Stochastic Processes: PROBLEMS

1. Let X be a random variable with Gamma(α) density:

$$f(x) = \frac{1}{\Gamma(\alpha+1)} x^\alpha e^{-x}, \quad x > 0,$$

for some $\alpha > 0$. Compute the rate function.

2. Consider the recursion

$$X_{n+1} = \rho X_n + \xi_n, \quad n = 0, 1, 2, \dots$$

where ρ is a number with $|\rho| < 1$, and $\xi_0, \xi_1, \xi_2, \dots$ are i.i.d. random variables with zero mean and finite variance. Also $X_0 = 0$. Compute an approximate expression for

$$P\left(\left| \frac{X_1 + \cdots + X_n}{n} \right| > \delta \right)$$

for $\delta > 0$, in the following two cases:
a) ξ_n is Normal with mean zero and variance 1,
b) ξ_n is any random variable with mean zero and variance 1.
Assume that n is sufficiently large.

3. There are two boxes, A and B, containing N balls in total. Some are in box A and some in B. The balls change position (from A to B or from B to A) according to the following rule: A ball remains in its present box for an amount of time that is exponentially distributed with parameter $\lambda > 0$. When the time expires, the ball changes position. This process repeats endlessly. All balls behave completely independently from each other.
a) Justify the fact that X_t, the number of balls in box A at time t is a Markov process as a function of the continuous parameter $t \geq 0$.
b) Assume that $X_0 = 0$ (no balls in box A at time 0). Compute, for any $t \geq 0$, the probabilities

$$P(X_t = k), \quad k = 0, 1, \dots, N.$$

4. Two particles, a, b, perform independent random walks in the two-dimensional integer lattice and in discrete time (up, down, left or right, at unit steps with equal probabilities, 1/4 each). Let X_n be

the position of particle a at step n. Similarly, let Y_n be the position of particle b at step n. For large n, and any $\delta > 0$, find an approximate expression for the probability

$$P(|X_n - Y_n| > n\delta)$$

where, for any two vectors $x = (x_1, x_2)$, $y = (y_1, y_2)$, the quantity $|x - y|$ stands for $|x - y| = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. Finally, for $\delta = 1/2$, and $n = 10000$, express this probability in the form $10^{-\epsilon}$ (find the exponent ϵ).

5. Consider an experiment with two outcomes (for example, a coin). Let μ be the distribution $\mu(1) = 0.3$, $\mu(2) = 0.7$. The experiment is performed $n = 10^6$ times, independently, and the empirical distribution is computed. In other words, the fraction of times that outcome 1 occurs is computed, and so is the fraction of time that outcome 2 occurs. It is observed that the first fraction is 0.31, and the second is 0.69. Would you accept or reject the hypothesis that the true distribution is μ, with a 99% degree of confidence? (Hint: Use Sanov's theorem.)