Explanatory Notes 2 for Bayesian Inference

Rejection sampling

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The proof of correctness of the rejection sampling algorithm (SZ notes, page 19) requires further elaboration.

Let us recall the algorithm. We wish to simulate a random variable with given density f. We assume that we know how to simulate a random variable from another density g which has the property that there exists a constant $M > 0$ such that

$$f(x) \le Mg(x), \qquad \text{for all } x \text{ (in the support of } f). \tag{1}$$

Define

$$p(x) = \frac{f(x)}{Mg(x)}.$$

Let $Z_1, Z_2, \ldots$ be i.i.d. random variables with common density g. Let $\xi_1, \xi_2, \ldots$ be i.i.d. coin tosses, conditionally on the $Z_1, Z_2, \ldots$. In other words,

$$P(\xi_n = 1|Z_1, Z_2, \ldots) = P(\xi_1 = 1|Z_n) = p(Z_n)$$
$$P(\xi_n = 0|Z_1, Z_2, \ldots) = P(\xi_0 = 1|Z_n) = 1 - p(Z_n).$$

Let

$$N = \min\{n \ge 1 : \ \xi_n = 1\}.$$

The claim is:

Claim 1. *Z_N has density f.*

Proof. First observe that

$$P(\xi_n = 1) = E[P(\xi_n = 1|Z_n)] = E[p(Z_n)]$$
$$= \int p(z)g(z)dz = \int \frac{1}{M} f(z)dz = \frac{1}{M},$$

where the integral is taken over the support of g which, by assumption (1), includes the support of f. We then have

$$\begin{aligned}
P(Z_N \in dy) &= \sum_{n=1}^{\infty} P(Z_N \in dy, N = n) \\
&= \sum_{n=1}^{\infty} P(Z_n \in dy, N = n) \\
&= \sum_{n=1}^{\infty} P(Z_n \in dy, \xi_1 = \cdots = \xi_{n-1} = 0, \xi_n = 1) \\
&= \sum_{n=1}^{\infty} P(Z_n \in dy) P(\xi_1 = \cdots = \xi_{n-1} = 0, \xi_n = 1 | Z_n = y) \\
&= \sum_{n=1}^{\infty} g(y)dy \ P(\xi_1 = 0) \cdots P(\xi_{n-1} = 0) \ P(\xi_n = 1 | Z_n = y) \\
&= \sum_{n=1}^{\infty} g(y)dy \ (1 - M^{-1})^{n-1} \ p(y) \\
&= \sum_{n=1}^{\infty} f(y)dy \ \frac{1}{M} \ (1 - M^{-1})^{n-1} \\
&= f(y)dy \frac{1}{M} \sum_{n=1}^{\infty} (1 - M^{-1})^{n-1} = f(y)dy \frac{1}{M} M = f(y)dy,
\end{aligned}$$

as required. □

Apology. If you don't like the notation $P(\cdots \in dy)$ then replace it by $P(\cdots \in A)$ for some set A and rewrite the calculation.