Bayesian Inference and Computational Methods 2009

Homework 4

1. A Markov chain $\{X_n\}_{n\geq 0}$ takes values in the state space $S = \{0, 1, 2\}$ and has transition matrix
$$P = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}.$$
   (a) Find the matrix giving the 2-step transition probabilities, and the matrix giving the 3-step transition probabilities. If the chain is in state 1 at time $n = 5$, what is the probability that it is in state 0 at time $n = 7$?

   (b) Suppose that $\mathbf{P}(X_0 = i) = 1/3$ for $i = 0, 1, 2$. Find the distributions of the random variables X_1, X_2, X_3.

   (c) Verify that the chain is *irreducible* (all states *intercommunicate*), and find the unique stationary distribution for the chain. Verify also that the transition matrix P does *not* possess the *detailed balance* property.

   (d) Use the results of part (b) to comment on the rate of convergence to the stationary distribution.

2. A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel, then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4.

   (a) Find the transition matrix for the Markov chain which corresponds to the driver's movements between A, B, and C.

   (b) Verify that the chain is irreducible, and find the stationary distribution of the driver's location.

   (c) Verify also that the transition matrix of the chain possesses the *detailed balance* property, and note that this gives a simpler method of determining the stationary distribution.

   (d) In the long term, what proportion of his journeys terminate at a hotel?

3. A Markov chain $\{X_n\}_{n\geq 0}$ takes values in the state space $S = \{1, 2\}$ and has transition matrix
$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}.$$
   (a) Show by induction that the matrix of n-step transition probabilities is given by
$$P^n = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix} + \left(-\frac{1}{2}\right)^n \begin{pmatrix} 1/3 & -1/3 \\ -2/3 & 2/3 \end{pmatrix}.$$
   (b) Deduce that
$$\lim_{n\to\infty} p_{ij}^{(n)} = \begin{cases} 2/3 & \text{if } j = 1 \\ 1/3 & \text{if } j = 2, \end{cases}$$
   independently of the state i.

   (c) Use the *detailed balance equations* to find the stationary distribution $\boldsymbol{\pi}$ associated with this chain and verify that, for all i, j, we have $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$.

4. A lecturer possesses 2 umbrellas which she employs when going from her home to the office and vice versa. If she is at home (the office) at the beginning (end) of the day and it is raining, then she will take an umbrella with her from the office (home), provided there is one to be taken. If it is not raining, then she never takes an umbrella. Finally, assume that, independent of the past, it rains at the beginning (end) of a day with probability $p = 1/3$.

   Consider the Markov chain with state space $S = \{0, 1, 2\}$ where the chain is in state k if there are k umbrellas at the lecturer's present location.

   (a) Determine the transition matrix P for this chain. Verify that P possesses the *detailed balance* property, and hence (or otherwise) find the stationary distribution for the chain.

   (b) Suppose that the chain's initial distribution is its stationary distribution. What is the probability that on any given day the lecturer will get wet going between home and the office?

5. A finite non-negative $N \times N$ matrix is called *doubly stochastic* if each of its rows and each of its columns sum to 1. Show that if a Markov chain has a doubly stochastic transition matrix, then the distribution $N^{-1}(1, 1, \dots, 1)$ is a stationary distribution for the chain.

6. Two urns each contain N balls. Of the total of $2N$ balls, N are black and N are white. From time to time a ball is chosen at random from urn 1 and swapped with a ball chosen at random from urn 2. Let X_n denote the number of black balls in urn 1 after the nth swap.

   (a) Write down the transition matrix for the Markov chain $\{X_n\}_{n\geq 0}$.

   (b) Explain, without detailed calculations, why this transition matrix necessarily possesses the *detailed balance* property.

   (c) Use the detailed balance equations to show that the stationary distribution for this chain is given by $\boldsymbol{\pi} = (\pi_0, \pi_1, \dots, \pi_N)$, where

   $$\pi_j = k\binom{N}{j}^2$$

   and where the constant k is to be chosen so that $\sum_{j\in S} \pi_j = 1$.
   [*Recall that, since the detailed balance equations have a solution which is unique up to a multiplicative constant, it is sufficient to verify that* $\boldsymbol{\pi}$ *as given above does indeed satisfy these equations. Elementary combinatorial arguments then show that* $k = \binom{2N}{N}^{-1}$.]

7. Consider further the simple random walk on the nonnegative integers, modified to have a reflecting barrier at 0 in such a way that $p_{01} = 1$. Use the detailed balance equations to show that the condition for the existence of a stationary distribution is still given by $p < 1/2$, and in this case determine the stationary distribution.

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Homework 4 – Solutions

1. (a) We have

$$P^2 = \begin{pmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{pmatrix} \qquad P^3 = \begin{pmatrix} 17/32 & 3/16 & 9/32 \\ 3/8 & 1/4 & 3/8 \\ 33/64 & 3/32 & 25/64 \end{pmatrix}.$$

[*Verify that these are stochastic matrices.*]
If the chain is in state 1 at time $n = 5$, the probability that it is in state 0 at time $n = 7$ is $p_{10}^{(2)} = 1/2$.

(b) Let $\boldsymbol{\mu}^{(n)} = (\mu_i^{(n)})_{i \in S}$ where $\mu_i^{(n)} = \mathbf{P}(X_n = i)$. Then $\boldsymbol{\mu}^{(n)} = \boldsymbol{\mu}^{(0)} P^n$ and so, if $\boldsymbol{\mu}^{(0)} = (1/3, 1/3, 1/3)$ we have

$$\begin{aligned} \boldsymbol{\mu}^{(1)} &= (7/12, 2/12, 3/12) \\ \boldsymbol{\mu}^{(2)} &= (25/48, 6/48, 17/48) \\ \boldsymbol{\mu}^{(3)} &= (91/192, 34/192, 67/192) \end{aligned}$$

[*Verify that in each case these probabilities sum to* 1.]

(c) The stationary distribution is given by $\boldsymbol{\pi} = (1/2, 1/6, 1/3)$. It is easily verified that P does *not* possess the detailed balance property.

(d) From (b) it is seen that the rate of convergence to the stationary distribution is fast.

2. (a) The required transition matrix is

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix}.$$

(b) The stationary distribution of the taxi driver's location is $\boldsymbol{\pi} = (3/7, 2/7, 2/7)$.

(c) Note that the detailed balance property is trivially verified.

(d) The long term proportion of his journeys which terminate at a hotel is $2/7 + 2/7 = 4/7$.

3. (a) It is easy to check that the required result is correct in the case $n = 1$ (indeed to start the induction it is sufficient to check $n = 0$). Suppose that the result is true for $n = k$. Then

$$\begin{aligned} P^{k+1} &= P^k P \\ &= \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} + \left(-\frac{1}{2}\right)^k \begin{pmatrix} 1/3 & -1/3 \\ -2/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix} + \left(-\frac{1}{2}\right)^k \begin{pmatrix} -1/6 & 1/6 \\ 1/3 & -1/3 \end{pmatrix} \\ &= \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix} + \left(-\frac{1}{2}\right)^{k+1} \begin{pmatrix} 1/3 & -1/3 \\ -2/3 & 2/3 \end{pmatrix} \end{aligned}$$

and so the result is true also for $n = k+1$. Hence the required induction follows.

(b) Since $(-1/2)^n \to 0$ as $n \to \infty$, the required limit result is immediate.

(c) The detailed balance equations here reduce to the single equation $\frac{\pi_1}{2} = \pi_2$. Hence, normalising, we obtain the stationary distribution $\pi = (2/3, 1/3)$.

4. (a) The transition matrix of the chain is given by

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2/3 & 1/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$

and its stationary distribution is given by $\boldsymbol{\pi} = (2/8, 3/8, 3/8)$.

(b) At the start of the day, with probability π_0 there is no umbrella at home and then the lecturer can only get wet if it rains on the journey to work; with probability π_1 there is 1 umbrella at home and then the lecturer cannot get wet on either the journey to work or the journey home; with probability π_2 there are 2 umbrellas at home and then the lecturer can only get wet if it fails to rain on her journey to work but does rain on her journey home. Hence the probability she gets wet on that day is

$$\pi_0 \times 1/3 + \pi_1 \times 0 + \pi_2 \times 2/3 \times 1/3 = 1/6.$$

5. Let the transition matrix be $P = (p_{ij})$. Then, since it is doubly stochastic, we have (in addition to the usual result for row sums) that

$$\sum_{i=1}^{N} p_{ij} = 1 \qquad \text{for all } j.$$

Thus

$$\begin{aligned} N^{-1}(1,1,..,1)P &= N^{-1}\left(\sum_{i=1}^{N} p_{i1}, \sum_{i=1}^{N} p_{i2}, \ldots, \sum_{i=1}^{N} p_{iN}\right) \\ &= N^{-1}(1,1,...,1), \end{aligned}$$

and so $N^{-1}(1,...,1)$ is a stationary distribution for the chain.

6. (a) The possible values for X_n are $0,1,...,N$. Suppose $X_n = i$, so that also after the nth swap there are $N-i$ black balls in the second urn. Then, for $i = 1,2,3,..,N-1$,

$$X_{n+1} = \begin{cases} i & \text{if either two white or two black balls are selected} \\ i+1 & \text{if a white ball is selected from urn 1 and a black ball is selected from urn 2} \\ i-1 & \text{if a black ball is selected from urn 1 and a white ball is selected from urn 2.} \end{cases}$$

Hence the transition matrix P of the chain is given as follows.
For $i = 1,2,3,..,N-1$,

$$\begin{aligned} p_{ii} &= \frac{N-i}{N} \times \frac{i}{N} + \frac{i}{N} \times \frac{N-i}{N} &&= \frac{2i(N-i)}{N^2} \\ p_{i,i+1} &= \frac{N-i}{N} \times \frac{N-i}{N} &&= \left(\frac{N-i}{N}\right)^2 \\ p_{i,i-1} &= \frac{i}{N} \times \frac{i}{N} &&= \left(\frac{i}{N}\right)^2 \\ p_{ii} &= 0 && \text{if } i \neq i-1 \text{ or } i \neq i+1. \end{aligned}$$

For $i = 0$ we have

$$p_{00} = 0; \qquad p_{01} = 1,$$

since a white ball must be selected from urn 1 and a black ball must be selected from urn 2.
For $i = N$ we similarly have

$$p_{NN} = 0; \qquad p_{N,N-1} = 1.$$

(b) Since the only transitions possible are between neighbouring states (or from a state to itself), it is immediate that the transition matrix P necessarily possesses the detailed balance property (see the lecture notes).

(c) Since the detailed balance equations have a solution which is unique (up to a multiplicative constant), for the given distribution to be stationary it is sufficient to check that it does satisfy the detailed balance equations; this is elementary algebra.

7. The detailed balance equations here become

$$\begin{aligned} \pi_0 &= \pi_1 q \\ \pi_i p &= \pi_{i+1} q, \qquad\qquad i \geq 1, \end{aligned}$$

which may be solved recursively to give

$$\begin{aligned} \pi_0 &= \pi_0 \\ \pi_i &= \pi_0 \frac{1}{q} \left(\frac{p}{q}\right)^{i-1}, \qquad\qquad i \geq 1. \end{aligned}$$

For $\boldsymbol{\pi}$ to be a stationary distribution, we require also $\sum_{i\geq 0} \pi_i = 1$, and it is clear that we can choose π_0 so that this is the case if and only if $p/q < 1$, i.e. $p < 1/2$. We then require

$$\begin{aligned} \pi_0^{-1} &= 1 + \frac{1}{q} \sum_{i\geq 1} \left(\frac{p}{q}\right)^{i-1} \\ &= 1 + \frac{1}{q-p}. \end{aligned}$$