Notes on HW1
Takis Konstantopoulos, Spring 2009

Hw1/Ex2

This problem is not well-defined. Let us examine a more general situation to see what we can do to provide a remedy.

The urn contains a random subset of distinct colours. Not only the elements of the set are random, but also its size.

Say there are n possible distinct colours. Call them $1, 2, \ldots, n$. Define ξ to be one of the $n!$ permutations of these colours.

Now let N be a random integer with values in $\{1, \ldots, n\}$. We are TOLD that $P(N = i) = 1/n$, for $i = 1, \ldots, n$, i.e. we have complete ignorance of what N is.

What we do is this: we define

$$Y = (\xi_1, \ldots, \xi_N),$$

so, for example, if $N = 2$, then $Y = (\xi_1, \xi_2)$.

Thus, to define Y, we need two random variables: one is ξ, the other is N. To find the law of Y we need the JOINT law of ξ and N. The problem only tells us the law of N. Now, it is REASONABLE to assume (but this is an ASSUMPTION!) that

$$P(\xi_1 = \alpha_1, \ldots, \xi_i = \alpha_i | N = i) = \frac{1}{n} \frac{1}{n-1} \cdots \frac{1}{n-i+1},$$

where $\alpha_1, \ldots, \alpha_i$ are distinct elements of $\{1, \ldots, n\}$. This is tantamount to assuming that ξ and N are independent. (Show this!)

Say colour 1 is the red colour. We are asked to compute a conditional probability of the form

$$P(N = 1 | 1 \in Y),$$

where "$1 \in Y$" is an abbreviation for the statement "$\xi_1 = 1$ or $\xi_2 = 1$ or $\cdots$ or $\xi_N = 1$". Having defined the joint law of N and ξ properly, we have no fear to make a mistake like Alice. So here we go:

$$P(N = 1 | 1 \in Y) = \frac{P(N = 1, 1 \in Y)}{P(1 \in Y)} = \frac{P(N = 1, 1 \in Y)}{\sum_{i=1}^{n} P(N = i, 1 \in Y)}$$

But

$$\begin{aligned}
P(N=i, 1 \in Y) &= P(1 \in Y | N=i) P(N=i) \\
&= P(\xi_1 = 1 \text{ or } \cdots \text{ or } \xi_i = 1 | N=i) \frac{1}{n} \\
&= P(\xi_1 = 1 \text{ or } \cdots \text{ or } \xi_i = 1) \frac{1}{n} \quad \text{[by independence between } N \text{ and } \xi] \\
&= \frac{i}{n} \frac{1}{n} \quad \text{[because the events between the or's are pairwise disjoint]}
\end{aligned}$$

Since $1+2+\ldots+n = n(n-1)/2$, we have that $\sum_{i=1}^{n} P(N=i, 1 \in Y) = n(n-1)/2n^2$. So

$$P(N=1 | 1 \in Y) = \frac{1/n^2}{n(n-1)/2n^2} = \frac{2}{n(n-1)}.$$

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If we have a choice with replacement, we need to change our probability space. Let $1, \ldots, n$ represent the different colours. We let $\xi_1, \ldots, \xi_n$ be i.i.d. colours, i.e., each ξ_i is uniformly distributed in $\{1, \ldots, n\}$ and the $\xi_1, \ldots, \xi_n$ are mutually independent. (This assumption is equivalent to the assumption that the random vector $(\xi_1, \ldots, \xi_n)$ is uniformly distributed in the set $\{1, \ldots, n\}^n$. (WHY?)

Let N be a random variable with values in $\{1, \ldots, N\}$. We are told that $P(N=i) = 1/n$, for all i.

As before, we look at the random variable $Y = (\xi_1, \ldots, \xi_N)$, and, as before, we are going to make the ASSUMPTION that ξ and N are independent. Hence

$$P(1 \in Y | N=i) = P(\xi_1 = 1 \text{ or } \cdots \xi_i = 1),$$

but now the events are not dijoint: $\{\xi_1 = \xi_2 = 1\} \neq \varnothing$. However, by independence

$$P(\xi_1 \neq 1, \ldots, \xi_i \neq 1) = ((n-1)/n)^i.$$

So

$$P(1 \in Y | N=i) = 1 - (1 - (1/n))^i.$$

And so

$$\begin{aligned}
P(N=1 | 1 \in Y) &= \frac{P(1 \in Y | N=1) \cancel{P(N=1)}}{\sum_{i=1}^{n} P(1 \in Y | N=i) \cancel{P(N=i)}} \\
&= \frac{1/n}{n - \sum_{i=1}^{n} (1 - (1/n))^i} = \frac{1/n}{n - n(1 - (1 - (1/n))^{n+1})} = \frac{1}{n(1 - (1/n))^{n+1}}
\end{aligned}$$

For large n this is about $1/en \approx 0.37/n$, which is much larger than the previous answer (about $2/n^2$). This is reasonable!

HW1/Ex3

One way to model the situation "there are n balls, some black, some white, but the proportion of the black balls is β" (for given $0 < \beta < 1$) is as follows.

We may take the point of view that each of the balls have been painted black with probability β, independently of one another. This means that the total number of black balls in the urn equals the total number of successes (success=black) in n independent coin tosses, whence the number of black balls has law $\text{BINOMIAL}(n, \beta)$.