Stochastic Processes: PROBLEMS, ANSWERS, AND SOLUTIONS

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I. PROBLEMS

1. Let X be a random variable with Gamma(α) density:
$$f(x) = \frac{1}{\Gamma(\alpha+1)} x^\alpha e^{-x}, \quad x > 0,$$
for some $\alpha > 0$. Compute the rate function.

2. Consider the recursion
$$X_{n+1} = \rho X_n + \xi_n, \quad n = 0, 1, 2, \dots$$
where ρ is a number with $|\rho| < 1$, and $\xi_0, \xi_1, \xi_2, \dots$ are i.i.d. random variables with zero mean and finite variance. Also $X_0 = 0$. Compute an approximate expression for
$$P\left(\left|\frac{X_1 + \cdots + X_n}{n}\right| > \delta\right)$$
for $\delta > 0$, in the following two cases:
a) ξ_n is Normal with mean zero and variance 1,
b) ξ_n is any random variable with mean zero and variance 1.
Assume that n is sufficiently large.

3. There are two boxes, A and B, containing N balls in total. Some are in box A and some in B. The balls change position (from A to B or from B to A) according to the following rule: A ball remains in its present box for an amount of time that is exponentially distributed with parameter $\lambda > 0$. When the time expires, the ball changes position. This process repeats endlessly. All balls behave completely independently from each other.
a) Justify the fact that X_t, the number of balls in box A at time t is a Markov process as a function of the continuous parameter $t \geq 0$.
b) Assume that $X_0 = 0$ (no balls in box A at time 0). Compute, for any $t \geq 0$, the probabilities
$$P(X_t = k), \quad k = 0, 1, \dots, N.$$

4. Two particles, a, b, perform independent random walks in the two-dimensional integer lattice and in discrete time (up, down, left or right, at unit steps with equal probabilities, $1/4$ each). Let X_n be the position of particle a at step n. Similarly, let Y_n be the position of particle b at step n. For large n, and any $\delta > 0$, find an approximate expression for the probability
$$P(|X_n - Y_n| > n\delta)$$
where, for any two vectors $x = (x_1, x_2)$, $y = (y_1, y_2)$, the quantity $|x - y|$ stands for $|x - y| = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. Finally, for $\delta = 1/2$, and $n = 10000$, express this probability in the form $10^{-\varepsilon}$ (find the exponent ε).

5. Consider an experiment with two outcomes (for example, a coin). Let μ be the distribution $\mu(1) = 0.3$, $\mu(2) = 0.7$. The experiment is performed $n = 10^6$ times, independently, and the empirical distribution is computed. In other words, the fraction of times that outcome 1 occurs is computed, and so is the fraction of time that outcome 2 occurs. It is observed that the first fraction is 0.31, and the second is 0.69. Would you accept or reject the hypothesis that the true distribution is μ, with a 99% degree of confidence? (Hint: Use Sanov's theorem.)

II. ANSWERS

Problem 1
The rate function is

$$h(t) = \begin{cases} t - (\alpha+1) + (\alpha+1)\log\left(\frac{\alpha+1}{t}\right), & t > 0 \\ +\infty, & t \le 0. \end{cases}$$

Problem 2
a) Let $S_n = X_1 + \cdots + X_n$. Then S_n is normal, and

$$P(|S_n/n| > \delta) \approx \sqrt{\frac{2}{\pi n}} \frac{1}{\delta(1-\rho)} e^{-n\delta^2(1-\rho)^2/2}.$$

b) Let h be the rate function of ξ_1. Let $s_n = \xi_1 + \cdots + \xi_n$. Then

$$P(|S_n/n| > \delta) \approx P(|s_n/n| > \delta(1-\rho)) \approx e^{-n \min[h(\delta(1-\rho)),\ h(-\delta(1-\rho))]}.$$

The results are asymptotic as $n \to \infty$, in the sense that the logarithms of the probabilities above, divided by n, have a limit as $n \to \infty$.

Problem 3
The exact answer is

$$P(X_t = k) = \frac{1}{2^N}\binom{N}{k}\left(1 - e^{-2\lambda t}\right)^k \left(1 + e^{-2\lambda t}\right)^{N-k}, \quad k = 0, 1, \ldots, N, \quad t \ge 0.$$

As $t \to \infty$ the probabilities converge:

$$\lim_{t\to\infty} P(X_t = k) = \frac{1}{2^N}\binom{N}{k}, \quad k = 0, 1, \ldots, N.$$

Problem 4

$$P(|X_n - Y_n| > n\delta) \approx \left(\frac{(2+\delta)^{2+\delta}(2-\delta)^{2-\delta}}{16}\right)^n.$$

For $\delta = 0.5$, $n = 10000$

$$P(|X_n - Y_n| > n/2) \approx \left(\frac{16}{2.5^{2.5}\ 1.5^{1.5}}\right)^n \approx 0.88 \approx 10^{-548.67}.$$

Problem 5
The hypothesis should be rejected.

III. SOLUTIONS

Problem 1
The rate function is given by

$$h(t) = \sup_\theta (t\theta - \log E \exp \theta X)$$

We compute the m.g.f. $\varphi(\theta) = E \exp \theta X$ of X by:

$$\varphi(\theta) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{\theta x} x^\alpha e^{-x} dx$$
$$= \frac{1}{\Gamma(\alpha+1)} \frac{1}{(1-\theta)^{\alpha+1}} \int_0^\infty y^\alpha e^{-y} dy = \frac{1}{(1-\theta)^{\alpha+1}}.$$

To perform the maximization (assuming $t > 0$), we solve the equation

$$\varphi'(\theta) = t\varphi(\theta),$$

arising by taking the derivative of $t\theta - \log E \exp \theta X$ with respect to θ and setting it equal to zero. This gives

$$\theta = 1 - \frac{\alpha+1}{t}.$$

Substituting, we obtain the answer. The complete solution is:

$$h(t) = \begin{cases} t - (\alpha+1) + (\alpha+1)\log\left(\frac{\alpha+1}{t}\right), & t > 0 \\ +\infty, & t \le 0. \end{cases}$$

□

Problem 2
It is easy to solve the recursion. We obtain:

$$X_n = \xi_{n-1} + \rho\xi_{n-2} + \rho^2\xi_{n-3} + \cdots + \rho^{n-2}\xi_1 + \rho^{n-1}\xi_0, \qquad n \ge 1.$$

Hence

$$S_n := X_1 + \cdots + X_n = \frac{1}{1-\rho}\sum_{k=1}^n (1-\rho^k)\xi_{n-k}.$$

Case (a): the ξ_n are i.i.d. normal with mean zero and variance 1.
Then S_n is normal with mean zero, and variance

$$\sigma_n^2 := ES_n^2 = \frac{1}{(1-\rho)^2}\sum_{k=1}^n (1-\rho^k)^2 = \frac{n}{(1-\rho)^2} - \frac{\rho(1-\rho^n)(2+\rho-\rho^{n+1})}{(1-\rho)^3(1+\rho)}.$$

Let N be a standard normal random variable. The standard normal approximation says that

$$P(N > u) \sim \frac{1}{\sqrt{2\pi}}\frac{1}{u}e^{-u^2/2}, \quad \text{as } u \to \infty,$$

where $f(u) \sim g(u)$ as $u \to \infty$ means $\lim_{u\to\infty}[f(u)/g(u)] = 1$. We apply this to the probability we are interested in:

$$P(|S_n/n| > \delta) = P(|\sigma_n N/n| > \delta) = 2P(N > n\delta/\sigma_n).$$

Since $u := n\delta/\sigma_n \to \infty$, as $n \to \infty$, the normal approximation applies and gives

$$P(|S_n/n| > \delta) \sim \sqrt{\frac{2}{\pi}} \frac{\sigma_n}{n\delta} \exp\left\{ -\frac{1}{2} \frac{n^2\delta^2}{\sigma_n^2} \right\} = \sqrt{\frac{2}{\pi}} \exp\left\{ -\frac{1}{2} \left[\frac{n^2\delta^2}{\sigma_n^2} + \log \frac{n^2\delta^2}{\sigma_n^2} \right] \right\}, \quad \text{as } n \to \infty.$$

To understand this better, write

$$\sigma_n^2 = an - b_n,$$
$$\text{where} \quad a = \frac{1}{(1-\rho)^2}, \quad b_n = \frac{\rho(1-\rho^n)(2+\rho-\rho^{n+1})}{(1-\rho)^3(1+\rho)},$$

and b_n converges to a constant, as $n \to \infty$. Then

$$\frac{n^2\delta^2}{\sigma_n^2} = \frac{n\delta^2}{a}\big(1 - \frac{b_n}{an}\big)^{-1} = \frac{n\delta^2}{a}\big(1 + \frac{b_n}{an} + o(1/n)\big),$$
$$\log \frac{n^2\delta^2}{\sigma_n^2} = \log \frac{n\delta^2}{a} + \frac{b_n}{an} + o(1/n),$$
$$P(|S_n/n| > \delta) \sim \sqrt{\frac{2}{\pi}} \exp\left\{ -\frac{1}{2} \left[\frac{n\delta^2}{a} + \log \frac{n\delta^2}{a} + \frac{b_n\delta^2}{a} + o(1) \right] \right\}.$$

This is a very precise answer for large n. But, for all practical purposes, the last two terms in the brackets are negligible, and so we can approximately write

$$P(|S_n/n| > \delta) \approx \sqrt{\frac{2}{\pi n}} \frac{1}{\delta(1-\rho)} e^{-n\delta^2(1-\rho)^2/2}.$$

Case (b): the ξ_n are general i.i.d.
We write

$$S_n = \hat{S}_n - V_n$$
$$\hat{S}_n = \frac{1}{1-\rho} \sum_{k=1}^{n} \xi_{n-k}, \quad V_n = \frac{1}{1-\rho} \sum_{k=1}^{n} \rho^k \xi_{n-k},$$

and show that (in an asymptotically logarithmic sense) V_n can be omitted. Let $\delta, \varepsilon > 0$.

$$P(|S_n/n| > \delta) = P(\hat{S}_n > n\delta + V_n) + P(\hat{S}_n < -n\delta + V_n)$$
$$\geq P(\hat{S}_n > n\delta + V_n, V_n < n\varepsilon) + P(\hat{S}_n < -n\delta + V_n, V_n > -n\varepsilon)$$
$$\geq P(\hat{S}_n/n > \delta + \varepsilon, V_n/n < \varepsilon) + P(\hat{S}_n/n < -(\delta + \varepsilon), V_n/n > -\varepsilon).$$

But $V_n/n \to 0$, as $n \to \infty$, almost surely. Hence $P(V_n/n < \varepsilon) \to 1$, and $PV_n/n > -\varepsilon) \to 1$, as $n \to \infty$. We obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log P(|S_n/n| > \delta) \geq \liminf_{n\to\infty} \frac{1}{n} \log P(|\hat{S}_n/n| > \delta + \varepsilon),$$

for all $\varepsilon > 0$, and so

$$\liminf_{n\to\infty} \frac{1}{n} \log P(|S_n/n| > \delta) \geq \liminf_{n\to\infty} \frac{1}{n} \log P(|\hat{S}_n/n| > \delta)$$
$$= \liminf_{n\to\infty} \frac{1}{n} \log P\left(\frac{1}{n} \Big| \sum_{k=1}^{n} \xi_k \Big| > \delta(1-\rho) \right) = -\min[h(\delta(1-\rho)), h(-\delta(1-\rho))],$$

where

$$h(x) := \sup_\theta \left[\theta x - \log\varphi(\theta)\right], \quad x \in \mathbb{R},$$
$$\varphi(\theta) := E\exp(\theta\xi_1)$$

To obtain an upper bound, we use Chernoff's inequality. Let $c_k = 1-\rho^k$. For all $\theta > 0$,

$$\begin{aligned} P(S_n/n > \delta) = P\bigg(\sum_{k=1}^n c_k\xi_k > n\delta(1-\rho)\bigg) &\le \exp(-n\theta\delta(1-\rho))E\exp\theta\sum_{k=1}^n c_k\xi_k \\ &= \exp\bigg\{-n\theta\delta(1-\rho)+\sum_{k=1}^n\log\varphi(\theta c_k)\bigg\} \\ &= \exp\big\{-n\theta\delta(1-\rho)+n\log\varphi(\theta)\big\}\exp\bigg\{\sum_{k=1}^n\log\varphi(\theta c_k)-n\log\varphi(\theta)\bigg\}. \end{aligned}$$

Minimizing the first exponential with respect to θ, we have

$$P(S_n/n>\delta) \le e^{-nh(\delta(1-\rho))}\exp\bigg\{\sum_{k=1}^n\log\varphi(\theta c_k)-n\log\varphi(\theta)\bigg\}.$$

This inequality is true for all $\theta > 0$. So

$$\limsup_{n\to\infty}\frac{1}{n}\log P(S_n/n>\delta) \le -h(\delta(1-\rho)) + \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\big(\log\varphi(\theta c_k)-\log\varphi(\theta)\big).$$

By the continuity of φ, and the fact that $c_n = 1-\rho^n \to 1$, we have

$$\lim_{n\to\infty}[\log\varphi(\theta c_n)-\log\varphi(\theta)] = 0,$$

and so the Cesàro limit is also zero:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\big(\log\varphi(\theta c_k)-\log\varphi(\theta)\big) = 0.$$

Thus,

$$\limsup_{n\to\infty}\frac{1}{n}\log P(S_n/n>\delta) \le -h(\delta(1-\rho)).$$

Similarly,

$$\limsup_{n\to\infty}\frac{1}{n}\log P(S_n/n<-\delta) \le -h(-\delta(1-\rho)).$$

This means that

$$\limsup_{n\to\infty}\frac{1}{n}\log P(|S_n/n|>\delta) \le -\min[h(\delta(1-\rho)),h(-\delta(1-\rho))].$$

We proved that $\limsup$ and $\liminf$ coincide. Hence

$$\lim_{n\to\infty}\frac{1}{n}\log P(|S_n/n|>\delta) = -\min[h(\delta(1-\rho)),h(-\delta(1-\rho))].$$

The answer then to the problem is that

$$P(|S_n/n| > \delta) \approx e^{-n\min[h(\delta(1-\rho)),h(-\delta(1-\rho))]},$$

where h is the rate function of ξ_1.
Let's compare the two approximations:
If ξ_1 is normal(0,1), then $h(\delta) = \delta^2/2$. Hence the last approximation of Case (b) gives

$$P(|S_n/n| > \delta) \approx e^{-n\delta^2(1-\rho)^2/2}.$$

This is slightly worse than the standard normal approximation of Case (a), but, asymptotically, they are equivalent (their logarithms, that is). □

Problem 3
a) The Markov property is based on the fact that the random variables involved are exponentially distributed, and independent of each other.
b) Consider the motion of a *specific* ball, say ball i ($1 \le i \le N$). Let ξ_t^i be its position at time t:

$$\xi_t^i = \begin{cases} 1, & \text{if ball } i \text{ is in box } A \text{ at time } t, \\ 0, & \text{if it is in } B. \end{cases}$$

It is clear that

$$\{\xi_t^1, t \ge 0\}, \ldots, \{\xi_t^N, t \ge 0\}$$

are independent processes. (The balls move completely independently of each other.) Furthermore, they are identical in distribution. Also, their sum is

$$X_t = \xi_t^1 + \cdots + \xi_t^N,$$

that is, their sum is the process we are interested in. Let

$$p_t = P(\xi_t^1 = 1)$$

be the probability that ball 1 is in box A at time t. Then, for each t, X_t has binomial distribution. So

$$P(X_t = k) = \binom{N}{k} p_t^k (1-p_t)^{N-k}.$$

To compute p_t notice that ξ^1 itself is Markovian. Hence

$$\begin{aligned} p_{t+\delta} &= P(\xi_{t+\delta}^1 = 1 \mid \xi_t^1 = 0)P(\xi_t^1 = 0) + P(\xi_{t+\delta}^1 = 1 \mid \xi_t^1 = 1)P(\xi_t^1 = 1) \\ &= (\lambda\delta + o(\delta))(1-p_t) + (1 - \lambda\delta + o(\delta))p_t, \end{aligned}$$

as $\delta \downarrow 0$. Dividing by δ, and letting it go to zero, we obtain the differential equation

$$\frac{dp_t}{dt} = \lambda - 2\lambda p_t,$$

with initial condition $p_0 = 0$ (given by the problem). The solution is

$$p_t = \frac{1}{2}\big(1 - e^{-2\lambda t}\big).$$

The complete answer to the problem is:

$$P(X_t = k) = \frac{1}{2^N}\binom{N}{k}\big(1 - e^{-2\lambda t}\big)^k \big(1 + e^{-2\lambda t}\big)^{N-k}, \quad k = 0, 1, \ldots, N, \quad t \geq 0.$$

Notice that

$$\lim_{t\to\infty} P(X_t = k) = \frac{1}{2^N}\binom{N}{k},$$

as should be expected! □

Problem 4

There are two ways we can proceed: the hard way, and the easy way.

THE HARD WAY:

Let e_1, e_2 be the standard unit vectors in $\mathbb{R}^2$, i.e. $e_1 = (1, 0)$, $e_2 = (0, 1)$. Note that $X_n - Y_n$ is a random walk itself, i.e. $X_n - Y_n = \xi_1 + \xi_2 + \cdots + \xi_n$, where the vectors ξ_j are the combined steps. These are i.i.d. random variables with values and probabilities as below:

$\xi_n =$	0	$2e_1$	$2e_2$	$-2e_1$	$-2e_2$	$e_1 + e_2$	$e_1 - e_2$	$-e_1 + e_2$	$-e_1 - e_2$
with prob.	$\frac{4}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$.

We compute the moment generating function of ξ_n (with components ξ_n^1, ξ_n^2):

$$\begin{aligned}\varphi(\theta_1, \theta_2) &= E\exp(\theta_1\xi_n^1 + \theta_2\xi_n^2) \\ &= \frac{1}{16}\big[4 + e^{2\theta_1} + e^{2\theta_2} + e^{-2\theta_1} + e^{-2\theta_2} + 2e^{\theta_1+\theta_2} + 2e^{\theta_1-\theta_2} + 2e^{-\theta_1+\theta_2} + 2e^{-\theta_1-\theta_2}\big].\end{aligned}$$

To find the rate function

$$h(x_1, x_2) = \sup_{(\theta_1,\theta_2)\in\mathbb{R}^2}\big[\theta_1 x_2 + \theta_2 x_2 - \log\varphi(\theta_1, \theta_2)\big]$$

we need to solve the equations

$$\frac{\partial\varphi}{\partial\theta_1}(\theta_1, \theta_2) = x_1\varphi(\theta_1, \theta_2), \qquad \frac{\partial\varphi}{\partial\theta_2}(\theta_1, \theta_2) = x_2\varphi(\theta_1, \theta_2).$$

And so on.................... But, there is...

AN EASY WAY:

We look at each of the coordinates separately. Let $X_n = (X_n^1, X_n^2)$, $Y_n = (Y_n^1, Y_n^2)$. Then

$$\begin{gathered}P(|X_n - Y_n| > n\delta) = P(|X_n^1 - Y_n^1| > n\delta \text{ or } |X_n^2 - Y_n^2| > n\delta) \\ = P(|X_n^1 - Y_n^1| > n\delta) + P(|X_n^2 - Y_n^2| > n\delta) - P(|X_n^1 - Y_n^1| > n\delta, |X_n^2 - Y_n^2| > n\delta) \\ = P(A_n^1) + P(A_n^2) - P(A_n^1 \cap A_n^2).\end{gathered}$$

Let

$$p(n) = P(A_n^1) = P(A_n^2), \quad q(n) = P(A_n^1 \cap A_n^2).$$

Hence $P(|X_n - Y_n| > n\delta) = 2p(n) - q(n)$, and so

$$\frac{1}{n}\log P(|X_n - Y_n| > n\delta) = \frac{1}{n}\log p(n) + \frac{1}{n}\log\big(2 - \frac{q(n)}{p(n)}\big).$$

By the Large Deviation Principle for 1-dimensional random walk,

$$\lim_{n\to\infty} \frac{1}{n} \log p(n) = -h(\delta),$$

where $h(\delta) > 0$ will be computed below. On the other hand, it can be deduced that

$$\frac{q(n)}{p(n)} = P(A_n^2 \mid A_n^1)$$

also obeys a Large Deviation Principle, and so it converges to zero (exponentially fast). Hence $\frac{1}{n} \log \big(2 - \frac{q(n)}{p(n)}\big) \to 0$. We thus conclude that

$$\lim_{n\to\infty} \frac{1}{n} \log P(|X_n - Y_n|) = \lim_{n\to\infty} \frac{1}{n} \log p(n) = -h(\delta),$$

where $h(\delta)$ is the rate function of the first component ξ_n^1 of the increment vector ξ_n. As before, we see that

$$\begin{array}{llllll} \xi_n^1 = & 0 & 1 & -1 & 2 & -2 \\ \text{with prob.} & \frac{6}{16} & \frac{4}{16} & \frac{4}{16} & \frac{1}{16} & \frac{1}{16}. \end{array}$$

The moment generating function is

$$\varphi(\theta) = E \exp(\theta \xi_n^1) = \frac{1}{16}\big[6 + 4e^{\theta} + 4e^{-\theta} + e^{2\theta} + e^{-2\theta}\big].$$

To compute $h(\delta)$ we solve

$$\begin{aligned} & \varphi'(\theta) = \delta\varphi(\theta) \\ \Leftrightarrow\quad & 4e^{\theta} - 4e^{-\theta} + 2e^{2\theta} - 2e^{-2\theta} = \delta\big[6 + 4e^{\theta} + 4e^{-\theta} + e^{2\theta} + e^{-2\theta}\big] \\ \Leftrightarrow\quad & 4y - 4y^{-1} + 2y^2 - 2y^{-2} = 6\delta + 4\delta y + 4\delta y^{-1} + \delta y^2 + \delta y^{-2}, \quad \text{where } y := e^{\theta} \\ \Leftrightarrow\quad & \Pi(y) := (2-\delta)y^4 + 4(1-\delta)y^3 - 6\delta y^2 - 4(1+\delta)y - (2+\delta) = 0. \end{aligned}$$

The above polynomial has a double root at $y = -1$. Hence, by Euclidean division by $(y+1)^2$, we find

$$\Pi(y) = (y+1)^2\big[(2-\delta)y^2 - 2\delta y - (2+\delta)\big].$$

The quadratic in brackets has roots

$$y = \frac{\delta \pm 2}{2-\delta}.$$

Only the positive one is acceptable, and this gives

$$y = \frac{2+\delta}{2-\delta}, \qquad \text{and so } \theta = \log \frac{2+\delta}{2-\delta}.$$

Substituting this θ into $\delta\theta - \log \varphi(\theta)$ we obtain

$$h(\delta) = \delta \log \frac{2+\delta}{2-\delta} - 2 \log \frac{4}{(2-\delta)(2+\delta)} = \log\left(\frac{(2+\delta)^{2+\delta}(2-\delta)^{2-\delta}}{16}\right).$$

And finally we have the approximation (a very good one for n large!)

$$P(|X_n - Y_n| > n\delta) \approx \left(\frac{(2+\delta)^{2+\delta}(2-\delta)^{2-\delta}}{16}\right)^n,$$

so, for $\delta = 0.5$, $n = 10000$

$$P(|X_n - Y_n| > n/2) \approx \left(\frac{16}{2.5^{2.5}\ 1.5^{1.5}}\right)^n \approx 0.88^n = 0.88^{10000} \approx 10^{-548.67}.$$

$\square$

Problem 5

Suppose we perform the ideal experiment, tossing a coin, with probability of heads equal to $p = 0.3$. If μ_n denotes the fraction of 1's in n trials, then, by Sanov's theorem,

$$P(\mu_n > x) \approx e^{-nh(x)}$$
$$\text{where} \quad h(x) = x \log\left(\frac{x}{p}\right) + (1-x)\log\left(\frac{1-x}{1-p}\right),$$

whenever $x > p$. Similarly,

$$P(\mu_n < y) \approx e^{-nh(y)},$$

whenever $y < p$. With $x = 0.31$ and $y = 0.29$ we find

$$h(0.31) = 0.31 \log\left(\frac{0.31}{0.30}\right) + 0.69 \log\left(\frac{0.69}{0.70}\right)$$
$$\approx 0.31 \times 0.03279 + 0.69 \times (-0.01439) \approx 2.36616 \times 10^{-4}.$$

This gives

$$P(\mu_n > 0.31) \approx e^{-10^6 \times 2.36616 \times 10^{-4}} \approx e^{-237} \approx 10^{-103}.$$

Similarly,

$$h(0.29) = 0.29 \log\left(\frac{0.29}{0.30}\right) + 0.71 \log\left(\frac{0.71}{0.70}\right)$$
$$\approx 0.29 \times (-0.03390) + 0.71 \times 0.01418 \approx 2.39641 \times 10^{-4}.$$

This gives

$$P(\mu_n < 0.29) \approx e^{-10^6 \times 2.39641 \times 10^{-4}} \approx e^{-240} \approx 10^{-104}.$$

In other words,

$$P(|\mu_n - 0.3| > 0.1) \approx 10^{-103}.$$

We thus reject the hypothesis that the true value of p is 0.3. $\square$