Malthusian parameter), such that

$$e^{-\alpha t} Z_t \xrightarrow{\text{a.s.}} W$$

  for some random variable $W > 0$.
- More generally,

$$e^{-\alpha t} Z_t^{\phi} \xrightarrow{\text{a.s.}} m_\phi W$$

  for a constant $m_\phi > 0$.
- Hence

$$Z_t^{\phi} / Z_t \xrightarrow{\text{a.s.}} m_\phi .$$

Fix a characteristic ψ.

Define $\tau_n := \inf\{t : Z_t^\psi \geq n\}$ and $T_n := \mathcal{T}_{\tau(n)}$.

Main example : $\psi = 1$.
T_n has n nodes (if birth times are a.s. distinct).

Remark The height of T_n is of order $\log n$.

Thus lower, and more branched, than a conditioned Galton–Watson tree. (Height order $\sqrt{n}$.)

Fringe trees

Theorem

(i). *(Annealed version.) The random fringe tree* T_n^* *converges in distribution to the random tree* $\overline{\mathcal{T}} = \mathcal{T}_\tau$*, where* $\tau \sim \mathrm{Exp}(\alpha)$ *is a random time, independent of* $\mathcal{T}$*.*

(ii). *(Quenched version.) For every finite tree* T*,*

$$\mathbb{P}(T_n^* = T \mid T_n) = \frac{|\{v : T_{n;v} = T\}|}{|T_n|} \xrightarrow{\text{a.s.}} \mathbb{P}(\overline{\mathcal{T}} = T).$$

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Remark (ii) is a LLN type result. Is there a corresponding CLT? I.e., is $|\{v : T_{n;v} = T\}|$ asymptotically normal?

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(ii). *(Quenched version.) For every finite tree* T*,*

$$\mathbb{P}(T_n^* = T \mid T_n) = \frac{|\{v : T_{n;v} = T\}|}{|T_n|} \xrightarrow{\text{a.s.}} \mathbb{P}(\overline{\mathcal{T}} = T).$$

Remark (ii) is a LLN type result. Is there a corresponding CLT? I.e., is $|\{v : T_{n;v} = T\}|$ asymptotically normal?

Partial result: Sometimes, but not always!

Extended fringe trees

Define a sin-tree $\tilde{T}$ as follows:

- Start with a copy of the branching process, starting with o born at time 0.
- Give o an infinite line of ancestors, $o^{(1)}, o^{(2)}, \dots$, each having a modified life history where one child is distinguished, and called *heir*, and the probability is weighted by a factor $e^{-\alpha\xi}$, where ξ is the time the heir is born.
- Let the heir of $o^{(i)}$ be $o^{(i-1)}$. This defines the (negative) birth times of the ancestors. Let all other children of the ancestors start new copies of $\mathcal{T}$.

Theorem

(i). *(Annealed.) The extended fringe tree of T_n converges in distribution to $\tilde{T}$.*

(ii). *(Quenched.) This holds also conditioned on T_n, a.s.*

Random recursive tree

Example

Children born with independent Exp(1) waiting times, i.e., according to a Poisson process with rate 1. The branching process is the Yule process.

T_n is the random recursive tree. The next node is added as a child to a uniformly chosen node.

General preferential attachment tree

Example

Let $(w_k)_0^\infty$ be a sequence of weights with $w_k \geq 0$ and $w_0 > 0$.

Grow a tree by choosing the mother of each new node randomly with probability proportional to w_d where d is the outdegree (number of existing children).

T_n where the waiting time for child k is $\mathrm{Exp}(w_{k-1})$.

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Linear case: $w_k = \chi k + \rho$.

Binary search tree

Example

Each individual gets two children, one left and one right; each after an Exp(1) time (independent).

m-ary search tree (with external nodes)

Example

$m \geq 2$ fixed.

A newborn has 0 "keys". It get $m-1$ keys after independent waiting times $Y_1, \ldots, Y_{m-1}$ with $Y_i \sim \text{Exp}(i)$. When the last key arrives, m children are born.

$\psi(t)$ is the number of keys at time t.

Fragmentation trees

Example

Start with an object of mass $x_0 > 0$; break it ito b pieces with masses $V_1 x_0, \dots, V_b x_0$, where $(V_1, \dots, V_b)$ is a random vector with $V_i \geq 0$ and $\sum_i V_i = 1$. Continue recursively with each piece of mass $\geq x_1$, using a new copy of $(V_1, \dots, V_b)$ each time.

Regard the fragments of masses $\geq x_1$ seen during the process as nodes in the *fragmentation tree*.

CMJ process: An individual has b children, born at times $\xi_1, \dots, \xi_b$ with $\xi_i := -\log V_i$.

The fragmentation tree is the tree $\mathcal{T}_{\log(x_0/x_1)}$.

THE END