Explanatory Notes 3 for Bayesian Inference

Simulating normal random variables

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The beautiful geometric method for simulating normal random variables rests on the following observation:

Lemma 1. *If (X, Y) are two independent standard normals then so are*

$$X' := X \cos\alpha + Y \sin\alpha$$
$$Y' := -X \sin\alpha + Y \cos\alpha$$

Proof. Because the map $(X, Y) \mapsto (X', Y')$ is linear, the new variables are also jointly normal. Clearly, they have zero mean, while

$$EX'^2 = EX^2 \cos^2\alpha + EY^2 \sin^2\alpha + 2EXY \sin\alpha\cos\alpha$$
$$= \cos^2\alpha + \sin^2\alpha + 0 = 1.$$

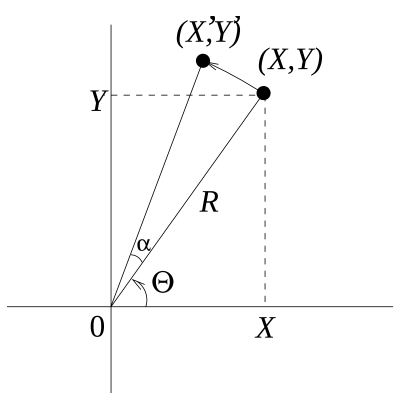
Similarly,

$$EY'^2 = 1,$$

and

$$EX'Y' = 0.$$

So the X', Y' are uncorrelated and, being jointly normal, this means that they are independent with unit variance each. □



This means that if we rotate the point (X, Y) on the Euclidean plane by any angle then its distribution does not change. This implies that if Θ denotes the angle formed between the vector and the positive horizontal axis then the distribution of Θ is the same as the distribution of any translation of it by any angle. So Θ is uniform (between 0 and 2π).

Next define

$$R := \sqrt{X^2 + Y^2}$$

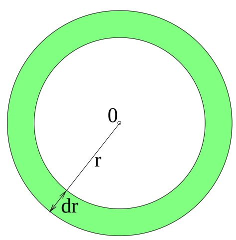
We have

Lemma 2. *R^2 is exponentially distributed with parameter 2.*

Proof. Notice that the joint density of (X, Y) is

$$f(x, y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

and so it is constant on circles centred at the origin: its value on the circle with radius r is $\frac{1}{2\pi} e^{-r^2/2}$. The probability that R falls in the ring between two circles centred at the origin with radii r and $r + dr$, when dr is small, is about the value of f on the circle of radius r times the area of the ~~disk~~ ring; this area is about $2\pi r dr$.



Hence

$$P(r < R < r + dr) \approx r e^{-r^2/2} dr. = \frac{1}{2} e^{-r^2/2} d(r^2).$$

Hence

$$P(u < R^2 < u + du) \approx \frac{1}{2} e^{-u/2} du.$$

This means that R^2 has density $\frac{1}{2} e^{-u/2}$, as claimed. □

Exercise 1. *Show that Θ and R^2 are independent.*

If we know R and Θ then certainly we know X and Y:

$$X = R \cos \theta$$
$$Y = R \sin \theta.$$

We can easily simulate R^2 by setting

$$R^2 = -2\log U,$$

where U is uniform between 0 and 1. We can simulate Θ by setting

$$\Theta = 2\pi V,$$

where V is independent of U, also uniform between 0 and 1.
Hence

$$X = \sqrt{-2\log U}\cos(2\pi V)$$
$$Y = \sqrt{-2\log U}\sin(2\pi V)$$

gives two i.i.d. standard normals.

Exercise 2. *Show that the product XY of two i.i.d. standard normal has the same law as TS where T is exponential with mean 1 and S has the arcsine law:*

$$P(S \in ds) = \frac{1}{\pi\sqrt{1-s^2}}\, ds, \quad -1 < s < 1.$$

Conclude that the density of XY is

$$\frac{1}{\pi}K_0(|u|) = \frac{1}{\pi}\int_1^\infty \frac{e^{-|u|t}}{\sqrt{t^2-1}}dt, \quad -\infty < u < \infty,$$

a function which is known as the 0-th order modified Bessel function of the second kind. In R, this function is called by the command `besselK(x,0)`*.*
You can experiment as follows

```
n=100000
x=rnorm(n); y=rnorm(n)
cx=abs(x)<2; cy=abs(y)<2
X=x[cx]; Y=y[cy]
hist(X*Y,breaks=100)
plot((1/pi)*besselK(seq(0.01,3,0.01),0),type='l')
```

Compare the histogram with the plot of the Bessel function.