Bayesian Inference and Computational Methods 2009

Homework 1

1. Player Y has a pack of 4 cards (Ace and Queen of clubs, Ace and Queen of Hearts) from which he deals a random selection of 2 to player X.

   (a) What is the probability that X receives both Aces conditional on receiving at least 1 Ace?

   (b) Suppose now that Y deals X two cards from the pack of 4, after which X says *I have an Ace.*

      i. Discuss whether the above information is sufficient to calculate the conditional probability

      $$\mathbf{P}(X \text{ has 2 Aces} \,|\, X \text{ says } \textit{I have an Ace}).$$

      ii. If it is not, what other information would be required in order to calculate this conditional probability?

2. An urn is known to contain n differently coloured balls where n can be any integer in the set $\{1, 2, 3\}$. Your prior information tells you that n is equally likely to be any of these values.

   (a) A ball is drawn randomly from the urn and is found to be red.

      i. Alice argues that, since the probability of the red ball being drawn conditional on there being n balls in the urn is $1/n$,

      $$\mathbf{P}(n = 1 \,|\, \text{red ball drawn}) = \frac{\frac{1}{3} \times \frac{1}{1}}{\frac{1}{3} \times \frac{1}{1} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{3}}$$

      and calculates the posterior probabilities of n being 1, 2, and 3 as $6/11$, $3/11$ and $2/11$ respectively. She then expresses her surprise that the her beliefs regarding n have changed having observed only the colour of a single draw from the urn. Explain the fallacy in her argument and why the above the information alone does not define a posterior probability for n.

      ii. Bertie assumes that the n balls placed in the urn are drawn uniformly at random from a large stock of differently coloured balls. Calculate Bertie's posterior probabilities for $n = 1, 2, 3$.

      iii. Under what circumstances would Alice's posterior probabilities be correct?

   (b) Suppose now that two balls are drawn from the urn *with replacement* and the event that both are the same colour is observed. Calculate the posterior probabilities for $n = 1, 2, 3$ in this case.

3. (*More balls and urns*) Five balls are drawn uniformly randomly from a very large population of black and white balls where the proportion of black balls is $1/3$. You do not know the colours of the balls selected. The balls are then placed in an urn.

   (a) Give suitable prior probabilities for the number of black balls in the urn.

   (b) You now select two balls uniformly at random from the urn with replacement. They are both white. Calculate the posterior probabilities for the number of black balls in the urn.

   (c) Suppose that the two balls were selected from the urn without replacement and were both white. Calculate your posterior probabilities for the number of black balls in the urn for this case.

4. (a) A fair coin is tossed n times where n can take the values $1, 2, 3, 4, 5$ with equal probability. Suppose that 2 heads result from the n tosses. Determine the posterior distribution (i.e. work out the probability function) of n and identify the value of n that is *a posteriori* most likely.

   (b) Suppose now the coin is instead tossed repeatedly until m tails are obtained where the value m is first selected from a *Geometric*$(1/3)$ distribution, i.e. the probability function of m is $(2/3)^{m-1}(1/3)$. Suppose that 2 heads are obtained in the sequence. What is the posterior distribution of m given this information? (It is sufficient to write an expression involving infinite sums!)

5. In a sequence of n independent trials each has a success with unknown probability $p \in (0, 1)$.

   (a) Write down the distribution of the total number X of successes.

   (b) A Bayesian approach is used to estimate p. A $\beta(a, b)$ prior distribution is used. (Note that, in particular, $a = b = 1$ defines a $U(0, 1)$ distribution.) A total of k successes are observed. Determine the posterior distribution of p, together with its mode, its mean and its standard deviation. Assuming that n and k are large, show that these quantities are relatively insensitive to the choice of a and b (provided that these quantities are not large).

   (c) Suppose $a = b = 2$ and that $n = 200$, $k = 70$. Plot the prior and posterior distributions for p and determine an equal-tailed 95% Bayesian credible interval for p.

6. The lifetime T days of a component is known to have an $\text{Exp}(\lambda)$ distribution, where λ is unknown, but modelled as having an $\text{Exp}(2)$ prior distribution. In a random sample of 5 such components, their *total* lifetime is observed to be 3.0 days.

   (a) Find the posterior distribution of λ. Sketch it. Determine also an equal-tailed 95% Bayesian credible interval for λ.

   (b) Write down, as a function of λ, the probability that a further component will have a lifetime of at least 0.25 days. Using simulation, or otherwise, estimate the expected value of this probability under the posterior distribution found above.

   (c) Suppose now that a further 2 components are observed and found to have lifetimes in excess of 1 day. Find the new posterior distribution of λ and again determine also an equal-tailed 95% Bayesian credible interval for λ.

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Homework 1 – Solutions

1. Player Y has a pack of 4 cards (Ace and Queen of clubs, Ace and Queen of Hearts) from which he deals a random selection of 2 to player X.

   (a) We have

   $$\mathbf{P}(\text{both aces}\,|\,\text{at least one ace}) = \frac{\mathbf{P}(\text{both aces})}{\mathbf{P}(\text{at least one ace})} = \frac{1/6}{5/6} = \frac{1}{5}.$$

   (b) The given information is insufficient to calculate the required condition probability since it is not clear, prior to the start of the experiment, under exactly what circumstances X will make the statement *I have an Ace.*

2. (a) Note that n can be thought of as an unknown parameter which takes each of the values $1, 2, 3$ with prior probability $1/3$.

      i. Alice argument assumes that each urn is known to contain a red ball.

      ii. Under Bertie's assumption each urn contains a red ball with the same probability, and hence Bertie's posterior probabilities are equal to the prior probabilities, i.e. $1/3$ for each value of n.

      iii. See above—Alice's posterior probabilities would be correct if it were known in advance that each urn contained a red ball.

   (b) When there are n balls in the urn the probability that both balls drawn are the same colour is $1/n$. Since each value of n is *a priori* equally likely, it follows that the posterior probabilities for $n = 1, 2, 3$ are proportional to $1/n$, i.e. $6/11$, $3/11$ and $2/11$ respectively.

3. Again the number n of black balls in the urn ($n = 0, 1, 2, 3, 4, 5$) can be thought of as an unknown parameter.

   (a) An appropriate prior distribution for n is $\text{Bin}(5, \frac{1}{3})$, i.e.

   $$\pi(n) = \binom{5}{n}\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^{5-n}.$$

   (b) When there are n black balls in the urn the probability that both balls drawn are white is $\left(\frac{5-n}{5}\right)^2$. Hence the posterior distribution for the number n of black balls in the urn is given by the posterior probability function

   $$\begin{aligned}\pi(n\,|\,\text{both balls drawn white}) &\propto \binom{5}{n}\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^{5-n}\left(\frac{5-n}{5}\right)^2\\ &\propto \binom{5}{n}2^{-n}(5-n)^2.\end{aligned}$$

   (The normalising constant for the last expression is $1/92.8125$.)

   (c) In this case when there are n black balls in the urn the probability that both balls drawn are white is $\left(\frac{5-n}{5}\right)\left(\frac{4-n}{4}\right)$. Hence the posterior distribution for the number n of black balls in the urn is given by the posterior probability function

   $$\begin{aligned}\pi(n\,|\,\text{both balls drawn white}) &\propto \binom{5}{n}\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^{5-n}\left(\frac{5-n}{5}\right)\left(\frac{4-n}{4}\right)\\ &\propto \binom{3}{n}\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^{3-n},\end{aligned}$$

   so that the posterior distribution for the number of black balls in the urn is $\text{Bin}(3, \frac{1}{3})$—a result which may also be seen directly.

4. (a) Here the number n ($n = 1, 2, 3, 4, 5$) of tosses may be thought of as an unknown parameter, with prior probability function

$$\pi(n) = \text{constant}.$$

Given that the number of tosses is n, the probability of obtaining two heads is $\binom{n}{2}2^{-n}$ if $n \geq 2$ and 0 if $n = 1$. Hence the posterior distribution of n is given by the probability function

$$\pi(n \,|\, \text{two heads obtained}) \propto \begin{cases} \binom{n}{2}2^{-n}, & n = 2, 3, 4, 5 \\ 0, & n = 1. \end{cases}$$

Normalising, we obtain that this posterior distribution is given by

n	1	2	3	4	5
$\pi(n \,\|\, \text{data})$	0	$\frac{4}{21}$	$\frac{6}{21}$	$\frac{6}{21}$	$\frac{5}{21}$

so that the values of n which are *a posteriori* most likely are 3 and 4.

(b) Here m should be thought of as an unknown parameter, with prior probability function

$$\pi(m) = \left(\frac{2}{3}\right)^{m-1}\left(\frac{1}{3}\right).$$

Given that the coin is tossed until m tails are obtained, the probability of obtaining two heads is $\binom{m+1}{2}2^{-(m+2)}$ (*why?*). Hence the posterior distribution of m is given by the probability function, for $m \geq 1$,

$$\begin{aligned} \pi(m \,|\, \text{two heads obtained}) &\propto \left(\frac{2}{3}\right)^{m-1}\left(\frac{1}{3}\right)\binom{m+1}{2}\frac{1}{2^{m+2}} \\ &\propto m(m+1)3^{-m}. \end{aligned}$$

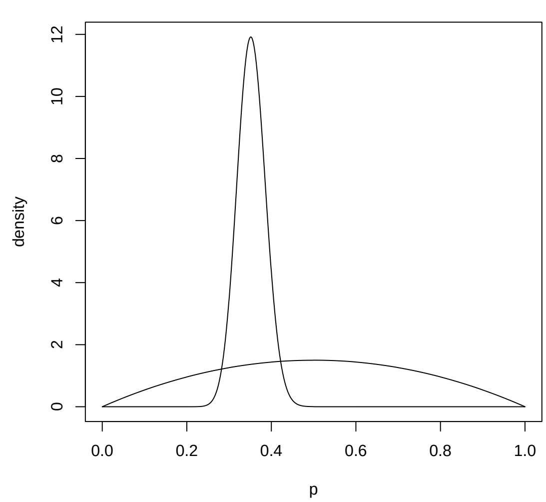
Here the normalising constant k is given by $k^{-1} = \sum_{m \geq 1} m(m+1)3^{-m} = 9/4$.

5. (a) $\text{Bin}(n, p)$.

(b) As in the lecture notes the posterior distribution of p is $\beta(a + k, b + n - k)$ and has mode $(a + k - 1)/(a + b + n - 2)$, mean $(a + k)/(a + b + n)$ and standard deviation

$$\left(\frac{(a + k)(b + n - k)}{(a + b + n)^2(a + b + n + 1)}\right)^{1/2}.$$

(c) The plot below shows the densities of the prior ($\beta(2, 2)$) and posterior ($\beta(72, 132)$) densities. The equal-tailed 95% Bayesian credible interval for p is given by the 0.025 and 0.975 quantiles of the $\beta(72, 132)$ distribution, i.e. by $[0.289, 0.420]$.



6. (a) The prior distribution for λ is given by the density $\pi(\lambda) = 2e^{-2\lambda}$ for $\lambda > 0$. The likelihood function is $L(\lambda; \boldsymbol{t}) = \lambda^5 e^{-3.0\lambda}$. Hence the posterior distribution of λ is given by the density

$$\pi(\lambda \,|\, \boldsymbol{t}) \propto \lambda^5 e^{-\lambda(2.0+3.0)},$$

i.e. by a $\Gamma(6,5)$ distribution. This may be sketched in, e.g., **R**. An equal-tailed 95% Bayesian credible interval for λ is given by $[0.44, 2.33]$.

(b) The probability that a further component will have a lifetime of at least 0.25 days is $e^{-0.25\lambda}$. Possible **R** code to estimate the expected value of this probability under the $\Gamma(6,5)$ posterior distribution is

```
lambda = rgamma(10000,6,5)        #10000 simulations of lambda
mean(exp(-0.25*lambda))           #sample mean of required probabilities
```

In fact, the expected probability may also be calculated by integrating $e^{-0.25\lambda}$ with respect to the $\Gamma(6,5)$ density to obtain the exact answer $(5/5.25)^6 = 0.7462$.

(c) The new likelihood function is $L(\lambda; \boldsymbol{t}) = \lambda^5 e^{-3.0\lambda}(e^{-\lambda})^2$, so that the new posterior distribution of λ is given by the density

$$\pi(\lambda \,|\, \boldsymbol{t}) \propto \lambda^5 e^{-7.0\lambda},$$

i.e. by a $\Gamma(6,7)$ distribution. An equal-tailed 95% Bayesian credible interval for λ is given by $[0.31, 1.66]$.