Chapter III

The Optimal Stopping Problem

§1. The Problem of Optimal Choice

We start with the following problem. Suppose that we scan n objects in random succession and that from these objects we wish to choose the best one. After an examination of each object in turn, we must either accept or reject that particular one; it is inadmissible to return to an object previously rejected.

The latter condition, of course, is not always a realistic limitation. It is realistic, for example, if we are concerned with an automobile tourist who wishes to stop over in the most comfortable or the least expensive hotel along his route but has no intention of backtracking (assuming that the traveler is apprised of the number of hotels, but knows nothing of their quality). Or consider the astute bride-to-be who wishes to make an unerring choice of the best of all the suitors proposing marriage to her. With this second interpretation our postulate of being unable to return to a previously rejected object is adequately justified On the other hand, the stipulation that the decision-maker has prior knowledge of the total number of objects n appears rather artificial in this case.

We now make a more precise statement of the problem. There exist n objects, ordered in some definite manner according to their quality. We might think of these objects as represented, for example, by points on a line, where points further to the right correspond to "better" objects. We denote by a_1 the first object we come to. Inasmuch as the objects are inspected in random sequence, the probabilities of any of the existing n points turning out to be the point a_1 are identical. From precisely the same point of view the point a_2 has equal probability of being any of the remaining n − 1 points. Continuing to number the objects in the order in which we meet them, we ultimately arrive at a certain set $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$, where any of the n! conceivable permutations appears with equal probability. This permutation becomes gradually known, after the second test we know only the relative position of a_1 and a_2, whereas after the kth test we know the relative position of $a_1, a_2, \ldots, a_k$ (the reader might think of indicator lights flashing one after the other at the points $a_1, a_2, \ldots, a_k, \ldots, a_n$). The problem is to discriminate the rightmost of all the n points at the instant it first appears. It is required to indicate the method by which this result is achieved with maximum probability.

For a better understanding of the problem we consider some elementary selection techniques. We could, for example, decide on the first point a_1. Clearly, the probability of guessing the rightmost point in this case is equal to 1/n (and thus tends to zero as $n \to \infty$). The same result is obtained if we decide on a_2 or on a_3, etc.

It might seem at first glance that the probability of success in any system of selection would tend to zero as $n \to \infty$. But this is not the case. Let us suppose for simplicity that the number of points n is even. Let us assume that we pass over the first n/2 points, then choose the first point that falls to the right of all the preceding ones. Following this strategy, we are certain to achieve our goal if the best object happens to lie in the second half of the sequence $a_1, \ldots, a_n$ and the second best object lies in the first half. The probability of the two best objects being so arranged is equal to $[(n/2)/n] \cdot [(n/2)/(n-1)] > 1/4$. Hence, no matter how large n even is made, there exists a strategy for which the probability of success is greater than 1/4.

Let the allocation of the points $a_1, \ldots, a_k$ on the line be already known (see Fig. 20, where k = 4). We wish to determine the

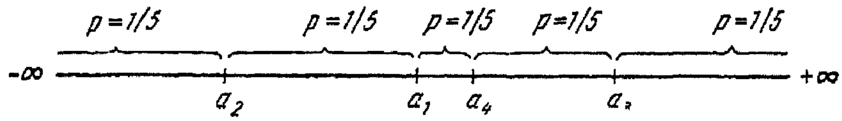

Fig. 20

probability of the next point a_{k+1} falling in each of the $k+1$ intervals partitioned off on the line by the points $a_1, \ldots, a_k$. The occurrence of a_{k+1} in a fixed interval corresponds to a definite permutation of the $k-1$ points $a_1, \ldots, a_k, a_{k+1}$ Inasmuch as all the points are equally probable, the probability of any such permutation is the reciprocal of the number of all permutations of $k-1$ elements and is equal to $1/(k+1)!$. Similarly, the probability of the permutation of the points $a_1, \ldots, a_k$ corresponding to their given position on the line is equal to $1/k!$. Consequently, the conditional probability of the point a_{k-1} falling in any of the $k+1$ intervals, given the condition that the relative position of the points $a_1, \ldots, a_k$ is known, is equal to $[1/(k+1)!]/(1/k!) = 1/(k+1)$, no matter how the points $a_1, \ldots, a_k$ are arranged. Thus, *the next point observed has equal probability of occurring in any of the intervals into which the line is divided by the existing points, regardless of the order in which these points have appeared.*

If the next point a_k turns out to be to the left of some previously inspected point, it is clearly not the rightmost. Consequently, it is only necessary to make our choice from among the points a_k situated to the right of the previous points $a_1, \ldots, a_{k-1}$. We call these points *maximal points*. It is clear that the point a_1 is always maximal, just as the rightmost of all the points $a_1, \ldots, a_n$ is maximal. This sought-after point is the last maximal point to be counted.

When the next maximal point a_k occurs, it is necessary to make a decision, either to choose that point or to wait until later. At this time the relative position of the points $a_1, \ldots, a_k$, of which a_k is the rightmost, is known. Since it is only possible now to choose from among the points $a_k, a_{k+1}, \ldots, a_n$, the decision rests solely on the prediction regarding the relative position of the points $a_k, a_{k+1}, \ldots, a_n$. With the stipulation that the points $a_1, \ldots, a_k$ are known, nothing affects this decision other than the conditional probabilities of the various permutations of the points $a_k, a_{k+1}, \ldots, a_n$. We will show that the conditional probabilities actually depend only

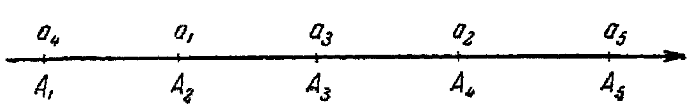

Fig. 21

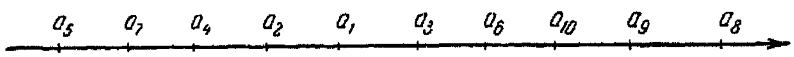

Fig. 22

on the index k and are not influenced in any way by the relative position of the points $a_1, \ldots, a_{k-1}$. In this way we establish the fact that *when a maximal point a_k makes its appearance, a particular decision must be made solely on the basis of the index k of that point* (with due regard, of course, for the number n of all the points).

The points $a_1, \ldots, a_n$ are numbered in the order of their occurrence on the line. We renumber the points $a_1, \ldots, a_k$ that have already occurred in the order of their position on the line from left to right: $A_1, \ldots, A_k$. Since the point a_k is maximal, a_k coincides with A_k (Fig. 21). Assignment of the relative arrangement of the points $a_1, \ldots, a_k$ is equivalent to assignment of the order of occurrence of the points $A_1, \ldots, A_k$. The fact that any permutation of the points $A_k, a_{k+1}, \ldots, a_n$ is independent of the order in which the points $A_1, \ldots, A_{k-1}$ occurred is established with the observation that the individual events, namely the relative position of the points $A_k, a_{k+1}, \ldots, a_n$, as well as their position relative to the points $A_1, \ldots, A_{k-1}$, are independent of the order of accession of the points $A_1, \ldots, A_{k-1}$. This is implied by the fact that each point in turn, as established earlier, has equal probability of falling within any of the intervals into which the line is divided by the preceding points. Specifically, the point a_{k+1} has a probability $1/(k+1)$ of occurring in any of the intervals $(-\infty, A_1)$, (A_1, A_2), ..., $(A_k, +\infty)$, regardless of the accession order of the points $A_1, \ldots, A_{k-1}$, the point a_{k+2} has a probability $1/(k+2)$ of occurring in any of the intervals delimited by the points $A_1, \ldots, A_k$ and a_{k+1}, regardless of the accession order of the points $A_1, \ldots, A_{k-1}$, etc. Multiplying these probabilities, we deduce that the probability of any permutation of the points $A_1, \ldots, A_k, a_{k+1}, \ldots, a_n$ (associated with the natural order of the points $A_1, \ldots, A_k$) is equal to

$$\frac{1}{k+1} \cdot \frac{1}{k+2} \cdots \frac{1}{n}$$

regardless of the accession order of the points $A_1, \ldots, A_{k-1}$. This proves our original assertion.

For example, let n = 10, and let the points $a_1, \ldots, a_{10}$ be arranged as in Fig. 22. Then the maximal points are a_1, a_3, a_6, and a_8. When the point a_1 occurs, it is necessary to make a decision with regard only for the fact that its index is equal to one; when the point a_3 occurs, the decision is based solely on the fact that its index is equal to three (providing, of course, that we have not stopped earlier); etc.

Thus, in order to make the optimal decision,* it is sufficient to analyze only the indices of the maximal points. We designate these indices in increasing order x(0), x(1), x(2), ... As already mentioned, x(0) = 1. The indices x(1), x(2), ... are random, just as the number of these indices is random. None of the indices exceeds n. The last (largest) of the indices x(i) is the index of the rightmost point a_k, and it must be guessed with the highest possible probability. The guessing procedure rests on the fact that when the next random variable x(i) comes up, it is required solely on the basis of its value either to declare this x(i) as the last one or to wait until later. (In particular, it is not required for the optimal choice to know what were the previous indices of the maximal points x(0), ..., x(i − 1) or how many of these indices there were.)

In order to translate the problem completely into the language of the sequence {x(i)}, we look further for the probabilistic law governing this random sequence. We show first of all that the random variables x(0), x(1), ... form a Markov chain. This means that the conditional probability of the event x(i + 1) = l, with the provision that the values of all the preceding random variables x(0), ..., x(i) are known, actually depends only on the value of k acquired by the immediately preceding variable x(i).† Thus, let it be known that x(0) = 1, x(1) = b, ..., x(i) = k. This is equivalent to saying that the maximal of the points $a_1, a_2, \ldots, a_k$ are the points a_1, $a_b, \ldots, a_k$. In other words, it is known that the point a_k is maximal, and something is known also about the relative position of the points $a_1, a_2, \ldots, a_{k-1}$. The event x(i + 1) = l now means that the points $a_{k+1}, \ldots, a_{l-1}$ are situated to the left of a_k and that the point a_l is to the right of it. Consequently, if it is known that x(0) = 1,

*Inasmuch as there only exists a finite number of selection strategies, there is certainly an optimum among them.

†More precisely, this is the definition of a homogeneous Markov chain. In the general inhomogeneous case the indicated conditional probability also depends on the time i. Inhomogeneous Markov chains are not discussed in the present book.

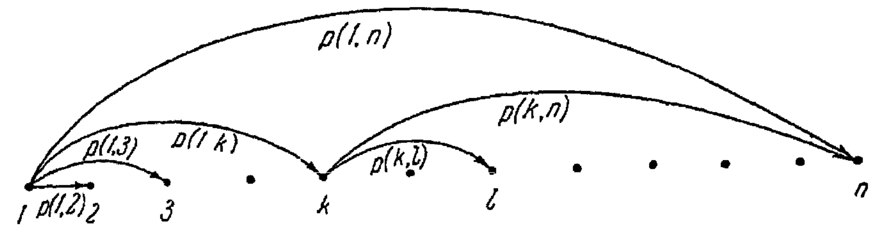

Fig. 23

$x(1)=b, \ldots, x(i)=k$, then the event $x(i+1)=l$ may be described in terms of the relative position of the points $a_k, a_{k+1}, \ldots, a_l, \ldots, a_n$. We found out above, however, that if the point a_k is maximal, the conditional probability of any event referred to the relative position of the points $a_k, \ldots, a_n$, given the condition that something is known regarding the relative position of the points $a_1, a_2, \ldots, a_{k-1}$, in fact depends only on the index k. Thus the conditional probability

$$P\{x(i+1)=l/x(0)=1,\ x(1)=b,\ \ldots,\ x(i)=k\},$$

apart from l, depends only on k (and possibly on the total number of points n). It is called the t r a n s i t i o n p r o b a b i l i t y of the Markov chain and is designated $p(k, l)$.

The variables $x(0), x(1), \ldots$ assume the values $1, 2, \ldots, n$. This set of values (called the p h a s e s p a c e) is conveniently represented in the form of points along which a particle executes a random walk (Fig. 23). At the initial instant the particle is located at the point 1, then it jumps to the point j with probability $p(1, j)$. In general, if the particle happens to be situated at the point k at some particular time, then in the next step it has a probability $p(k, l)$ of transferring to the point l, regardless of how it arrived at the point k. In our case $\sum_l p(k, l)$ can turn out to be less than one. It is reasonable to interpret the difference $1-\sum_l p(k, l)$ as the extinction probability of the particle. The transition of the particle from k to l means that the next maximal point after the maximal point a_k will be the maximal point a_l. Extinction of the particle implies that there are no more maximal points.

Let us calculate the transition probabilities $p(k, l)$. By definition of the conditional probability

$$p(k, l) = \frac{P\{x(i) = k,\ x(i+1) = l\}}{P\{x(i) = k\}} \quad (k, l = 1, \ldots, n).$$

Clearly, $p(k, l) = 0$ for $l \leq k$ (only left-to-right jumps are possible in Fig. 23). For $l > k$ the event $\{x(i) = k, x(i+1) = l\}$ implies that the points a_k and a_l (where a_l is to the right of a_k) are the furthest to the right of all the points $a_1, \ldots, a_l$. The probability of this event, considering the equal likelihood of all the points, is equal to $1/l(l-1)$. By complete analogy $\mathbf{P}\{x(i) = k\} = 1/k$. Consequently,

$$p(k, l) = \frac{k}{l(l-1)} \quad (1 \leqslant k < l \leqslant n).$$

We proceed now to formulate the optimal choice procedure.

As already mentioned, this method may be obtained by indicating for each index k whether to stop with this number or to look further. It is clearly sufficient to specify the subset Γ of the indices at which it is required to stop. The set of numbers 1, 2, ..., n has 2^n subsets (including the empty subset and the total set). Each of these corresponds to a certain strategy, and it is our problem to decide upon the best of these 2^n strategies.

Of course, there are many other strategies in addition to the procedures listed above. For example, we denote by ξ the first of the values $x(0), x(1), x(2), \ldots$ greater than or equal to k [so that $\xi = x(i)$ for $x(0) < \ldots < x(i-1) < k$, $x(i) \geq k$]. We could conceive of a strategy such that it is prescribed to stop with the number following ξ, i.e., with $x(i+1)$. Strategies of this type are clearly nonoptimal, but we will use them in studying the best selection procedure.

Let us calculate the conditional probability $q(k)$ of payoff by stopping at the point $x(i) = k$:

$$q(k) = 1 - \sum_{l=k+1}^{n} p(k, l) = 1 - \sum_{l=k+1}^{n} \frac{k}{l(l-1)}$$

$$= 1 - k \sum_{l=k+1}^{n} \left(\frac{1}{l-1} - \frac{1}{l}\right) = \frac{k}{n} \quad (1 \leqslant k \leqslant n)$$

For comparison we find the conditional payoff probability $q'(k)$ if in the same situation exactly one more step is taken, i.e., if one stops with the number $x(i+1)$. According to the total probability formula

$$q'(k) = \sum_{l=k+1}^{n} p(k, l)\, q(l) = \sum_{l=k-1}^{n} \frac{k}{l(l-1)} \frac{l}{n} = \frac{k}{n}\left(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n-1}\right)$$

$$= q(k)\left(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n-1}\right) \qquad (k < n),$$

$$q'(k) = 0 \qquad (k = n).$$

Since the sum $1/k + \ldots + 1/(n-1)$ decreases monotonically with increasing k, the ratio $q'(k)/q(k)$ also decreases monotonically, going to zero for $k = n$. Consequently, the condition $q'(k) \leq q(k)$ is satisfied by some interval k_n, $k_n + 1$, ..., n of the series of numbers 1, 2, ..., n.

We will show that the set $\Gamma = k_n, \ldots, n$ *corresponds to the optimal strategy* (in other words, that *scanning must be continued as long as $x(i) < k_n$, and stopped the first time $x(i) \geq k_n$*).

We assume below that the number of objects $n \geq 3$. For $n = 1$ there is in general no choice, and for $n = 2$ there are equal chances of success in stopping with either of two objects. It is at once apparent that in both of these cases the set $\Gamma = \{k_n, \ldots, n\}$ leads to an optimal strategy, but the next argument is inapplicable to these cases, because $k_n = 1$ for $n = 1$ or 2.

For $n \geq 3$

$$q'(1) = q(1)\left(1 + \frac{1}{2} + \ldots + \frac{1}{n-1}\right) > q(1)$$

and, therefore, $k_n > 1$. This means that a strategy requiring us to stop with the number $1 = x(0)$ is nonoptimal; in fact, this type of strategy is successful with probability $q(1)$, whereas a strategy calling for the choice of the number $x(1)$ yields a payoff with a greater probability $q'(1)$.

Thus, we can only seek the optimal selection technique among those strategies which require going past the first index. Since $p(1, k) > 0$ for $2 \leq k \leq n$, we have a positive probability $p_A(k)$, applying such a strategy A, of waiting until any index k. Let us suppose that the strategy A prescribes us to stop with a number $k < k_n$. Then the strategy A', which coincides with A as long as the particle arrives at k but requires exactly one more step after arriving at k, is clearly better than A. In fact, with the strategy A' the probability of success will be greater than with the strategy A by an amount $p_A(k)[q'(k) - q(k)]$. Consequently, the optimal strategy excludes stopping at the points $1, \ldots, k_n - 1$.

We now show by induction from a larger to a smaller value of k that at points of the interval $\{k_n + 1, \ldots, n\}$ the optimal strategy A requires stopping at once. Clearly, at points of this interval we have the strict inequality $q'(k) < q(k)$. If the strategy A required passing over the number n, the strategy A' prescribing stopping at the point n and otherwise coinciding with A would increase the probability of success relative to A by an amount $p_A(n)$, and the strategy A would not be optimal. Hence, for $k = n$ our assertion is true. Suppose now that it has already been proved for the points $k+1, k+2, \ldots, n$ $(k \geq k_n + 1)$. If A had prescribed passing over the number k, then the strategy A' requiring that we stop at the point k and otherwise coinciding with A would have been better than A. Actually, on arriving at the point k, the strategy A' would in fact prescribe stopping at once, while the strategy A, according to the induction hypothesis, would prescribe stopping after precisely one more step. Therefore, the probability of success would be greater for A' than for A, by an amount $p_A(k)[q(k) - q'(k)]$, and the strategy A would not be the optimal one. Consequently, A also requires stopping at the point k.

We have established the fact that the optimal strategy A forbids stopping at the points $1, \ldots, k_n - 1$ and, conversely, requires stopping at the points $k_n + 1, \ldots, n$. If for $k = k_n$ we have the strict inequality $q'(k_n) < q(k_n)$, the induction process can be continued to $k = k_n$ and the fact verified that the strategy A also requires stopping at the point k_n. But if for some n it turns out that $q'(k_n) = q(k_n)$, then, applying the same arguments, it is immaterial how we arrive at the point k_n. For convenience in this case we attach the point k_n to the set Γ.*

Thus, the best method of selection consists in passing over the first $k_n - 1$ objects and then choosing the first object that is better than all the preceding ones.

The number k_n is the smallest integer for which $q'(k) \leq q(k)$, i.e., for which

$$\frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{n-1} \leqslant 1.$$

Therefore, k_n is found from the double inequality

$$\frac{1}{k_n} + \ldots + \frac{1}{n-1} \leqslant 1 < \frac{1}{k_n - 1} + \frac{1}{k_n} + \ldots + \frac{1}{n-1}. \qquad (1)$$

We now determine the probability of success using the optimal strategy. We first calculate the probability s_m that the first object coming after the first $k_n - 1$ rejected objects and better than all the preceding objects will have the index m. This event means that the rightmost of all the points $a_1, \ldots, a_m$ will be a_m and that the next one to it will be any of the points $a_1, \ldots, a_{k_n - 1}$. By virtue of the equal weight of the objects, the probability of this event is $(1/m) \cdot (k_n - 1)/(m - 1)$. Consequently,

$$s_m = \frac{k_n - 1}{m(m-1)}.$$

The conditional probability of success in this case is equal to $q(m) = m/n$. Hence, in the large, the probability of a correct decision

* Actually, the equation $q'(k) = q(k)$ holds only for $n = 2$ and $k = 1$. Thus, of the numbers $k, k+1, \ldots, n-1$, exactly one is divisible by the highest power of the number two not in excess of $n - 1$, so that after reduction of the sum

$$s = \frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{n-1}$$

to a common denominator an odd number is obtained in the numerator. For $n > 2$ the denominator will be even, hence the sum s will be different from one.

is equal to

$$p_n = \sum_{m=k_n}^{n} s_m q(m) = \sum_{m=k_n}^{n} \frac{k_n - 1}{m(m-1)} \cdot \frac{m}{n}$$

$$= \frac{k_n - 1}{n}\left(\frac{1}{k_n - 1} + \frac{1}{k_n} + \ldots + \frac{1}{n-1}\right). \tag{2}$$

For example, for n = 10 we have the following table:

TABLE 1

k	$\frac{1}{k}$	$\frac{1}{k}+\ldots+\frac{1}{n-1}$	k	$\frac{1}{k}$	$\frac{1}{k}+\ldots+\frac{1}{n-1}$
9	0.111	0.111	4	0.250	0.996
8	0.125	0.236	3	0.333	1.329
7	0.143	0.379	2	0.500	. . .
6	0.167	0.546	1	1.000	. . .
5	0.200	0.746			

It is apparent from this table that $k_n = 4$. Consequently, it is necessary first to reject three objects, then to choose the first object better than all the preceding ones. The probability of success in this case is

$$p_{10} = 0.3 \cdot 1.329 = 0.399.$$

Similar calculations are easily carried out for any n, as long as it is not too large. We now derive relations giving a better approximation for k_n and p_n for large values of n. For any $m \geq 2$ we have

$$\ln(m+1) - \ln m = \int_m^{m+1} \frac{dx}{x} < \frac{1}{m} < \int_{m-1}^{m} \frac{dx}{x} = \ln m - \ln(m-1).$$

Summing these inequalities from m = k to m = n − 1, we deduce that

$$\ln\frac{n}{k} < \frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{n-1} < \ln\frac{n-1}{k-1}.$$

It follows from these estimates and from the inequalities (1) that

$$\ln \frac{n}{k_n} < 1 < \ln \frac{n-1}{k_n-2},$$

whence

$$\frac{n}{e} < k_n < \frac{n}{e} + \left(2 - \frac{1}{e}\right).$$

Inasmuch as no more than two integers can fall within an interval of length $2 - 1/e$, the above inequalities permit k_n to be found for any n with an error no greater than one. For large n an error of one in the calculation of k_n has little effect on the probability of correct choice.

It is evident from the inequalities (1) that the sum $1/(k_n - 1) + 1/k_n \ldots + 1/(n-1)$ differs from unity by less than $1/(k_n - 1)$. Since $k_n \to \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \left(\frac{1}{k_n-1} + \frac{1}{k_n} + \ldots + \frac{1}{n-1}\right) = 1.$$

From Eq. (2), therefore, we find

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{k_n - 1}{n} = \frac{1}{e} \approx 0\ 368.$$

§2. Optimal Stopping Problem for a Markov Chain

In the preceding section we solved the optimal choice problem by the construction of a special Markov chain. We now investigate the general problem of the optimal stopping of an arbitrary Markov chain.

Let a certain particle (or system) exist at each instant of time in one of the states formed by a finite or denumerable set E (*phase space*). If the particle is found at some instant in the state x, then after a unit of time it is found in the state y with a probability p(x, y) (regardless of when and by what route it arrived at the point x). We say then that we have specified a *Markov chain with transition probabilities* p(x, y).

The probabilities p(x, y) may be any nonnegative numbers obeying the condition

$$\sum_y p(x, y) \leqslant 1 \qquad (x \in E).$$

If $\sum_y p(x, y) < 1$, for some x, then the variable $q(x) = 1 - \sum_y p(x, y)$ represents the extinction probability in the next step for a particle situated at x. An extinct particle cannot be recreated, hence the chain in this case is terminated once and for all.

Examples of Markov chains are the random walk on a lattice studied in Chapt. I and the sequence of indices of the maximal points in the optimal choice problem In the former example the chain sometimes fails to terminate, and in the latter the probability is one that it will terminate no later than the nth step.

We denote by x(n) the position of the particle at the instant n. Let us suppose that we observe the path x(0), x(1), ..., x(n), ..., and can at any instant n stop the migrating particle. If at the time of stopping the particle is situated at the point x, we acquire a payoff f(x), where f is a known function. If we do not stop the process (either because it succeeded in terminating itself or because we wait an infinitely long time), the payoff is zero. We wish to inquire how to optimize the payoff.

Let us refine the statement of the problem. We first of all describe the class of possible stopping times τ. The time τ, generally speaking, is random because it depends on the random path of the particle. However, it is not an arbitrary integer-valued random variable. As a matter of fact, at the time τ we do not know how the process would behave after τ, and we have to solve the problem knowing the process prior to the time τ. We therefore consider only those integer-valued random variables τ for which the occurrence or nonoccurrence of the event $\{\tau = t\}$ is uniquely determined according to the values of x(0), x(1), ..., x(t). These random times are called the Markov times (the Markov times for a Wiener process have already been discussed in Chapt. II, §4).

The sum $\sum_{t=0}^{\infty} \mathrm{P}_x\{\tau = t\}$ can be less than one (and even equal to zero). Instead of saying that τ is indeterminate for the corresponding paths of the particle, we sometimes write $\tau = \infty$.

A typical Markov time is the time of first visit to some subset Γ of the set E (there are, of course, other Markov times, for example, $\tau = 5$ or $\tau = \tau_1 + 2$, where τ_1 is a Markov time, etc.).

If the time τ is chosen (in other words, if the strategy of the person stopping the process is given), the gain turns out to be a random variable $f(x(\tau))$. It is required to choose τ such that the mean value $\mathbf{M}_x f(x(\tau))$ is as large as possible (as usual, $\mathbf{M}_x$ indicates the expectation for the initial position of the particle at the point x).* In order for the expectation $\mathbf{M}_x f(x(\tau))$ to have meaning for any τ, certain restrictions must be imposed on the function f. It is sufficient to demand that f be bounded.

In summary, the problem is stated as follows. A Markov chain with transition probabilities p(x, y) and a bounded function $f(x)$ are given on a finite or denumerable set E. It is required to: 1) calculate the variable $v(x) = \sup_{\tau} \mathbf{M}_x f(x(\tau))$, where τ represents all the possible Markov times, 2) find the Markov time τ_0 for which $\mathbf{M}_x f(x(\tau_0)) = v(x)$

By analogy with the theory of games, the variable v(x) is called the value of a game, and the Markov time τ_0 is called the optimal strategy.

In order to gain better insight into the problem, we consider some special cases and examples.

If $f \leq 0$ over the entire phase space E, the problem has a trivial solution, clearly, $\tau_0 = \infty$ (i.e., never stopping the process) may be adopted as the optimal strategy, and $v(x) = 0$. We exclude this uninteresting case right away and assume that $\sup_x f(x) > 0$.

We next consider a random walk on a one-dimensional lattice. As we know (see Chapt. I, §1), in this type of random walk the particle has a probability one of sooner or later visiting any state x. Consequently, here $v(x) = c$, where $c = \sup_x f(x)$, because it is permissible to wait until the particle attains a state in which $f(x)$ is arbitrarily close to c. If the value of c is attained on a subset Γ

*In calculating the expectation $\mathbf{M}_x f(x(\tau))$, the summation extends only over the elementary events for which τ is finite (see the footnote on page 41).

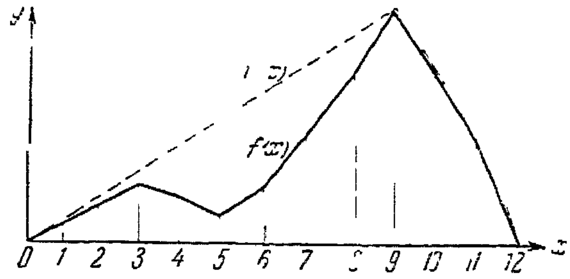

Fig. 24

of points of the lattice, then in order to obtain an optimal strategy it is sufficient to set τ_0 equal to the time of first visit to Γ If, on the other hand, c is not attained at any point, an optimal strategy does not exist, even though it might be possible to obtain a payoff arbitrarily close to c.

It is clear that the same pattern will be observed in any Markov chain in which the particle has a probability one of occupying all states (such chains are called recurrent).

We next examine a homogeneous random walk on a line segment with absorption at the ends (Fig. 24). The particle has a probability 1/2 of jumping from the states 1-11 to the nearest point to the right or left, but on arriving at the state 0 or 12 it always stays there. A graph of the function $f(x)$ is shown in Fig. 24 (the adjacent points of the graph are joined for clarity).

Inasmuch as it is impossible to exit from the points 0 and 12, we have $v(0) = f(0) = 0$, $v(12) = f(12) = 0$. At these points there is nothing to wait for, and the process may be stopped at once. Similarly, it must be stopped immediately in the state 9; in this state $f(x)$ attains an absolute maximum, hence any continuation of the process can only diminish the payoff. Consequently, $v(9) = f(9)$. At the point 5, where $f(x)$ has a relative minimum, conversely, it is unfavorable to stop, even after one step it is possible to obtain a payoff greater than $f(x)$. Therefore, $v(5) > f(5)$. What is the situation in the other states? At the point 3, for example, where $f(x)$ has a relative maximum, a postponement by one or two steps clearly diminishes the average payoff. If one waits longer, there is hope of arriving in the domain of a second or higher peak, where the payoff would be considerably greater than $f(3)$. But then there is the danger of becoming trapped at the point 0 and of gaining nothing.

Moving ahead, we point out that the value of the game v(x) in this example is the least concave function greater than or equal to $f(x)$. In other words, in order to generate the graph of v(x), it is necessary to run a line above the graph of $f(x)$ between the points 0 and 12 [in Fig. 24 the graph of v(x) is indicated by a dashed line]. The optimal strategy is to stop the chain at the time τ_0 of first arrival of the particle at the point where $f(x) = v(x)$.

It will be shown that the problem has an analogous solution in the general case of a chain with a finite number of states. The role of the concave functions in this case is taken by the class of excessive functions associated with the given Markov chain.

The optimal choice problem analyzed in §1 is a special case of our general problem. In fact, in §1 we constructed a Markov chain x(i) with states 1, 2, ..., n, and the problem was one of stopping this chain with maximum probability at the instant immediately prior to termination. If the particle is situated in the state k, the chain terminates in the next instant with a probability q(k) = k/n. Consequently, the probability of success with the strategy τ is equal to

$$\sum_{k=1}^{n} \mathsf{P}_1 \{x(\tau) = k\} \cdot \frac{k}{n} = \mathbf{M}_1 \frac{x(\tau)}{n} = \mathbf{M}_1 q(x(\tau)).$$

[The subscript 1 attached to P and $\mathbf{M}$ indicates that the path x(0), x(1), ... is initiated at the point 1.] Therefore, the optimal choice problem reduces to the optimal stopping problem for the payoff function $f(x) = q(x)$ and the initial state x = 1.

§3. Excessive Functions

We begin our investigation of the optimum stopping problem for an arbitrary Markov chain with a study of the payoff functions f for which the optimal strategy consists in stopping the process at once. Clearly, these must be functions f that satisfy the following inequality at any Markov time τ:

$$f(x) \geqslant \mathbf{M}_x f(x(\tau)) \qquad (x \in E). \tag{3}$$

Since the number of Markov times, in general, is infinite, it would be difficult to test the condition (3) directly for every Markov time τ. As we shall see, it is sufficient for (3) to hold for $\tau = \infty$ and $\tau = 1$; then this condition will be satisfied for all other Markov times.

For $\tau = \infty$ the condition (3) leads to the inequality

$$f(x) \geqslant 0 \qquad (x \in E). \tag{4}$$

For $\tau = 1$ it becomes the condition

$$f(x) \geqslant Pf(x), \tag{5}$$

where P denotes an operator operating according to the formula $Pf(x) = \sum_y p(x, y) f(y)$ (the one-step shift operator).

We are well acquainted with the requirements (4) and (5) from Chapt. I; they comprised the definition of the excessive function for a symmetric random walk on a lattice. It is logical to introduce analogous definitions in the case of an arbitrary Markov chain as well. Nonnegative functions f for which $Pf \leq f$ are called excessive functions.

We will prove that if a function f is excessive, the inequality (3) is satisfied for any Markov time τ.*

This statement has already been proved for a random walk on a lattice in Chapt. I, §6. Of course, τ was interpreted there as the time of first visit to a certain set, but, as is readily observed, the arguments are wholly applicable to arbitrary Markov times as well. The fundamental notion of the proof was to represent the excessive function f as the sum of a constant, for which (3) is obvious, and the potential

$$G\varphi(x) = \varphi(x) + P\varphi(x) + P^2\varphi(x) + \dots$$
$$= \mathbf{M}_x[\varphi(x(0)) + \varphi(x(1)) + \dots] \tag{6}$$

of the nonnegative function $\varphi = f - Pf$. For the potential the inequality (3) was derived from the relation

$$\mathbf{M}_x G\varphi(x(\tau)) = \mathbf{M}_x[\varphi(x(\tau)) + \varphi(x(\tau+1)) + \dots], \tag{7}$$

* This fact (in a more general situation) was established by Hunt [5].

the right side of which is less than or equal to the right side of Eq. (6).

In the case of an arbitrary Markov chain the series (6) can diverge. We cope with this difficulty by introducing a "correction factor" $\alpha < 1$ and then letting α tend to unity.

Setting $\varphi(x) = f(x) - \alpha Pf(x)$, $0 < \alpha < 1$, we write the obvious identity

$$f = \varphi + \alpha P\varphi + \alpha^2 P^2\varphi + \ldots + \alpha^n P^n \varphi + \alpha^{n+1} P^{n+1} f,$$

where, by virtue of (5), once again $\varphi \geq 0$. Utilizing the fact that $0 \leq P^n f = P^{n-1}(Pf) \leq P^{n-1} f$, so that $\alpha^n P^n f \to 0$ as $n \to \infty$, we obtain a representation of f as an infinite series:

$$\begin{aligned} f(x) &= \varphi(x) + \alpha P\varphi(x) + \alpha^2 P^2 \varphi(x) + \ldots \\ &= \mathbf{M}_x [\varphi(x(0)) + \alpha\varphi(x(1)) + \alpha^2 \varphi(x(2)) + \ldots] \end{aligned} \quad (8)$$

[in the general case the relation $P^n\varphi(x) = \mathbf{M}_x\varphi(x(n))$ is derived in exactly the same manner as for a random walk on a lattice]. Just as the relation (6) is implied by (7), it also follows from (8) that

$$\mathbf{M}_x \alpha^\tau f(x(\tau)) = \mathbf{M}_x [\alpha^\tau \varphi(x(\tau)) + \alpha^{\tau+1}\varphi(x(\tau+1)) + \ldots] \quad (9)$$

(we leave the verification of this relation to the reader). We infer from a comparison of (8) and (9) that

$$f(x) \geqslant \mathbf{M}_x \alpha^\tau f(x(\tau)).$$

In order to obtain the inequality from (3) from this, all we need is to let α tend to one.*

The following more general property of excessive functions is proved analogously: *If f is excessive and $\tau' \geq \tau$ are two Markov times, then*

$$\mathbf{M}_x f(x(\tau)) \geqslant \mathbf{M}_x f(x(\tau')) \qquad (x \in E). \quad (10)$$

*Passing to the limit too hastily in the argument of the expectation can result in invalid equations. However, $\xi_\alpha \to \xi$ implies $\mathbf{M}\xi_\alpha \to \mathbf{M}\xi$ in the following two important cases:

1) when $|\xi_\alpha| < \eta$ for all α and $\mathbf{M}\eta < \infty$;

2) when $\xi_\alpha \geq 0$ and $\xi_\alpha \to \xi$, increasing monotonically.

For the proof of this property it is required to write Eq. (9) for each of the times τ and τ'. Since $\tau \leq \tau'$, the series (9) for τ will contain all the same terms as the series (9) for τ' and possibly some additional positive terms. Consequently, for $0<\alpha<1$

$$\mathbf{M}_x \alpha^{\tau} f(x(\tau)) \geqslant \mathbf{M}_x \alpha^{\tau'} f(x(\tau')).$$

As $\alpha \to 1$, we obtain Eq. (10) from the latter.

It is readily deduced from the inequality (10) that *if a function v is excessive and τ is the time of first visit to some set Γ, then the function*

$$h(x) = \mathbf{M}_x v(x(\tau))$$

is also excessive.

To prove this assertion, we denote by τ' the first time $t \geq 1$ at which the particle is situated in the set Γ. It is clear that $\tau' \geq \tau$, and hence that

$$\mathbf{M}_x f(x(\tau')) \leqslant \mathbf{M}_x f(x(\tau)) = h(x).$$

But if the first step has brought the particle from x to y, then under this condition $\mathbf{M}_x f(x(\tau'))$ is equal to $\mathbf{M}_y f(x(\tau)) = h(y)$. Consequently,

$$\mathbf{M}_x f(x(\tau')) = \sum_{y \in E} p(x, y) h(y) = Ph(x).$$

Thus, $Ph \leq h$.

§4. The Value of a Game

If the payoff function f is excessive, then, as we readily perceive, the value of the game v coincides with f.

We note in the general case that *if an excessive function g majorizes the payoff function f, it also majorizes the value of the game v.*

In fact, if $g \geq f$ and g is excessive, then for any strategy τ

$$\mathbf{M}_x f(x(\tau)) \leqslant \mathbf{M}_x g(x(\tau)) \leqslant g(x)$$

and, hence,

$$v(x) = \sup_{\tau} \mathbf{M}_x f(x(\tau)) \leqslant g(x)$$

We show next that **the value of the game v itself is an excessive function.**

Clearly, the function v is nonnegative; zero payoff can always be obtained with the strategy $\tau = \infty$.

In order to test the condition $Pv \leq v$, we formulate a strategy τ yielding an average payoff $\mathbf{M}_x f(x(\tau))$ arbitrarily close to $Pv(x)$, then we make use of the inequality $\mathbf{M}_x f(x(\tau)) \leq v(x)$.

We pick an arbitrary number $\varepsilon > 0$ and denote by $\tau_{\varepsilon,\ y}$ the strategy for which

$$\mathbf{M}_y f(x(\tau_{\varepsilon,\, y})) \geqslant v(y) - \varepsilon \qquad (y \in E).$$

(The existence of a Markov time $\tau_{\varepsilon,y}$ for any y follows from the actual definition of the value of a game.) Let the strategy τ consist in first making one step, then, if this step brings the particle to the state y, using the strategy $\tau_{\varepsilon,\ y}$. More precisely, if $x(1) = y$, then $\tau = 1 + \tau_{\varepsilon,y}$, where $\tau_{\varepsilon,y}$ is found from the trajectory $x(1), x(2), \ldots,$ beginning at the time 1 rather than the time 0. It is readily agreed that τ is a Markov time. For this τ we have

$$\mathbf{M}_x f(x(\tau)) = \sum_{y \in E} p(x,\ y) \mathbf{M}_y f(x(\tau_{\varepsilon,\, y}))$$

$$\geqslant \sum_{y \in E} p(x,\ y)[v(y) - \varepsilon] = Pv(x) - \varepsilon \sum_{y \in E} p(x,\ y) \geqslant Pv(x) - \varepsilon.$$

Consequently, $v(x) \geq Pv(x) - \varepsilon$ for any $\varepsilon > 0$, hence $Pv(x) \leq v(x)$. This proves the excessiveness of the function v.

Since one possible strategy is instantaneous stopping, $v(x) \geq f(x)$.

Summarizing, we have deduced the fact that **the value of a game v is the minimum excessive function greater than or equal to the payoff function f** (and is logically called the **excessive majorant of f**).

As a by-product, we have also proved the existence of an excessive majorant for any function f (this is not evident a priori).

This result permits the value of the game to be found by linear programming methods in the case of a finite number of states. In fact, the value of the game v(x) is the minimum function satisfying the following set of 3n linear inequalities:

$$\left.\begin{array}{l} v(x) \geqslant \sum\limits_{y \in E} p(x, y) v(y), \\ v(x) \geqslant f(x), \\ v(x) \geqslant 0, \end{array}\right\} (x \in E),$$

where n is the number of states of the Markov chain.

§5. The Optimal Strategy

We denote by Γ the set of all states x in which the payoff function f(x) is equal to its excessive majorant v(x). We call this set the support set (in Fig. 24 the support set comprises the points 0, 9, 10, 11, and 12; at these points the graph of the function f "supports" the line representing the function v).

Let a particle begin to move at a point x of the support set. Immediate stopping at this point yields a payoff equal to v(x), and it is not possible to give a better strategy. On the other hand, stopping at an initial state x outside Γ results in a payoff f(x) that is strictly less than the value of the game v(x). Therefore, had we known beforehand, first, that an optimal strategy existed and, second, that this strategy prescribed stopping or continuing to scan solely as a function of the position of the particle at the current instant (as is specifically the situation in the optimal choice problem), we could have inferred that the optimal strategy is given by the time τ of first visit of the particle to Γ. So far, however, we can only adopt this as a reasonably plausible hypothesis.

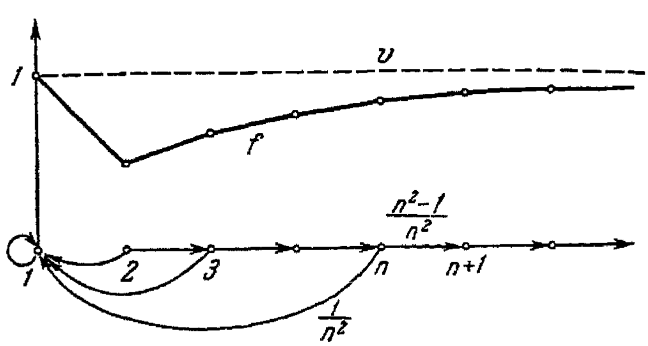

Fig. 25

Not always, however, does this hypothesis turn out to be true. Let us consider, for example, a Markov chain with an infinite number of states 1, 2, ..., n, ..., in which the particle has a probability $1/n^2$ of going from the point n to the point 1 and a probability $(n^2 - 1)/n^2$ of going from the same point to the point $n+1$ (Fig. 25). Let $f(n) = 1 - 1/n$ for $n > 1$, and let $f(1) = 1$. Obviously, in this situation it is always possible to expect a payoff arbitrarily close to, but never greater than, one, hence $v(n) = 1$. The support set Γ in this example consists of the single point 1. Inasmuch as $f(1) = 1$, for the time τ of first visit to Γ the average payoff $M_n f(x(\tau))$ is equal to the probability $\pi(n)$ of leaving n and arriving sometime at 1. The probability of the converse event, i.e., of the particle departing to infinity on the right, is equal to

$$\prod_{k=n}^{\infty} \frac{k^2-1}{k^2}. \tag{11}$$

Since

$$\prod_{k=n}^{m} \frac{k^2-1}{k^2} = \prod_{k=n}^{m} \frac{(k-1)(k+1)}{k \cdot k} = \frac{(n-1)(m+1)}{nm},$$

the infinite product (11) converges and is equal to $(n-1)/n$. Therefore, $\pi(n) = 1/n$, whereas $v(n) = 1$.

The violation of our hypothesis in this example is connected with the fact that the phase space is infinite. We will show that *the time τ_0 of first visit to the support set in the case of a finite phase space is an optimal strategy.*

Let us examine the average payoff

$$h(x) = M_x f(x(\tau_0)), \tag{12}$$

which corresponds to the strategy τ_0. It is required to prove that $h = v$. According to the actual definition of the value of a game, $h \le v$. Inasmuch as $x(\tau_0) \in \Gamma$ while f and v coincide on Γ, the function f may be replaced in Eq. (12) by the excessive function v; then it follows from this formula that h is also excessive (see §3). Since v is the minimum excessive function majorizing f, in order to obtain the inverse inequality $h \ge v$, it is sufficient to verify that $h \ge f$.

At points of the support set Γ we have $h(x) = f(x)$, because the strategy τ_0 prescribes immediate stopping at these points. Let us assume that the inequality $h(x) < f(x)$ is satisfied somewhere outside Γ. We denote by a the point at which the difference $f(x) - h(x)$ attains a maximum. Then the function $h_1(x) = h(x) + [f(a) - h(a)]$ majorizes f, coincides with f at the point a, and, as the sum of the excessive function $h(x)$ and the positive constant $f(a) - h(a)$, is also excessive. Consequently, h_1 majorizes v, and $f(a) = h_1(a) \geq v(a)$. This means that the point a chosen outside the support set Γ belongs to Γ. The ensuing contradiction reveals that the inequality $h(x) < f(x)$ is inadmissible. The optimality of the strategy τ_0 is thus proved.

We turn next to the case of a Markov chain with a denumerable phase space. Here, as we are aware, stopping at the time of first visit to the reference set Γ can prove to be a highly inauspicious strategy. It can be shown, however, that if we adopt in place of the set $\Gamma = \{x:\ f(x) = v(x)\}$ an "ε-support" set $\Gamma_\varepsilon = \{x:\ v(x) - f(x) \leq \varepsilon\}$ and investigate the time τ_ε of first visit to Γ_ε, we have for any $\varepsilon > 0$

$$\mathbf{M}_x f(x(\tau_\varepsilon)) \geqslant v(x) - \varepsilon. \tag{13}$$

Consequently, the ε-support sets enable one to find strategies affording a payoff arbitrarily close to the value of the game.

The proof of the inequality (13) follows the same plan, with slight modifications, as in the case of a finite phase space, when $\varepsilon = 0$. Inasmuch as $f(x) \geq v(x) - \varepsilon$ on Γ_ε, we have

$$\mathbf{M}_x f(x(\tau_\varepsilon)) \geqslant \mathbf{M}_x v(x(\tau_\varepsilon)) - \varepsilon \mathbf{P}_x\{\tau_\varepsilon < \infty\} \geqslant \mathbf{M}_x v(x(\tau_\varepsilon)) - \varepsilon.$$

The function $h(x) = \mathbf{M}_x v(x(\tau_\varepsilon))$ is excessive, along with v. We will show that $h(x) \geq f(x)$. If $\sup[f(x) - h(x)] = c > 0$, the function $h(x) + c$ is excessive and majorizes $f(x)$. Consequently, $h(x) + c \geq v(x)$ for all x. Since $c > 0$, there exists a state a in which $f(a) - h(a) > 0$ and simultaneously $f(a) - h(a) > c - \varepsilon$. Then $f(a) = f(a) - h(a) + h(a) \geq c - \varepsilon + v(a) - c = v(a) - \varepsilon$, hence $a \in \Gamma_\varepsilon$. At points belonging to Γ_ε, however, the functions h and v are equal, so that $h(a) = v(a) \geq f(a)$. This contradicts the inequality $f(a) - h(a) > 0$. Therefore, c cannot be positive, hence $h(x)$ majorizes $f(x)$. But then the excessive function $h(x)$ also majorizes v(x), and consequently

$$\mathbf{M}_x f(x(\tau_\varepsilon)) \geqslant h(x) - \varepsilon \geqslant v(x) - \varepsilon.$$

§6. Application to a Random Walk with Absorption and to the Optimal Choice Problem

In a random walk along a line segment $[0, a]$ with absorption at the end points, a particle situated at any of the points 1, 2, ..., $a - 1$ has a probability 1/2 of shifting one unit to the left or right in a single step, but if it arrives at the point 0 or a, it always stays there (see Fig. 24, where $a = 12$).

The solution to the optimal stopping problem for this kind of Markov chain was given without proof at the end of §2. In correspondence with the general formulations of §§3-5, all that is needed for the justification of this solution is to verify that the excessive functions are nonnegative concave functions.

By definition, a function f is excessive if $f \geq 0$ and $Pf \leq f$. The condition $Pf \leq f$ reduces in the present case to the inequalities

$$\frac{f(x-1)+f(x+1)}{2} \leqslant f(x) \qquad (x = 1, 2, \ldots, a-1) \tag{14}$$

and the trivial relations

$$f(0) \leqslant f(0), \qquad f(a) \leqslant f(a).$$

The inequalities (14) signify that if the adjacent points of the graph of the function $f(x)$ are joined by segments, the vertex of the resulting polygonal curve at any interior point x will be situated no lower than the chord connecting the vertices at the points $x - 1$ and $x + 1$ (Fig. 26). Consequently, the condition $Pf \leq f$ is tantamount to concavity of the function $f(x)$, which it was required to prove.

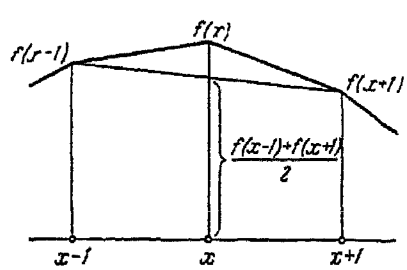

Fig. 26

Let us investigate how the concepts we have introduced operate in the optimal choice problem. As we know, this problem reduces to the optimal stopping of a chain characterized by states 1, 2, ..., n, transition probabilities

$$p(k,\ l)=\begin{cases}\dfrac{k}{l(l-1)} & \text{for}\quad l>k,\\ 0 & \text{for}\quad l\leqslant k,\end{cases}$$

and payoff functions $f(k)=k/n$ (see §2).

We find the excessive majorant v(k) of the payoff function $f(k)$ and the reference set $\Gamma=\{k\colon f(k)=v(k)\}$. By definition, v is the minimum function satisfying the inequalities $v\geq f$, $Pv\geq v$, and $v\geq 0$. In the given case these inequalities assume the form

$$v(k)\geqslant\frac{k}{n},$$

$$v(k)\geqslant\sum_{l=k-1}^{n}\frac{k}{l(l-1)}v(l),$$

$$k=1,\ 2,\ \ldots,\ n.$$

Hence, if $v(l)$, $l>k$, is already known, then

$$v(k)=\max\left\{\frac{k}{n},\ k\sum_{l=k+1}^{n}\frac{v(l)}{l(l-1)}\right\}.$$

We have obtained a recursion formula for the determination of v(k). On the basis of this formula we successively find

$$v(n)=\max\left\{\frac{n}{n}\right\}=1=f(n),$$

$$v(n-1)=\max\left\{\frac{n-1}{n};\ (n-1)\frac{1}{n(n-1)}\right\}$$

$$=\max\left\{\frac{n-1}{n},\ \frac{1}{n}\right\}=\frac{n-1}{n}=f(n-1),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$v(k)=\max\left\{\frac{k}{h},\ k\left[\frac{\frac{k+1}{n}}{(k+1)k}+\frac{\frac{k+2}{n}}{(k+2)(k+1)}+\cdots\right.\right.$$

$$\ldots+\frac{\frac{n}{n}}{n(n-1)}\Bigg]\Bigg\}=\max\left\{\frac{k}{n},\ \frac{k}{n}\left(\frac{1}{k}+\frac{1}{k+1}+\ldots+\frac{1}{n-1}\right)\right\}=\frac{k}{n}=f(k)$$

as long as the following inequality remains in force:

$$\frac{1}{k}+\frac{1}{k+1}+\ldots+\frac{1}{n-1}\leqslant 1. \qquad (15)$$

As soon as the sum $1/k+\ldots+1/(n-1)$ becomes greater than one with diminishing k, v(k) turns out to be strictly greater than $k/n=f(k)$. With a further reduction of k the sum $1/k+\ldots+1/(n-1)$ remains greater than one, so that at these points

$$v(k)\geqslant k\sum_{l=k+1}^{n}\frac{v(l)}{l(l-1)}\geqslant k\sum_{l=k+1}^{n}\frac{f(l)}{l(l-1)}$$

$$=\frac{k}{n}\left(\frac{1}{k}+\frac{1}{k+1}+\ldots+\frac{1}{n-1}\right)>\frac{k}{n}=f(k).$$

Hence, the support set Γ has the form $\{k_n, k_n+1,\ldots, n\}$, where k_n is the smallest integer satisfying the inequality (15). We are already familiar with this result.

For $k\geq k_n$ the value of the game is equal to $v(k)=f(k)=k/n$, and for $k<k_n$ it is calculated in succession according to the relation*

$$v(k)=k\sum_{k=l+1}^{n}\frac{v(l)}{l(l-1)}.$$

§7. Optimal Stopping of a Wiener Process

The optimal stopping problem can be analyzed not only for Markov chains, but also for processes involving a nondenumerable phase space and continuous time. We propose to investigate one of the most elementary processes of this type, namely, a *Wiener process* x(t) *on the interval* $[0,a]$ *with absorption at the points*. By definition, given any initial position x, $0\leq x\leq a$, the

*It is not difficult to show that v(k) is in fact independent of k for $k<k_n$ and is found from Eq. (2).

particle executes exactly the same motion as in an ordinary Wiener process on an infinite line until the first time it hits the end of the interval; on hitting the point 0 or the point a, the particle always becomes trapped at that point.*

Let the payoff function $f(x)$ be specified on the interval $[0, a]$. It is required to find the value of the game

$$v(x) = \sup_{\tau} \mathbf{M}_x f(x(\tau)) \qquad (0 \leqslant x \leqslant a),$$

where τ represents all the possible Markov times, and to formulate the particular Markov time τ_0 at which

$$\mathbf{M}_x f(x(\tau_0)) = v(x)$$

(i.e., to find the optimal strategy).

The process of interest here is the continuous analog of a symmetric random walk on a line segment with absorption at the ends, i.e., the problem discussed in §§2 and 6. We see that the solution of the problem in the continuous case remains the same, except that instead of concave functions of an integer-valued argument it is necessary to use concave functions specified on the entire interval $[0, a]$.

We recall that a function $f(x)$ given on the interval $[0, a]$ is called *concave* if the entire chord connecting any two points of the graph of the function f is situated no higher than the graph f (Fig. 27). We note that a *function concave on an interval is continuous inside the interval* and at the ends of the interval has finite limits no smaller than the values of the function at the end points (see the Appendix, §2). For example, in Fig. 27

$$\lim_{x \to 0} f(x) = f(0), \quad \lim_{x \to a} f(x) > f(a).$$

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* We are not concerned with a Wiener process on the entire infinite line, because in this case the particle has a probability one of hitting any point, and the optimal stopping problem has the same trivial solution as for a recurrent Markov chain.

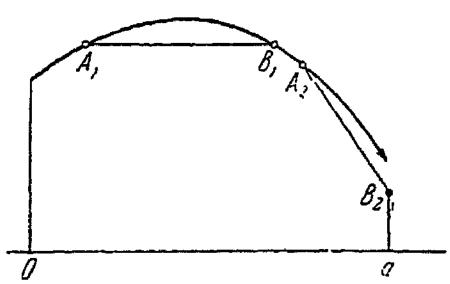

Fig. 27

The special role played by concave functions for our process is explained by the fact that **non-negative concave functions (and only those functions) satisfy the inequality**

$$\mathbf{M}_x f(x(\tau)) \leqslant f(x) \qquad (16)$$

at any Markov time τ. The proof of this statement is rather intricate, and we will save it for a special section (§8).

After having described the class of functions satisfying the condition (16), the value of the game and the optimal strategy are found in approximately the same manner as in §§4 and 5 for an arbitrary Markov chain.

We first calculate the probability $q(x) = q(x; x_1, x_2)$, on starting from x, of hitting the point x_1 before x_2, as well as the probability $p(x) = p(x; x_1, x_2)$ of hitting x_2 before x_1 $(0 \leq x_1 \leq x \leq x_2 \leq a)$. It follows from the results of Chapt. II that the function q(x) is a solution of the Dirichlet problem on the interval $[x_1, x_2]$ and assumes a value of one at the point x_1 and a value of zero at the point x_2. Inasmuch as the Laplace equation $\Delta q = 0$ assumes the form $q'' = 0$ in the one-dimensional case, all of its solutions are linear, i.e., they have the form $q(x) = cx + d$. Determining the values of the constants c and d from the boundary conditions $q(x_1) = 1$, $q(x_2) = 0$, we obtain

$$q(x;\ x_1,\ x_2) = \frac{x_2 - x}{x_2 - x_1},$$

$$p(x;\ x_1,\ x_2) = 1 - q(x;\ x_1,\ x_2) = \frac{x - x_1}{x_2 - x_1}. \qquad (17)$$

We now find the value of the game v(x), regarding the payoff function $f(x)$ for the time being only as bounded, but not necessarily continuous.* We note that if g(x) is a nonnegative concave function majorizing $f(x)$, then for any τ

$$\mathbf{M}_x f(x(\tau)) \leqslant \mathbf{M}_x g(x(\tau)) \leqslant g(x)$$

and, hence, g(x) majorizes v(x).

* The boundedness of the function c (along with measurability, which we agreed earlier not to discuss) guarantees the existence of the expectation $M_X f(x(\tau))$.

The function v(x) itself is nonnegative (as there exists a strategy $\tau = \infty$ yielding zero payoff) and is also concave. Thus, we let $[x_1, x_2]$ be some subinterval contained in $[0, a]$ and let τ_1 and τ_2 be strategies yielding average payoffs greater than $v(x_1) - \varepsilon$ and $v(x_2) - \varepsilon$ in the respective initial states x_1 and x_2 (the existence of these strategies for any $\varepsilon > 0$ ensues from the actual concept of upper bounds). Let us examine the strategy τ, whereby we first wait for the first hitting time at one of the points x_1 or x_2, then use the corresponding strategy τ_1 or τ_2. Here, according to Eq. (17),

$$\begin{aligned} M_x f(x(\tau)) &= \frac{x_2 - x}{x_2 - x_1} M_{x_1} f(x(\tau_1)) + \frac{x - x_1}{x_2 - x_1} M_{x_2} f(x(\tau_2)) \\ &\geqslant \frac{x_2 - x}{x_2 - x_1}\left[v(x_1) - \varepsilon\right] + \frac{x - x_1}{x_2 - x_1}\left[v(x_2) - \varepsilon\right] \\ &= \frac{x_2 - x}{x_2 - x_1} v(x_1) + \frac{x - x_1}{x_2 - x_1} v(x_2) - \varepsilon, \end{aligned}$$

hence

$$v(x) \geqslant \frac{(x_2 - x)\, v(x_1) + (x - x_1)\, v(x_2)}{x_2 - x_1} - \varepsilon \qquad (x_1 \leqslant x \leqslant x_2).$$

Inasmuch as ε can be an arbitrarily small positive number, the above inequality is also true for $\varepsilon = 0$. Since the function

$$\frac{(x_2 - x)\, v(x_1) + (x - x_1)\, v(x_2)}{x_2 - x_1}$$

is linear on $[x_1, x_2]$ and coincides with v at the points x_1 and x_2, this means that the graph of v on the interval $[x_1, x_2]$ does not pass any lower than the chord spanning it. Consequently, the function v(x) is concave.

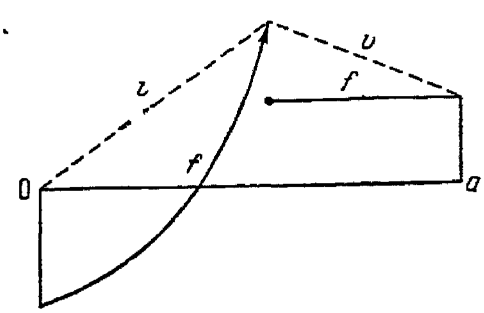

Fig. 28

Therefore, the value of the game is the minimum nonnegative concave function greater than or equal to the payoff function f, or, more concisely, **the value of the game v is the nonnegative concave majorant of the function f** (see Fig. 28, which illustrates a discontinuous function f).

We next show that **if the function f is continuous, then**, as in the discrete case, **an optimum strategy is to stop the process at the time τ_0 of first visit to the support set Γ**, on which $f(x) = v(x)$. We note that this statement no longer holds for a discontinuous payoff function f. Thus, in the example illustrated in Fig. 28 the set Γ comprises the single point a. This means that if we wait for arrival in Γ, we will never obtain a payoff greater than $f(a)$, whereas $v(x)$ is much larger than $f(a)$ at some points.

We first verify the fact that continuity of the payoff function f implies continuity of the value of the game v. Inasmuch as the function v is concave, it is continuous at all interior points of the interval $[0, a]$, and $\lim_{x\to 0} v(x) \geqslant v(0)$, $\lim_{x\to a} v(x) \geqslant v(a)$. We examine the point 0 for definiteness and show that

$$\lim_{x\to 0} v(x) \leqslant v(0). \tag{18}$$

We set $c(u) = \max_{0\leqslant x\leqslant u} f(x)$, $0 \leq u \leq a$. It is apparent that the function $c(u)$ is continuous, along with $f(x)$. For $x(\tau) < u$ the payoff, clearly, cannot exceed the value of $c(u)$, while for $x(\tau) \geq u$ it cannot exceed $c(a)$. Moreover, the inequality $x(\tau) \geq u$ for $x = x(0) < u$ can only occur in the event that the particle arrives from the point x at u before it arrives at the point 0. The probability of this event, according to Eq. (17), is equal to x/u. Conse-

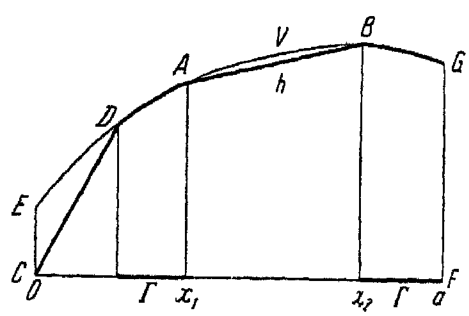

Fig. 29

quently, for $0 < x < u$ and any τ

$$M_x f(x(\tau)) \leqslant c(u) P_x \{x(\tau) < u\} + c(a) \cdot \frac{x}{u}.$$

If $c(u) \geq 0$, the first term here does not exceed $c(u)$, but if $c(u) < 0$, it does not exceed 0; hence, in any case

$$M_x f(x(\tau)) \leqslant \max[c(u), 0] + c(a)\frac{x}{u},$$

and, consequently,

$$v(x) \leqslant \max[c(u), 0] + c(a)\frac{x}{u}.$$

Letting $x \to 0$ here, we obtain

$$\lim_{x\to 0} v(x) \leqslant \max[c(u), 0] \qquad (u > 0),$$

and then letting u tend to 0, we find

$$\lim_{x\to 0} v(x) \leqslant \max[c(0), 0] = \max[f(0), 0].$$

Inasmuch as $0 \leq v(0)$ and $f(0) \leq v(0)$, the inequality (18) is proved.

Since both of the functions f and v are continuous, the support set Γ, comprising those points x at which $f(x) = v(x)$, is closed (a priori, Γ can also be an empty set). Let τ be the time of first visit to Γ, and let

$$h(x) = M_x f(x(\tau))$$

be the average payoff for the strategy τ. Since $f = v$ on Γ,

$$h(x) = M_x v(x(\tau)). \tag{19}$$

We see now that the function h defined by Eq. (19), like v, is concave, continuous, and nonnegative. In fact, if $x = x(0) \in \Gamma$, then $\tau = 0$, and $h(x) = v(x)$. The points x not belonging to the closed set Γ form a system of intervals, the ends of which either belong to Γ or coincide with one of the points 0, a (Fig. 29). If the end points x_1 and x_2 of such an interval belong to Γ, then on the interval $[x_1, x_2]$, according to Eqs. (17), the function h is equal to

$$h(x) = \frac{x_2 - x}{x_2 - x_1} v(x_1) + \frac{x - x_1}{x_2 - x_1} v(x_2). \tag{20}$$

It is apparent from this formula that the graph of h on the interval $[x_1, x_2]$ is the chord AB subtending the points of the graph of the function v. If, on the other hand, one end of the interval (x_1, x_2) coincides with an end of the segment $[0, a]$ and does not belong to Γ, the value of v at the corresponding point x_1 or x_2 in Eq. (20) is replaced by zero, and the graph of h is a line segment such as CD in Fig. 29. This segment may also be called a chord of the graph of v, provided the vertical sections CE and FG are included in this graph. Thus, the graph of h is obtained from the graph of v by "cutting off the convexities" with chords on some systems of intervals. It is geometrically evident that this operation again produces a graph of a continuous concave nonnegative function (see the Appendix).

Inasmuch as v is the smallest of the nonnegative concave functions majorizing f, for the proof of the inequality $h \geq v$ (and, hence, of the optimality of the strategy τ) it is sufficient to verify the fact that $h \geq f$. Let us assume that the difference $f - h$ acquires a positive value somewhere. Then the continuous function $f - h$ must reach its maximum value $c > 0$ at some point x_0. The nonnegative concave function $h(x) + c$ majorizes f, which means that it also majorizes v. Consequently, $h(x_0) + c \geq v(x_0)$, which combines with the equation $c = f(x_0) - h(x_0)$ to yield the relation $f(x_0) \geq v(x_0)$. Therefore, $x_0 \in \Gamma$, whence $h(x_0) = v(x_0) = f(x_0)$ and $c = f(x_0) - h(x_0) = 0$. This contradicts the premise that $c > 0$. Thus the optimality of the strategy τ is proved.

We conclude with a few remarks about the multidimensional case. Consider the optimal stopping problem of an l-dimensional Wiener process in a closed domain G with absorption at the boundary. The value of the game is found as in the one-dimensional case, except that the nonnegative concave functions must be replaced by nonnegative functions f satisfying the two following conditions:

1) For any l-dimensional sphere $S \subset G$ with center x the mean value of f on S does not exceed $f(x)$.

2) For any $x \in G$ and any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f(y) \geqslant f(x) - \varepsilon,$$

provided only that

$$|y - x| < \delta, \ y \in G.$$

[We point out that the condition 1 is a special case of the inequality $M_x f(x(\tau)) \leq f(x)$ when τ is the time of first exit from S.] The conditions 1 and 2 form the definition adopted in modern potential theory for a *superharmonic* function in the domain G.* Consequently, it may be stated that the value of the game is the nonnegative superharmonic majorant of the payoff function. As far as the optimal strategy is concerned, it far from always exists. In any case, however, it is possible to formulate ε-optimal strategies by means of ε-support sets, as was done for denumerable Markov chains at the end of §6.

We note further that inasmuch as for $l \geq 3$ a Wiener path no longer has a probability one of entering any arbitrary domain, for $l \geq 3$ the optimal stopping problem is important in the special case when G is the entire space (see the footnote on page 113).

§8. Proof of the Fundamental Property of Concave Functions

It remains for us to prove that in the case of a Wiener process on an interval $[0, a]$ with absorption at the end points the class of functions $f(x)$, $x \in [0, a]$, satisfying the condition

$$f(x) \geqslant M_x f(x(\tau)) \tag{21}$$

for any Markov time τ coincides with the class of nonnegative concave functions.

This is a very simple matter in one direction. Letting $\tau = \infty$ in (21), we find that $f \geq 0$. Moreover, let the subinterval $[x_1, x_2]$ be

* If the function f is continuous and has continuous second partial derivatives, 2 is fulfilled automatically, and 1 reduces to the inequality $\Delta f \leq 0$, where Δ is the Laplace operator (cf. the derivation of the equation $\Delta f = 0$ in Chapt. II, §4).

contained in $[0, a]$, and let τ be the time of first exit of $x(t)$ from $[x_1, x_2]$. According to Eq. (17), for this τ

$$\mathbf{M}_x f(x(\tau)) = f(x_1)\frac{x_2 - x}{x_2 - x_1} + f(x_2)\frac{x - x_1}{x_2 - x_1}$$

for $x_1 \leq x \leq x_2$. Consequently, the graph of the function $\mathbf{M}_x f(x(\tau))$ for $x \in [x_1, x_2]$ is a line segment connecting the points with abscissas x_1 and x_2 on the graph of the function $f(x)$. It follows from the inequality (21), therefore, that any chord of the graph of f does not lie higher than this graph, i.e., the function f is concave.

It is a much more complex task to show that every concave nonnegative function satisfies the condition (21), although basically the argumentation remains the same as in the derivation of the condition (21) in the discrete case for excessive functions. We analyze the proof in six parts.

1°. We define an operator P_t $(t > 0)$ on bounded functions $f(x)$, $0 \leq x \leq a$, by the formula

$$P_t f(x) = \mathbf{M}_x f(x(t)) = \int_0^a f(y)\,\mu_t(dy), \tag{22}$$

where $\mu_t(\Gamma) = \mathbf{P}_x\{x(t) \in \Gamma\}$, and we let

$$P_\infty f(x) = \lim_{t\to\infty} P_t f(x).$$

By virtue of the Markov property, the process $y(s) = x(s + t)$ for any fixed $t > 0$ is a Wiener process with absorption at the end points and an initial distribution $\mu_t(\Gamma)$. Therefore, applying Eq. (22) twice, we write

$$\mathbf{M}_x f(y(s)) = \int_0^a \mathbf{M}_y f(x(s))\,\mu_t(dy) = \int_0^a P_s f(y)\,\mu_t(dy) = P_t P_s f(x).$$

On the other hand,

$$\mathbf{M}_x f(y(s)) = \mathbf{M}_x f(x(t + s)) = P_{t+s} f(s).$$

Consequently, the operators P_t are multiplied according to the rule

$$P_t P_s = P_{t+s}, \tag{23}$$

We recall for comparison that in the case of a discrete Markov chain

$$\mathbf{M}_x f(x(n)) = P^n f(x),$$

where P is the one-step shift operator. Consequently, in the discrete-time case Eq. (23) reduces to the ordinary rule for the multiplication of powers. We note that families of operators P_t $(t > 0)$ which are multiplied according to Eq. (23) are called one-parameter subgroups.

It is immediately evident from the definition of the operator P_t that if $f \geq 0$, then $P_t f \geq 0$ also (the operator P_t is positive). Applying this property to the difference function $f - g$, we deduce that if $f \geq g$, then $P_t f \geq P_t g$ also (the operator P_t preserves the inequality between functions).

We next calculate $P_\infty f(x)$. We know that a particle leaving any point of the interval has a probability one of sooner or later arriving at an end point of the interval, where it will always remain. Consequently, as $t \to \infty$ the measure μ_t of the interval $(0, a)$ tends to zero, while the measure μ_t of the points 0 and a tends to $\mathbf{P}_x\{x(\tau) = 0\}$ and $\mathbf{P}_x\{x(\tau) = a\}$, where τ is the time of first exit of the path from the interval $(0, a)$. Therefore,

$$P_\infty f(x) = f(0) \cdot \mathbf{P}_x\{x(\tau) = 0\} + f(a)\mathbf{P}_x\{x(\tau) = a\}$$

or, according to Eq. (17),

$$P_\infty f(x) = f(0)\frac{a-x}{a} + f(a)\frac{x}{a}.$$

It is apparent from the above expression that $P_\infty f$ is a linear function whose values at the points 0 and a coincide with the values of f at those points (Fig. 30).

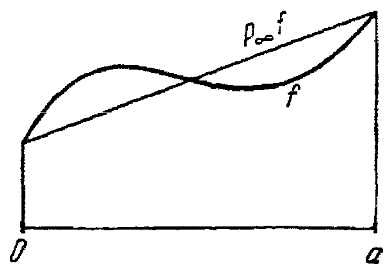

Fig. 30

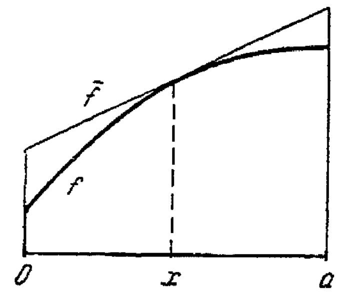

Fig. 31

2°. For linear functions f

$$P_t f = f. \tag{24}$$

According to part 1°, if f is linear, $f = P_\infty f$. Passing to the limit in the identity*

$$P_t(P_s f) = P_{t+s} f$$

as $s \to \infty$, we arrive at Eq. (24). It is easily shown that the converse is also true (we leave the proof of this to the reader).

3°. If the function f is concave,

$$P_t f \leqslant f.$$

For $x=0$ and $x=a$ the probability is one that $x(t) = x(0)$, so that

$$P_t f(x) = \mathbf{M}_x f(x(t)) = \mathbf{M}_x f(x(0)) = f(x).$$

Let x be an interior point of the interval $[0, a]$. Inasmuch as the function f is concave, it is possible to formulate a linear function $\bar{f}$ such that $\bar{f}(x) = f(x)$ at the given point x, and at all other points $f \geq \bar{f}$ (Fig. 31) (the proof of this property of concave functions is given in § 2 of the Appendix). According to part 2°,

$$P_t \bar{f} = \bar{f}.$$

Since $\bar{f} \geq f$ on the entire interval, while at the point x the values of

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*See the footnote on page 104.

$\bar{f}$ and f are equal, we have

$$P_t f(x) \leqslant P_t \bar{f}(x) = \bar{f}(x) = f(x).$$

4°. Let α be some number from the interval (0, 1). We agree to call functions h represented in the form

$$h(x) = \int_0^\infty \alpha^t P_t g(x)\, dt = \mathbf{M}_x \int_0^\infty \alpha^t g(x(t))\, dt,$$

where $g \geqq 0$, α-potentials [the α-potentials play the same role in the continuous case as series of the form (8) in the discrete case].

We now show that if $f \geq 0$, $P_t f \leq f$ for all t, and the function f is continuous at interior points of the interval $[0, a]$, then no matter what α is, $0 < \alpha < 1$, and the function f may be represented as the limit of nondecreasing α-potentials.

Making use of the identity $P_t P_s = P_{t+s}$, we write

$$\int_0^s \alpha^t P_t f\, dt = \int_0^\infty \alpha^t P_t f\, dt - \int_s^\infty \alpha^t P_t f\, dt$$

$$= \int_0^\infty \alpha^t P_t f\, dt - \int_0^\infty \alpha^{s+t} P_{s+t} f\, dt = \int_0^\infty \alpha^t P_t (f - \alpha^s P_s f)\, dt,$$

or

$$\frac{1}{s} \int_0^s \alpha^t P_t f\, dt = \int_0^\infty \alpha^t P_t g\, dt, \tag{25}$$

where

$$g = \frac{f - \alpha^s P_s f}{s}; \tag{26}$$

these integrals converge, since $|\alpha| < 1$ and $|P_t f(x)| = |\mathbf{M}_x f(x(t))|$ is bounded by the number $\sup_x |f(x)|$. Inasmuch as $0 \leq P_t f \leq f$ and

$0<\alpha<1$, it follows from (26) that $g \geq 0$. Consequently, an α-potential stands on the right side of Eq. (25). We can establish the fact that this α-potential, without decreasing, converges to f as $s \to 0$ if we verify that

$$\lim_{t \to 0} P_t f = f \tag{27}$$

and that $\alpha^t P_t f$ is a nonincreasing function of the argument t. The required monotonicity of $\alpha^t P_t f$ follows from the sequence of relations

$$\alpha^{t+u} P_{t+u} f \leqslant \alpha^t P_{t-u} f = \alpha^t P_t (P_u f) \leqslant \alpha^t P_t f$$
$$(u > 0).$$

In order to demonstrate (27), we recall that

$$P_t f(x) = \mathbf{M}_x f(x(t)).$$

As $t \to 0$, the probability is one that $x(t) \to x$, because the paths of the process $x(t)$ are continuous. This means that $f(x(t)) \to f(x)$ also with probability one at those points x where f is continuous, i.e., at all interior points of $(0, a)$. But if the random variable $f(x(t))$ converges with probability one to a constant $f(x)$, its expectation converges to the expectation of the constant $f(x)$, i.e., to the number $f(x)$ itself [passage to the limit in the argument of the expectation is legitimate, insofar as the random variable $f(x(t))$ is bounded for any t by the same number $k = \sup_x |f(x)|$]. Consequently,

$$\lim_{t \to 0} \alpha^t P_t f(x) = \lim_{t \to 0} \alpha^t \cdot \lim_{t \to 0} \mathbf{M}_x f(x(t)) = f(x)$$
$$(0 < x < a).$$

As for the points $x=0$ and $x=a$, where f can suffer a discontinuity, there $P_t f(x) = f(x)$ for all t; Eq. (27) follows at once from this.

5°. If $h(x)$ is an α-potential and τ is any Markov time,

$$\mathbf{M}_x \alpha^{\tau} h(x(\tau)) \leqslant h(x).$$

We have the condition

$$h(x) = \mathrm{M}_x \int_0^\infty \alpha^t g(x(t))\, dt,$$

where $g \geq 0$. Therefore,

$$h(x) \geqslant \mathrm{M}_x \int_\tau^\infty \alpha^t g(x(t))\, dt = \mathrm{M}_x \alpha^\tau \int_0^\infty \alpha^s g(x(\tau + s))\, ds = \mathrm{M}_x \alpha^\tau \int_0^\infty \alpha^s g(y(s))\, ds, \tag{28}$$

where $y(s) = x(\tau + s)$. According to the strong Markov property, the process $y(s)$ under the condition $\tau = t$, $x(\tau) = y$ is exactly the same process as $x(s)$ beginning at the point y.* Hence

$$\mathrm{M}_x (\alpha^\tau \int_0^\infty \alpha^s g(y(s))\, ds \mid \tau = t,\ x(\tau) = y) = \alpha^t \mathrm{M}_y \int_0^\infty \alpha^s g(x(s))\, ds = \alpha^t h(y).$$

Denoting by $F(t, y)$ the joint distribution function of the pair of random variables τ and $x(\tau)$, we then write

$$\mathrm{M}_x \alpha^\tau \int_0^\infty \alpha^s g(y(s))\, ds = \int_0^\infty \int_0^a \alpha^t h(y)\, dF(t, y) = \mathrm{M}_x \alpha^\tau h(x(\tau)).$$

Subsituting this value into Eq. (28), we obtain the desired result.

6°. Finally, we prove that a nonnegative concave function f satisfies the condition (21). It follows from the continuity of a concave function inside the interval $[0, a]$ and parts 3° and 4° that for any $\alpha \in (0, 1)$ f is the limit of a nondecreasing sequence of α-potentials $h_1, h_2, \ldots, h_n, \ldots$. According to part 5°, for any Markov time τ

$$\mathrm{M}_x \alpha^\tau h_n(x(\tau)) \leqslant h_n(x) \leqslant f(x).$$

* The intuitively justified but rather loose arguments presented here with regard to what happens under the condition $\tau = t$, $x(\tau) = y$, which has a probability zero, can be translated into completely rigorous form.

Since h_n converges monotonically to f, it is permissible to pass to the limit in this inequality as $n \to \infty$ in the argument of M_x. Thus

$$M_x \alpha^\tau f(x(\tau)) \leqslant f(x)$$

for any positive $\alpha < 1$. Passing to the limit again as $\alpha \to 1$, we obtain $\mathbf{M}_x f(x(\tau)) \leq f(x)$.

To what extent does the given proof extend to a multidimensional Wiener process? As already stated at the end of §7, in general the role of the concave functions is taken by superharmonic functions. Defining the operator P_t as before by the formula

$$P_t f(x) = \mathbf{M}_x f(x(t)),$$

we call nonnegative functions f satisfying the conditions

$$\begin{gathered} P_t f \leqslant f, \\ \lim_{t \to 0} P_t f = f \end{gathered} \tag{29}$$

e x c e s s i v e   f u n c t i o n s (cf. the definition of excessive functions for Markov chains in §3). In essence, we began in the present section by demonstrating that nonnegative concave functions are excessive, then we established the fact that excessive functions satisfy the inequality

$$\mathbf{M}_x f(x(\tau)) \leqslant f(x)$$

for any Markov time τ. The reader can easily verify that this second half of the proof has a completely general character and is equally applicable to the multidimensional case. On the other hand, the proof that superharmonic nonnegative functions are excessive in the multidimensional case is more complicated than in the one-dimensional case (on more than one occasion we made use of the special properties of concave functions, for example, their continuity at interior points of the interval). Moreover, the one-dimensional nature of the problem made it possible for us in §7 to circumvent the problems associated with the measurability of the value of a game.

PROBLEMS

Choosing One of the Two Best Objects

Suppose that it is required to choose one of the two best objects (it being immaterial exactly which one) among n objects. As in the case analyzed in §1, the problem reduces to the optimal stopping of a certain Markov chain x(0), x(1), x(2), ... In §1 the elements of this chain stand for the indices of the maximal objects (points), i.e., objects better than all those already scanned. It is clear that maximality in the new problem must be given a weaker interpretation, regarding an object a_k as "maximal" if it is the best or second best of all the objects $a_1, a_2, \ldots, a_k$ already scanned. This is a small matter, however The value of x(i) must indicate not only the order number (index) of the corresponding "maximal" object, but also whether this object is the best (i.e., maximal in the previous sense) or second best. The phase space of the chain x(i) is therefore conveniently represented as two parallels rows of n points, regarding the upper row as representative of objects better than all the preceding ones and the lower row as representative of objects inferior to just one of the preceding ones (Fig. 32).

1. Find the transition probabilities of the chain x(i).

Answer. Regardless of whether the points k and l are situated in the upper or lower row,

$$p(k, l) = \frac{k(k-1)}{l(l-1)(l-2)} \qquad (l > k)$$

[for $l=2$, k = 1 the fraction is to be assumed equal to $k/[l(l-1)]$].

2. Find the probability of success (payoff function) f for stopping of the chain at a given point.

Answer. Letting the subscript 1 refer to points in the upper, the subscript 2 to points in the lower row, we have

$$f_1(k) = \frac{k(2n-k-1)}{n(n-1)},$$

$$f_2(k) = \frac{k(k-1)}{n(n-1)}.$$

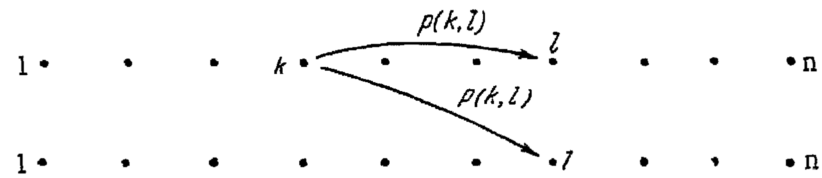

Fig. 32

Arguments similar to those used in §6 indicate that the value of the game v is found successively according to the relation

$$v_j(k) = \max\left\{ f_j(k),\ \sum_{l=k+1}^{n} p(k,\ l)\,[v_1(l) + v_2(l)] \right\} \tag{30}$$

(v_1 corresponds to upper points, v_2 to lower points). We denote by Γ_j the set of points of the jth row in which the functions f and v coincide ($j = 1, 2$).

3. The set Γ_2 has the form $\{m_2, m_2+1, \ldots, n\}$, where m_2 is the smallest integer greater than or equal to $(2n+1)/3$.

The set Γ_1 also contains all the numbers $m_2, m_2+1, \ldots, n$.

Hint. Verify the fact that

$$\sum_{l=k+1}^{n} p(k,\ l)\,[f_1(l) + f_2(l)] = \frac{2k(n-k)}{n(n-1)}$$

and apply Eq. (30).

We denote by B_k the set comprising Γ_2 plus the points $k+1, k+2, \ldots, n$ of the upper column ($k < m_2$), and we let τ_k denote the time of first visit to B_k.

4. If $f_1(k) < \mathbf{M}_x f(x(\tau_k))$, k does not belong to Γ_1. If $k+1, k+2, \ldots, n$ belong to Γ_1 and $f_1(k) \geq \mathbf{M}_k f(x(\tau))$, k belongs to Γ_1.

Hint. Given any initial state, stopping at the time of first visit to $\Gamma_1 \cup \Gamma_2$ is an optimal strategy (see §5).

5. Find the distribution of $x(\tau_k)$ for an initial state k.

Hint. Describe the event $x(\tau_k) = l$ in terms of the objects $a_{k+1}, a_{k+2}, \ldots, a_l$. For $k < l < m_2$ we have $\mathbf{P}_k\{x(\tau_k) = l\} = \frac{k}{l(l-1)}$

for points l of the upper row, and for $m_2 \leq l \leq n$ we have $P_k\{x(\tau_k)=l\}=\frac{k}{m_2-1}\,p(m_2-1,\,l)$ for points l of both rows.

6. The set Γ_1 has the form $\{m_1, m_1+1, \ldots, n\}$, where m_1 is the smallest positive integer for which

$$\left(\frac{1}{m_1}+\frac{1}{m_1+1}+\cdots+\frac{1}{m_2-2}\right)\leqslant\frac{3m_2-2m_1-4}{2(n-1)}.$$

7. If the number of objects n grows indefinitely,

$$\lim\frac{m_1}{n}=\alpha,\quad \lim\frac{m_2}{n}=\frac{2}{3},$$

where α is the root of the equation $\alpha - \ln\alpha = 1 + \ln(3/2)$ and is smaller than one ($\alpha \approx 0.347$).

8. The probability of success with an optimal strategy tends to $\alpha(2-\alpha) \approx 0.574$ as $n \to \infty$.

Hint. The distribution at the time of first visit to the set $\Gamma_1 \cup \Gamma_2$ for any initial state $s < m_1$ will be the same as the distribution of $x(\tau_k)$ for $k = m_1 - 1$ in Problem 5.

Further Generalization of the Choice Problem

Now let it be required to choose with maximum probability one of the first s objects in order of quality [for a total number of objects n $(s<n)$]. The phase space of the chain x(i) consists in this case of s rows involving n points, and the arrival of the particle at a point k of the jth row means that the object a_k is ranked in jth place according to quality in the group $a_1, a_2, \ldots, a_k$. We denote by $f_j(k)$ the payoff function (probability of success) for stopping the chain at the point k of the jth row, by $v_j(k)$ the value of the game at that point, and by Γ_j the part of the support set Γ located in the jth row.

9. The transition probabilities $p(k, l)$ of the chain x(i) do not depend on which rows the points indexed by k and l are located in.

It is easily shown that

$$v_j(k)=\max\left\{f_j(k),\ \sum_{l=k+1}^{n} p(k,\ l)\sum_{i=1}^{s} v_i(l)\right\} \tag{31}$$

[cf. Eq. (30)].

10. The function $f_j(k)$ increases monotonically with respect to the argument k and decreases monotonically with respect to the argument j.

11. The double sum in Eq. (31) decreases monotonically with increasing k.

Hint. This sum is equal to the expected payoff for the optimal strategy if stopping is forbidden at the first k objects.

12. The set Γ_j has the form $m_j, m_{j+1}, \ldots, n$, where $1 \le m_1 \le m_2 \le m_s \le n$.

13. Calculate $\sum_{j=1}^{s} f_j(k)$.

Hint. We introduce the following symbolic events:

$A=\{a_k$ is one of the s best objects$\}$,

$B_j=\{a_k$ is the jth in quality of the objects $a_1, a_2, \ldots, a_k\}$.

Then

$$\sum_{j=1}^{s} f_j(k)=\sum_{j=1}^{s} \mathbf{P}\{A/B_j\}=k\sum_{j=1}^{s}\mathbf{P}\{A/B_i\}\,\mathbf{P}\{B_i\}=k\mathbf{P}(A)=\frac{ks}{n}.$$

14. In the notation of Problem 12, for $s\ge 2$

$$\lim_{n\to\infty}\frac{m_s}{n}=\sqrt[s-1]{\frac{s}{2s-1}}.$$

Hint. After computing

$$f_s(k)=\frac{k(k-1)\ldots(k-s+1)}{n(n-1)\ldots(n-s+1)}$$

and

$$p(k, l)=\frac{k(k-1)\ldots(k-s+1)}{l(l-1)\ldots(l-s)},$$

use Eq. (31) and Problem 12 (cf. the cases s = 1 and s = 2). For the calculation of the sum in Eq. (31) use the identity

$$\sum_{l=k+1}^{\infty}\frac{1}{(l-1)(l-2)\ldots(l-s)}=\frac{1}{s-1}\cdot\frac{1}{(k-1)(k-2)\ldots(k-s+1)},$$

which is valid for $s \geq 2$.

Optimal Rule for the Stopping of a Sequence of Independent Random Variables

Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent random variables, which take values from a certain number set X, and let $f(k, x)$ ($k = 1, 2, \ldots, n$; $x \in X$) be a nonnegative function. We identify ξ_1 first, then ξ_2, ξ_3, etc. The observations may be terminated at any time k. The gain in this case is $f(k, \xi_k)$. It is required to find the optimal stopping rule such that the average payoff is maximized.

As in the optimal choice problem, it is possible by retrograde induction to formulate the value of the game v(k, x) and to verify that the optimal strategy is to stop at the time of first visit of the point (k, ξ_k) to the support set Γ consisting of those pairs (k, x) for which $f(k, x) = v(k, x)$.

The statement of the problem is preserved intact for dependent random variables, but the solution is greatly complicated by the fact that the optimal stopping rule, in general, requires inclusion of all the values observed, rather than the last one only. It is interesting that the optimal stopping problem was in fact first formulated for the dependent case. In particular, A. Cayley posed the following problem in 1874 (see [20] and the solution in [21]):

"A lottery is arranged as follows: There are k tickets representing *a*, b, c, ... pounds, respectively. A person draws once; looks at his ticket; and, if he pleases, draws again (out of the remaining k − 1 tickets); looks at his ticket; and, if he pleases, draws again (out of the remaining $k-2$ tickets); and so on, drawing in all not more than n times; and he receives the value of the last drawn ticket. Suppose he regulates his drawings in the manner most advantageous to him according to the theory of probabilities, what is the value of his expectations?"

For his solution of the problem Cayley formulated an algorithm incorporating retrograde induction and calculated the answer for the case $k=4$, $a=1$, $b=2$, $c=3$, $d=4$, and $n=1, 2, 3, 4$.

The choice of one of the first s objects according to quality (see Problems 9-12) is reduced as follows to a choice from among a sequence of independent random variables.*

15. If an object a_k occupies the jth place in quality in the group $a_1, a_2, \ldots, a_k$, we put

$$\xi_k = \begin{cases} j, & 1 \leqslant j \leqslant s, \\ s+1, & s+1 \leqslant j. \end{cases}$$

The random variables $\xi_1, \xi_2, \ldots, \xi_n$ are independent, and the probability $f(k, j)$ of success in choosing the object a_k under the condition $\xi_k = j$ is equal to

$$f(k, j) = \begin{cases} f_j(k), & 1 \leqslant j \leqslant s, \\ 0, & s+1 \leqslant j, \end{cases}$$

where $f_j(k)$ is the function from Problem 10.

16. If $f(k, x)$ is a nondecreasing function of the argument k, and $f > 0$, there exists an integer-valued function $m(x)$, $x \in X$, such that the set Γ is specified by the inequalities $m(x) \leq k \leq n$. If, in addition, $f(k, x)$ is a nonincreasing (nondecreasing) function of x, $m(x)$ is a nondecreasing (nonincreasing) function of x.

17. (See [23]). Let ξ_k be distributed uniformly on the interval [0, 1] and let $f(k, x) = x$.

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*See [22]; also obtained in this paper are some results that were presented in another manner in the preceding sets of problems.

Then

$$m(x)=n-k \qquad \text{for} \quad x_k \leqslant x < x_{k+1},$$

where the numbers x_k are found from the relations

$$x_{k+1}=\frac{1+x_k^2}{2}, \qquad x_0=0.$$

<u>Hint.</u> It can be shown by induction on k that

$$v(k, x)=\begin{cases} x_k, & 0 \leqslant x \leqslant x_k, \\ x, & x_k \leqslant x \leqslant 1. \end{cases}$$

18. In the preceding problem, as $k \to \infty$,

$$1-x_k \sim \frac{2}{k}.$$

<u>Hint.</u> Letting

$$x_k=1-\frac{2}{\alpha_k},$$

we find that

$$\alpha_{k+1}=\alpha_k+1+\frac{1}{\alpha_k-1}, \qquad \alpha_0=2.$$

We find in succession, therefore, that $\alpha_k \to \infty$, $\alpha_{k+1}-\alpha_k \to 1$, $\alpha_k/k \to 1$. A better estimate

$$k+\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)+1<\alpha_k \leqslant k+\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)+2$$

and further refinements may be found in the paper by Moser [23].

<u>Optimal Stopping of a General Markov Chain</u>

19. If in a chain with a denumerable infinity of states the support set Γ is accessible with probability one from any

state x, stopping at the time of first visit to Γ is an optimal strategy.

Hint. Investigate the time τ_ε of first visit to the ε-support set Γ_ε and let ε tend to zero.

20. A state a belongs to the support set Γ if and only if there exists an excessive function h everywhere greater than or equal to the payoff function f and coinciding with f at the point a.

21. (Method of successive approximations.*) Let f^+ be a function equal to the payoff function f wherever $f \geq 0$ and equal to zero wherever $f < 0$, and let the operator Q be given by the equation

$$Qf(x) = \max\{f(x),\ Pf(x)\}.$$

Then $Q^n f^+$ converges monotonically to the value of the game v as $n \to \infty$.

Hint. The function $Q^\infty f = \lim_{n\to\infty} Q^n f$ is the excessive majorant of f.

Fee for a Game

Suppose that after every transition from x to y a fee $\Phi(x, y)$ is collected. If for any initial state x the expectation of the fee up to the instant of termination of the chain ζ

$$F(x) = \mathbf{M}_x \sum_{t=1}^{\zeta-1} \Phi(x(t-1),\ x(t))$$

is finite, the optimal stopping problem reduces to the case in which there is no fee for the game.

22. For any Markov time τ

$$F(x) = \mathbf{M}_x \sum_{t=1}^{\tau} \Phi(x(t-1),\ x(t)) + \mathbf{M}_x F(x(\tau)).$$

Hint. Compare the proof of Eq. (24) from Chapt. I, §5.

*Proposed by A. D. Venttsel'.

23. The quantity

$$M_x\left[f(x(\tau))-\sum_{t=1}^{\tau}\Phi(x(t-1),\ x(t))\right]$$

attains its maximum value at the Markov time τ when and only when τ is the optimal strategy in the stopping problem for a chain $x(t)$ with the payoff function $f(x)+F(x)$.

Unbounded Payoff Functions

It was postulated in Chapt. III that the payoff function f was bounded. Now we lift this assumption, assuming that f is nonnegative [so that there always exists a finite or infinite expectation $M_x f(x(\tau))$]. We define the value of the game and the class of excessive functions in the same manner as in §§2 and 3, except that now we admit the value of $+\infty$ for these functions.

24. Any excessive function f is the limit of a nondecreasing sequence of bounded excessive functions.

Hint. Investigate $f_n(x)=\min\{n, f(x)\}$.

25. Extend the inequality $M_x f(x(\tau))\leq f(x)$ (τ is any Markov time) to excessive functions admitting the value $+\infty$.

26. The value of the game v is the excessive majorant of the payoff function f.

Hint. The function v is the limit of a nondecreasing sequence $\{v_n\}$, where v_n is the value of the game corresponding to the payoff function f_n of Problem 24.

27. The value of the game v can be infinite for a finite payoff function f.

Hint. Investigate a random walk on the integer points of the line $x\geq 0$ with absorption at zero and assume a payoff function $f(0)=1$, $f(k)=k$ ($k\geq 1$).

28. The average payoff for stopping at the time of first visit to the ε-support set Γ_ε optionally tends to the value of the game as $\varepsilon\downarrow 0$, when the value of the game is finite.

The Martin Boundary

The method of Martin (developed later by Doob [24]) provides a means for exhibiting the structure of the set of all excessive functions associated with a denumerable Markov chain.

Let x(t) be a Markov chain on a denumerable infinite phase space E, such that for any initial state x the probability of returning to x is smaller than one. We denote by g(x, y) the expectation of the number of hits on the point y for an initial state x (Green's function, cf. Chapt. I, §5).

29. Prove that

$$g(x, y) = \pi_y(x) g(y, y),$$

where $\pi_y(x)$ is the probability, on leaving x, of arriving sometime at y.

It follows from Problems 29 and 2 of Chapt. I that

$$g(x, y) < \infty$$

for any x, y.

Let us extend the definitions given in Chapt. I for the potential and harmonic function in the case of a symmetric random walk to the case of the chain x(t); the potential of a nonnegative function φ refers to the function

$$G\varphi = \varphi + P\varphi + P^2\varphi + \dots + P^n\varphi + \dots,$$

and a harmonic function is a function h for which Ph = h.

As in Chapt. I, §5, we establish the fact that

$$G\varphi(x) = \sum_{y \in E} g(x, y)\varphi(y),$$

that the potential is excessive, and that any excessive function f is described in the form $G\varphi + h$, where $\varphi = f - Pf$, $h = \lim_{n \to \infty} P^n f$ is a nonnegative harmonic function.

30. An excessive function f is a potential when and only when $P^n f \to 0$ as $n \to \infty$.

31. The minimum of an excessive function and a potential is a potential.

32. Any excessive function is the limit of a nondecreasing sequence of potentials.

Hint. We number the points of the space E and denote by B_n the set of the first n points. Then the functions

$$f_n = \min\{nG\chi_{B_n}, f\}$$

form the required sequence of potentials (χ_B is the characteristic function of the set B).

We assume in addition that for some state $0 \in E$ the probability $\pi_y(0)$ is positive for all $y \in E$.* Then $g(0, y) > 0$ also. According to Problem 32, for an excessive function f there exists a sequence of functions $\varphi_n \geq 0$ such that

$$f(x) = \lim_{n \to \infty} \sum_{y \in E} g(x, y)\varphi_n(y). \tag{32}$$

Introducing the Martin kernel

$$k(x, y) = \frac{g(x, y)}{g(0, y)} = \frac{\pi_y(x)}{\pi_y(0)}$$

(see Problem 29), we rewrite (32) in the form

$$f(x) = \lim_{n \to \infty} \sum_{y \in E} k(x, y)\mu_n(y), \tag{33}$$

where μ_n is a sequence of measures on E described by the equation

$$\mu_n(y) = g(0, y)\varphi_n(y). \tag{34}$$

*In general the same formulations are applicable to a Markov chain as are obtained when the set S of states accessible from the state 0 is bounded (clearly, it is impossible to go from S into $E \setminus S$).

In cases requiring emphasis of the fact that k(x, y) is regarded as a function of $x \in E$ for a fixed value of y we write $k_y(x)$ instead of k(x, y).

33. Different states $y \in E$ correspond to different functions $k_y(x)$.

Hint. The function $k_y(x) - Pk_y(x)$ has a nonzero value at the single point x = y.

34. The values of all the functions $k_y(x)$ at a given point $x \in E$ are bounded by the number $1/\pi_x(0)$.

Hint. Make use of Problem 29 and the inequality $\pi_y(0) \geq \pi_x(0)\, \pi_y(x)$.

Problem 33 shows that the functions $k_y(x)$ $(y \in E)$ stand in one-to-one correspondence with points y of the space E. We affix to the family of functions $\{k_y\}$ all the possible limits of these functions (in other words, we close the set of functions k_y, using coordinatewise convergence). According to Problem 34 here and Problem 4 of Chapt. I, the resulting set of functions K is compact. Identifying the points $y \in E$ with their corresponding functions k_y, we say that the space E is embedded in the compactum K. The set $B = K \setminus E$ is called the Martin boundary for the Markov chain x(t). The elements of the set B, like those of E, are represented either by the letter y or, if it is to be stressed that they are functions on the space E, by the symbol $k_y(x)$.

35. The function $k_y(x)$ is excessive for any $y \in K$.

If for every x the function p(x, y) has a nonzero value only for a finite number of values of y, then $k_y(x)$ is a harmonic function for $y \in B$.

Hint. Examine the case $y \in B$. If $y = \lim_{n \to \infty} y_n$ $(y_n \in E)$, then, according to the hint to Problem 33, for any $x \in E$ we have

$$k_y(x) = \lim_{n \to \infty} k_{y_n}(x) = \lim_{n \to \infty} Pk_{y_n}(x) = \lim_{n \to \infty} \sum_{z \in E} p(x, z)\, k_{y_n}(z)$$

$$\geqslant \sum_{z \in E} \lim_{n \to \infty} p(x, z)\, k_{y_n}(z) = Pk_y(x).$$

[It is easily verified that if the variables $u_n(z)$ are nonnegative and

$u_n(z) \to u(z)$, then

$$\lim_{n\to\infty} \sum_z u_n(z) \geqslant \sum_z u(z).]$$

If the sums are finite, the equality sign holds.

36. For the measures μ_n given by Eq. (34) the sequence $\mu_n(E)$ is bounded.

Hint. Set $x = 0$ in Eq. (33).

Let us continue the measures μ_n over the entire compactum K, setting $\mu_n(B) = 0$. Then Eq. (33) may be rewritten in the form

$$f(x) = \lim_{n\to\infty} \left[\sum_{y\in E} k(x, y)\mu_n(y) + \int_B k(x, y)\mu_n(dy)\right]$$

$$= \lim_{n\to\infty} \int_K k(x, y)\mu_n(dy), \tag{35}$$

where $k(x, y) = k_y(x)$ $(x \in E, y \in K)$.

In the actual structure of the compactum K the function k(x, y) is continuous with respect to y for any x. According to a theorem of Helly,* if $\{\mu_n\}$ is a sequence of measures on the compactum K, such that the values of $\mu_n(K)$ are bounded, it is possible to construct a measure μ on K and to pick out from $\{\mu_n\}$ a subsequence $\{\mu_{n_k}\}$ such that for any continuous function $F(y)$ $(y \in K)$

$$\lim_{k\to\infty} \int_K F(y)\mu_{n_k}(dy) = \int_K F(y)\mu(dy).$$

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* Helly proved this theorem for the case when K is a line segment. The proof is available in any standard text on probability theory (see, e.g., [10], Chapt. IV, §11.2). A general proof is easily obtained by comparing the following two facts: 1) In the Banach space C of all continuous functions on the compactum K any nonegative linear functional l is expressed as an integral over some finite measure ν; here $\|l\| = \nu(K)$ (see, e.g., [25], §56); 2) it is possible from every sequence of linear functionals with bounded norms to pick out a weakly convergent subsequence (see, e.g., [26], Chapt. III, §24).

In application to Eq. (35) this leads to the equation

$$f(x)=\int_K k(x,\ y)\mu(dy), \tag{36}$$

where μ is a finite measure on K depending on the excessive function f.

37. Any function f representable in the integral form (36) with $\mu(K)<\infty$ is excessive.

Hint. In the case of nonnegative functions it is permissible to change the order of summation and integration.

We denote by V the set of all excessive functions satisfying the condition $f(0)=1$. It is readily seen that V is a convex set (see the problems to Chapt. I).

38. Any excessive function, unless identically equal to zero, is specified in the form $cf(x)$, where $f\in V,\ c>0$.

Hint. It is required to verify the fact that if f is excessive and $f(0)=0$, then $f=0$ everywhere. This is easily deduced from the accessibility of all states from zero and the inequality $M_x f(x(\tau))\leq f(x)$ (τ is any Markov time).

39. All extremal points of the set V are included among the functions $k_y(x)$ $(y\in K)$.

Hint. Let f be an extremal point of the set V. Setting $x=0$ in (36), we find that $\mu(K)=1$. Inasmuch as K is a compact, there exists a point $z\in K$ such that $\mu(U)>0$ for any neighborhood U of the point z. If $\mu(U)<1$, it follows from the representation

$$f(x)=\mu(U)\frac{\int_U k(x,\ y)\mu(dy)}{\mu(U)}+\mu(K\setminus U)\frac{\int_{K\setminus U} k(x,\ y)\mu(dy)}{\mu(K-U)}$$

that

$$f(x)=\frac{1}{\mu(U)}\int_U k(x,\ y)\mu(dy)$$

(see Problem 37). Clearly, this is equally true for $\mu(U)=1$.

Shrinking U to the point z, we deduce that $f(x) = k_z(x)$.

40. For $y \in E$ the function $k_y(x)$ is an extremal point of the set V.

Hint. For $y \in E$

$$k_y(x) = G\varphi(x),$$

where $\varphi(x)$ has a nonzero point only at the one point y. If

$$k_y(x) = \alpha f_1(x) + \beta f_2(x),$$

where $f_1, f_2 \in V, \ \alpha > 0, \ \beta > 0, \ \alpha + \beta = 1$, then f_1 and f_2 are also potentials of some functions $\varphi_1 \geq 0$ and $\varphi_2 \geq 0$ (see Problem 31). It is easily verified that $\alpha\varphi_1 + \beta\varphi_2 = \varphi$, whence it follows that φ_1 and φ_2 are proportional to φ and, therefore, $f_1 = f_2 = k_y$.

It is readily seen that the subset H of harmonic functions from V is also a convex set.

41. All extremal points of the set H are included among the functions $k_y(x)$ $(y \in B)$.

Hint. From the representation of the excessive function in the form $G\varphi + h$ deduce the fact that an extremal point of the set H is also an extremal point of the set V.

We denote by B_e the set of points of the boundary B that correspond to extremal functions of H. According to a theorem of Choquet,* if H is a compact convex set in the space of sequences and B_e is the set of extremal points of H, then any element $h \in H$ is represented in the form of an integral of the extremal functions according to some finite measure ν on B_e.

Consequently, any positive harmonic function h is specified in the form

$$h(x) = \int_{B_e} k(x, y)\,\nu(dy). \tag{37}$$

*See, for example, [27]; in this paper the theorem is proved for any locally convex linear topological space.

Specifying the potential $G\varphi$ ($\varphi \geq 0$) in the form

$$G\varphi(x) = \sum_{y \in E} g(x, y)\varphi(y) = \sum_{y \in E} k(x, y)\nu(y),$$

where $\nu(y) = g(0, y)\varphi(y)$, we obtain the following representation for an arbitrary excessive function $f = G\varphi + h$:

$$f(x) = \sum_{y \in E} k(x, y)\nu(y) + \int_{B_e} k(x, y)\nu(dy) = \int_{E \cup B_e} k(x, y)\nu(dy).$$

It is inferred from another theorem of Choquet that the representation obtained for $f(x)$ is unique.

In essence we were dealing with Martin boundaries in the problems to Chapt. I, where we computed the set B_e for a symmetric random walk on a plane (see Problems 42-47). Another instructive example of the calculation of a Martin boundary is offered in the next set of problems.

Random Walk on a Free Group with a Finite Number of Generators [28]

A free group G with generators $a_1, a_2, \ldots, a_m$ is constructed as follows. We consider a word $a_{i_1}a_{i_2}\ldots a_{i_n}$ of arbitrary length n, where the indices assume values of $\pm 1, \pm 2, \ldots, \pm m$. Adjoining one word to another, we obtain the product of these words. The inverse element is defined by the relation $(a_{i_1}a_{i_2}\ldots a_{i_n})^{-1} = a_{-i_n}\ldots a_{-i_2}a_{-i_1}$. The identity element is the "word" e containing no letters. Two words specify one and the same element of a group when and only when one of them can be derived from the other by the insertion or deletion of a product of the form $a_j a_{-j}$ an arbitrary number of times. For every element there exists a uniquely defined notation comprising a minimum number of letters.

Let $p_1, \ldots, p_m, p_{-1}, \ldots, p_{-m}$ represent positive numbers which sum to unity. We assume that during unit time the word g is transformed with probability p_i to the word ga_i (if $g = a_{i_1} \ldots a_{i_n}$, then $ga_i = a_{i_1} \ldots a_{i_{n-1}}$ for $i = -i_n$). The Markov chain thus defined is called a random walk on the group G.

42. The probability $r(x)$, on starting from x, of returning at some time to this state is the same for all $x \in G$.

43. If $p_i \neq p_{-i}$ for at least one i, then $r(x) < 1$.

Hint. If $p_i > p_{-i}$, then there is a probability one, beginning with a certain time, that the number of occurrences of the letter a_i will exceed the number of occurrences of the letter a_{-i} (this follows from the irreversibility of an asymmetric random walk on a line; see Chapt. IV, §4).

44. If all the p_i are equal and the number of generators $m \geq 2$, then $r(x) < 1$.

Hint. The probability of a minimal notation of a word $x \neq e$ being lengthened by one letter is $(2m - 1)/2m$, the probability of its being shortened by one letter is $1/2m$, and the affair reduces to an asymmetric random walk on a half-line (see Chapt. IV, §4).

Subtler considerations reveal that $r(x) = 1$ in the unique case $m = 1$, $p_1 = p_{-1} = 1/2$. Henceforth we postulate that $r(x) < 1$ and use only the minimal notation of the elements of the group.

45. Express the Martin kernel $k(x, y) = k_y(x)$ in terms of the probabilities u_i of arriving sometime at a_i from e ($i = \pm 1, \pm 2, \ldots, \pm m$).

Answer. If $x = a_{i_1} \ldots a_{i_n}$, $y = a_{j_1} \ldots a_{j_s}$ and the letters from the first to the kth coincide in these two words, while $i_{k+1} \neq j_{k+1}$, then

$$k(x, y) = \frac{u_{-i_{k+1}} \ldots u_{-i_n}}{u_{j_1} \ldots u_{j_k}}. \tag{38}$$

46. The sequence $k_{y_1}(x), k_{y_2}(x), \ldots, k_{y_n}(x), \ldots$ converges for every $x \in G$ if and only if the number of letters coinciding from the beginning in the words $y_n, y_{n+1}, y_{n+2}, \ldots$ tends to infinity as $n \to \infty$.

Hint. Verify first that $u_i u_{-i} < 1$ ($i = 1, \ldots, m$).

By virtue of Problem 46, the points of a Martin boundary are uniquely identified with infinite words $y = a_{i_1} a_{i_2} \ldots a_{i_n} \ldots$, where $i_k + i_{k+1} \neq 0$ ($k = 1, 2, \ldots$). The Martin boundary B consists of all such words. By virtue of Problem 35, the function $k_y(x)$ is harmonic for $y \in B$.

47. The functions $k_y(x)$ for $y \in B$ are extremal points of the set H (see Problem 41).

<u>Hint.</u> Let $y = a_{j_1} a_{j_2} \dots a_{j_s} \dots$, and let

$$k_y(x) = \alpha f_1(x) + \beta f_2(x) \qquad (x \in G),$$

where $f_1, f_2 \in H$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$. We set $y_s = a_{j_1} a_{j_2} \dots a_{j_s}$ $(s = 1, 2, \dots)$. It follows from the inequality $f_i(x) \geq \mathbf{M}_x f_i(x(\tau))$ (see §3) that

$$f_i(x) \geqslant f_i(y_s)\, \pi_{y_s}(x) \qquad (i = 1, 2), \tag{39}$$

where $\pi_z(x)$ is the probability of arriving sometime from x at z $(x, z \in G)$. If the word x contains n letters and $n \leq s$, then, by virtue of (38),

$$k_y(x) = \pi_{y_s}(x)\, k_y(y_s). \tag{40}$$

Therefore, for $n \leq s$

$$k_y(x) = \alpha f_1(x) + \beta f_2(x) \geqslant \pi_{y_s}(x)\,[\alpha f_1(y_s) + \beta f_2(y_s)] = \pi_{y_s}(x)\, k_y(y_s) = k_y(x),$$

hence the equality sign is indeed valid in (39) for $n \leq s$. Combined with (40), this yields the proportionality

$$\frac{f_i(x)}{k_y(x)} = \frac{f_i(y_s)}{k_y(y_s)} \qquad (n \leqslant s),$$

whence it is readily inferred that $f_1(x) = f_2(x) = k_y(x)$. In the case considered, therefore, $B_e = B$.

48. All positive harmonic functions are obtained according to the relations

$$f(e) = \nu,$$

$$f(a_{i_1} \dots a_{i_n}) = \frac{\nu(i_1, \dots, i_n)}{u_{i_1} \dots u_{i_n}}$$

$$+ \sum_{k=0}^{n-1} \frac{u_{-i_{k+1}} \dots u_{-i_n}}{u_{i_1} \dots u_{i_k}} [\nu(i_1, \dots, i_k) - \nu(i_1, \dots, i_k, i_{k+1})],$$

where ν and $\nu(i_1, \ldots, i_n)$ are arbitrary nonnegative numbers satisfying the relations

$$\nu(i_1, \ldots, i_n) = \sum_{i_{n+1}=1}^{m} \nu(i_1, \ldots, i_n, i_{n+1}) + \sum_{i_{n+1}=-1}^{-m} \nu(i_1, \ldots, i_n, i_{n+1})$$

$$(n = 0, 1, 2, \ldots)$$

[for n = 0 we interpret $\nu(i_1, \ldots, i_n)$ as the number ν].

Hint. Use Eqs. (37) and (38).

We will show that

$$\lim_{\|x\|\to\infty} \|x\| g(x,\ 0) = \frac{3}{2\pi}. \tag{3}$$

Equation (1) means that g(x, 0), correct to a constant, is the Fourier coefficient with index $x = \{x_1, x_2, x_3\}$ (x_1, x_2, x_3 are integers) for the function $F(\theta)$.

We note that if a periodic function $H(\theta)$ (period 2π with respect to each argument) has continuous second derivatives, its Fourier coefficients

$$h(x) = \int_Q H(\theta)\, e^{i\theta x}\, d\theta \tag{4}$$

satisfy the condition

$$h(x) = O\left(\frac{1}{\|x\|^2}\right) \tag{5}$$

[here and elsewhere $O(\alpha)$ denotes a quantity that does not exceed the product of α multiplied by a certain constant]. Thus, let Δ be the Laplace operator in the space of the variables θ. According to Green's formula

$$\int_Q H \cdot \Delta e^{i\theta x}\, d\theta = \int_Q \Delta H \cdot e^{i\theta x}\, d\theta, \tag{6}$$

because, owing to the periodicity of the integrated functions, the surface integrals over opposite faces of the cube Q cancel one another. Since $\Delta e^{i\theta x} = -\|x\|^2 e^{i\theta x}$, it follows from (6) that

$$|h(x)| = \frac{1}{\|x\|^2}\left|\int_Q \Delta H \cdot e^{i\theta x}\, d\theta\right| \leqslant \frac{1}{\|x\|^2}\int_Q |\Delta H|\, d\theta, \tag{7}$$

and we arrive at the estimate (5).

The estimate (5) remains valid in the case when the derivatives of the function H have a singularity of not too high order at zero (and the function H is twice continuously differentiable at all other points of the cube Q). Specifically, it is sufficient to demand that the function H be bounded, its first partial derivatives equal to $O(1/\rho)$, and its second partial derivatives $\partial^2 H/\partial\theta_1^2$, $\partial^2 H/\partial\theta_2^2$, $\partial^2 H/\partial\theta_3^2$ equal to $O(1/\rho^2)$. In fact, we apply Green's formula to the domain $Q \setminus K$, where K is a small cube enclosing the point 0; the integral over its surface approaches zero by virtue of the estimate for the derivatives $\partial H/\partial\theta_1$, $\partial H/\partial\theta_2$, and $\partial H/\partial\theta_3$, and in the limit we obtain Eq. (6). As a result of the estimate for the second derivatives, the integrals in (7) converge, and we again arrive at Eq. (5).

The function in which we are interested, $F(\theta)$, has a higher-order singularity at zero. Differentiating Eq. (2) as many times as necessary and writing out the first two or three terms of the expansion of the sine or cosine in a Taylor series, we obtain for small ρ

$$\left.\begin{aligned} F(\theta) &= \frac{2}{\rho^2 + O(\rho^4)}, \\ \frac{\partial F}{\partial\theta_i} &= \frac{-4\theta_i + O(\rho^3)}{\rho^4 + O(\rho^6)}, \\ \frac{\partial^2 F}{\partial\theta_i^2} &= \frac{16\theta_i^2 - 4\rho^2 + O(\rho^4)}{\rho^6 + O(\rho^8)}. \end{aligned}\right\} \tag{8}$$

This singularity may be weakened by subtracting the function $2/\rho^2$ from $F(\theta)$, as the former behaves similarly to the latter near zero. It is readily deduced from Eq. (8) that the function $F(\theta) - (2/\rho^2)$ already meets the restrictions imposed on $H(\theta)$ in the preceding paragraph. For example,

$$\frac{\partial}{\partial\theta_i}\left(F - \frac{2}{\rho^2}\right) = \frac{-4\theta_i + O(\rho^3)}{\rho^4 + O(\rho^6)} + \frac{4\theta_i}{\rho^4} = \frac{O(\rho^7)}{\rho^8 + O(\rho^{10})} = O\left(\frac{1}{\rho}\right).$$

It is still impossible, however, to use the estimate (5), because the subtracted function $2/\rho^2$, if continued periodically beyond the limits of the cube Q, will not have continuous first and second derivatives on the face of this cube. In order to remove this obstacle, we multiply $2/\rho^2$ by a nonincreasing twice continuously differentiable function $S(\rho)$ equal to one for $0 < \rho \leq 1/2$ and equal to zero for $1 \leq \rho < \infty$. It is clear that the function $2S(\rho)/\rho^2$ will, as before, "extinguish" the singularity of the function $F(\theta)$ at zero, and in the integration over the cube Q this function may be regarded as periodic with period 2π, without disturbing its smoothness. Now, therefore, the estimate (5) is applicable to the function

$$H(\theta)=F(\theta)-\frac{2S(\rho)}{\rho^2}$$

and we find that the Fourier coefficients of the functions $F(\theta)$ and $2S(\rho)/\rho^2$ differ from one another by an amount $O(1/||x||^2)$. Thus,

$$g(x,0)=\frac{6}{(2\pi)^3}\int\limits_Q \frac{S(\rho)\,e^{i\theta x}}{\rho^2}\,d\theta+O\left(\frac{1}{||x||^2}\right). \tag{9}$$

We proceed now with the computation of the integral in Eq. (9). Inasmuch as the function S is equal to zero outside the cube Q, we are in a position to replace integration over the domain Q by integration over the entire space R^3. After this we rotate the coordinates axes θ_1, θ_2, and θ_3 so that the θ_1 axis will pass through the point $x=\{x_1, x_2, x_3\}$. The quantities ρ, $S(\rho)$, and $d\theta$ remain unchanged when this is done, and the scalar product $\theta x=\theta_1x_1+\theta_2x_2+\theta_3x_3$ goes over to $\theta_1||x||$, since the vector x in the new system has the coordinates $\{||x||, 0, 0\}$. Further, we replace $e^{i\theta_1||x||}$ by $\cos\theta_1||x|| + i\sin\theta_1||x||$; inasmuch as $A(\rho)/\rho^2$ is an even function of the argument θ_1, the integral containing the sine will be equal to zero. It turns out, therefore, that

$$\int\limits_Q \frac{S(\rho)}{\rho^2}e^{i\theta x}d\theta=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{S(\rho)\cos\theta_1||x||}{\rho^2}\,d\theta_1\,d\theta_2\,d\theta_3.$$

In the latter integral we transform to spherical coordinates according to the relations

$$\theta_1=\rho\cos\psi,\quad \theta_2=\rho\sin\psi\cos\varphi,\quad \theta_3=\rho\sin\psi\sin\varphi.$$

Recognizing that the Jacobian of the transformation is equal to

$\rho^2 \sin\psi$, we obtain

$$\int_Q \frac{S(\rho)\, e^{i\theta x}}{\rho^2}\, d\theta = \int_0^\infty d\rho \int_0^{2\pi} d\varphi \int_0^\pi S(\rho) \cos(\|x\|\rho\cos\psi) \sin\psi\, d\psi$$

$$= \frac{4\pi}{\|x\|} \int_0^\infty \frac{S(\rho)\sin(\|x\|\rho)}{\rho}\, d\rho = \frac{4\pi}{\|x\|} \int_0^\infty \frac{S\left(\frac{\lambda}{\|x\|}\right)\sin\lambda}{\lambda}\, d\lambda. \qquad (10)$$

Since the integral $\int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda$ converges and the function $S(\lambda/\|x\|)$ is monotonic in λ and bounded for all x by the same number, the integral obtained in Eq. (10) converges uniformly in x (see [19], p. 477). It is permissible, therefore, to pass to the limit in the integrand, and we obtain

$$\lim_{\|x\|\to\infty} \int_0^\infty \frac{S\left(\frac{\lambda}{\|x\|}\right)\sin\lambda}{\lambda}\, d\lambda = \int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda = \frac{\pi}{2}.$$

Returning to Eqs. (9) and (10), we find

$$\lim_{\|x\|\to\infty} \|x\| g(x, 0) = \frac{6}{(2\pi)^3} \cdot 4\pi \cdot \frac{\pi}{2} = \frac{3}{2\pi}.$$

§2. Certain Properties of Concave Functions

A function $f(x)$, $x \in [a, b]$, is called concave on the indicated interval if any chord joining two points of the graph of f lies entirely on or below this graph (Fig. 52). Analytically, for any values of $x_1 < x_2$ on the interval $[a, b]$ and any numbers p and q satisfying the conditions $p \geq 0$, $q \geq 0$, $p + q = 1$, the following inequality is fulfilled:

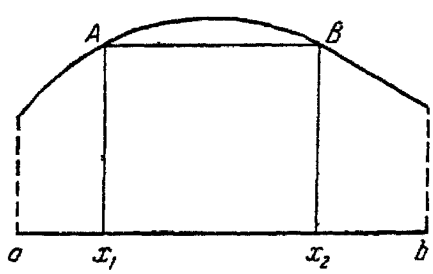

Fig. 52

$$f(px_1 + qx_2) \geqslant pf(x_1) + qf(x_2). \qquad (11)$$

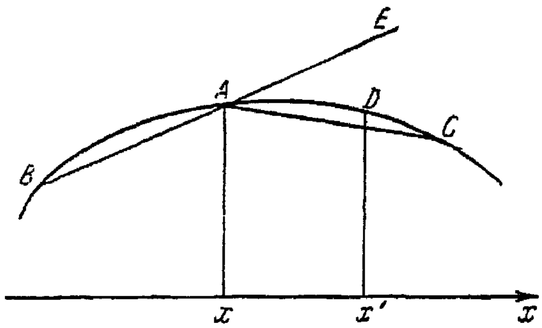

Fig. 53

The following properties of concave functions were used in Chapter III and now need to be proved.

I. The function f is continuous at all interior points of the interval $[a, b]$ and has finite limits as $x \downarrow a$ and $x \uparrow b$, where $f(a+0) \geq f(a)$, $f(b-0) \geq f(b)$.

First let x be an interior point of the interval, and let A be the corresponding point of the graph (Fig. 53). On the graph of f we pick points B and C to the left and right of A and investigate on the graph a variable point D with abscissa x' tending on the right to x. We draw the chord AC and the half-line AE representing the continuation of the chord BA. The point D cannot go above the line AE; otherwise, the chord BD would pass above the point A. On the other hand, after D goes to the left of C, it cannot drop below the chord AC. Hence, as $x' \downarrow x$ the point D will not emerge from the angle EAC, and its ordinate will tend to the ordinate of the point A. The function f is therefore right continuous at the point x. It is demonstrated analogously that the function f is left continuous at the point x.

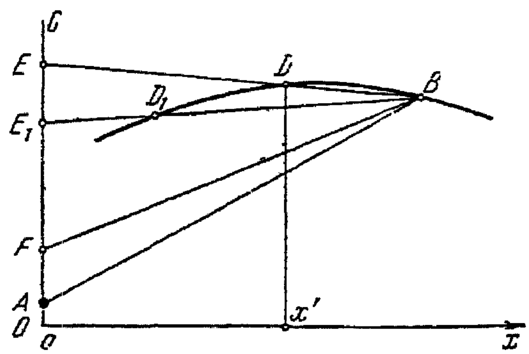

Fig. 54

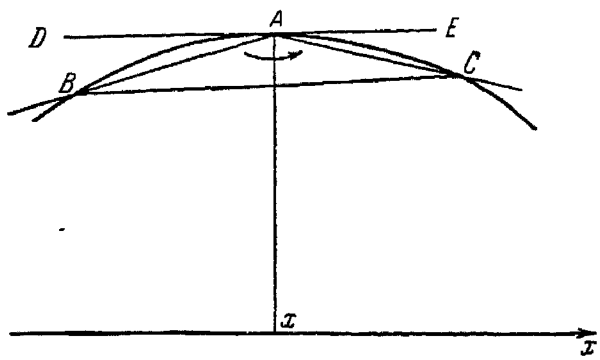

Fig. 55

We now investigate the left end point A of the graph of the function (the case of the right end point is analyzed analogously). On the graph we pick a point B distinct from A and draw the chord AB and vertical half-line AC (Fig. 54). Let D be a variable point on the graph, its abscissa x' tending on the right to a. We continue the chord DB until it intersects with the line AC at the point E. By the same arguments as in the preceding paragraph, a point D_1 situated to the left of D cannot lie above the segment ED. Therefore as $x' \downarrow a$ the point E moves along the line AC monotonically downward, without passing the point A. In the limit the point E occupies some position F, where OF $\geq$ OA. Inasmuch as the segments FE and ED shrink to zero as $x' \downarrow a$, the ordinate of the point D tends to the ordinate of the point F, hence $f(a+0) = \text{OF} \geq \text{OA} = f(a)$.

II. For any interior point x it is possible to choose a linear function $\bar{f}$ that coincides with f at the point x and is greater than or equal to f at all other points.

On the graph of the function f we pick variable points B and C to the left and right of a fixed interior point A (Fig. 55). Arguing as before, we are readily convinced that the half-line AB majorizes the graph of the function to the left of the point B, the line AC doing the same to the right of the point C, and that as B and C tend to A, these lines rise monotonically upward. Since the chord BC cannot pass above the point A, the angle BAC never exceeds 180° (the angles at the point A are measured counterclockwise). In the limit, therefore, the lines AB and AC occupy some positions AD and AE, the angle DAE again never exceeding 180°. If this angle is equal to 180°, the line DE is then the graph of the function $\bar{f}$ we are looking for. If, on the other hand, the angle DAE is smaller than 180°, any line passing through the point A outside the angle DAE will serve as the graph of $\bar{f}$.

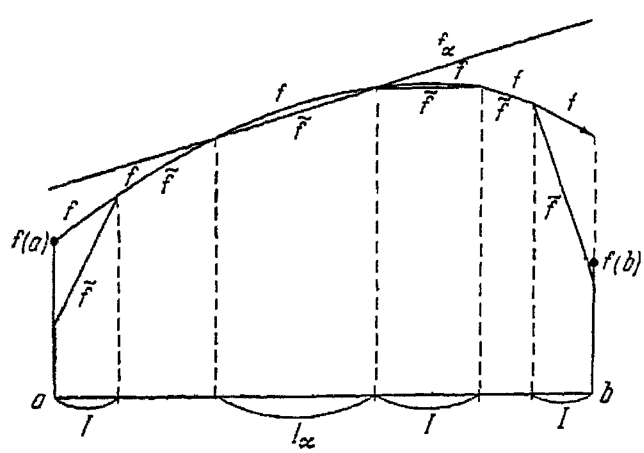

Fig. 56

III. Let us choose an arbitrary system of nonoverlapping segments I_α belonging to the interval $[a, b]$. On every segment I_α we replace the function f by a linear function f_α that coincides with f at the end points of the segment I_α, except that if an end point of I_α coincides with the point a [or point b], the function f_α can be either equal to or smaller than $f(a)$ [or $f(b)$] at the point a [or b]. At all other points we leave the function f unchanged. Then the resulting function $\tilde{f}$ is again concave on the interval $[a, b]$ (Fig. 56).

It follows from the foregoing considerations that $f_\alpha \geq f$ outside the segment I_α. Therefore, if $x \in I_\alpha$, then $f_\beta(x) \geq f(x) \geq f_\alpha(x) = \tilde{f}(x)$ for all $\beta \neq \alpha$, and if x does not belong to any of the segments I_α, then $f_\alpha(x) \geq f(x) \geq \tilde{f}(x)$ for all α. Hence, the function $\tilde{f}$ is a lower bound of the functions f and f_α (α spans all possible values). Inasmuch as the functions f and f_α are concave, all that is left to prove is that the lower bound $\tilde{f}$ of any family $\{f_\alpha\}$ of concave functions is also a concave function. For this it is sufficient to invoke the analytic condition of concavity of the functions (11) and note that for any α

$$f_\alpha(px_1 + qx_2) \geqslant pf_\alpha(x_1) + qf_\alpha(x_2) \geqslant p\tilde{f}(x_1) + q\tilde{f}(x_2).$$

§3. Solution of the Equation $p(s)p(t) = p(s + t)$

We need to show that any bounded solution of the functional equation

$$p(s)\,p(t) = p(s+t) \qquad (s,\ t > 0), \tag{12}$$

which was investigated in Chapt. IV, §2, has the form

$$p(t) = e^{-at}, \tag{13}$$

where $0 \leq a \leq +\infty$ (considering $e^{-\infty} = 0$).

We point out that if p(t) goes to zero at some point $t_0 > 0$, then, according to (12), p(t) = 0 for all $t \geq t_0$. Moreover, it follows from the relation

$$p\left(\frac{t}{2}\right)^2 = p(t) \tag{14}$$

that $p(t_0/2) = 0$, hence that p(t) = 0 for all $t \geq t_0/2$. Repeating this argument, we obtain p(t) = 0 for all t > 0, and Eq. (13) is valid with $a = +\infty$.

It now remains for us to consider the case when $p(t) \neq 0$ for all t > 0. Equation (14) implies that p(t) > 0 in this case, and we are entitled to set

$$f(t) = \ln p(t).$$

Now Eq. (12) goes over to the equation

$$f(s) + f(t) = f(s+t) \qquad (s,\ t > 0), \tag{15}$$

and the problem is reduced to one of finding all solutions of this equation that are bounded above.

It is readily deduced from Eq. (15) by induction that for any natural n

$$f(nt) = nf(t). \tag{16}$$

Picking the number a on the basis of the condition

$$f(t_1) = -at_1,$$

where t_1 is a fixed positive number, we obtain by means of Eq. (16)

$$f\left(\frac{t_1}{n}\right)=\frac{f(t_1)}{n}=-a\frac{t_1}{n}$$

Applying Eq. (16) once again, we find that for any natural numbers m and n

$$f\left(\frac{m}{n}t_1\right)=mf\left(\frac{t_1}{n}\right)=-a\frac{m}{n}t_1.$$

Consequently, for all $t>0$ commensurable with t_1 we have

$$f(t)=-at. \tag{17}$$

If it turned out for some $t_2>0$ that $f(t_2)\neq -at_2$, then, determining the number b from the condition $f(t_2)=-bt_2$, we would have found in fully analogous fashion that

$$f(t)=-bt$$

for all $t>0$ commensurable with t_2, where $b\neq a$. Let $b>a$ for definiteness. If s is commensurable with t_2, and $s+t$ with t_1, then

$$f(t)=f(s+t)-f(s)=-a(s+t)+bs=(b-a)s-at. \tag{18}$$

Inasmuch as numbers commensurable with a given number are densely distributed everywhere, s can be made as large as we like and $s+t$ as close to s as we like in the above equation. In this case t is small, and Eq. (18) gives arbitrarily large values for $f(t)$. We have thus arrived at a contradiction with the upper-boundedness requirement on $f(t)$. This means that Eq. (17) for the function $f(t)$ is valid for all $t>0$. Since $f(t)$ is bounded above and t can be as large a number as we like, in this equation $a\geq 0$.*

Returning to the function $p(t)=e^{f(t)}$, we obtain the representation (13) for it.

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*We note that Eq. (18) makes it possible to obtain arbitrarily large values for $f(t)$ when t varies over any predetermined interval. For the derivation of Eq. (17) from (15), therefore, it is sufficient to require that the function $f(t)$ be bounded above in some interval of variation of t (the number a in this case can be of any sign).

Literature Cited

The following books are recommended primarily for a familiarization with the more classical aspects of the theory of Markov processes:

1. W. Feller, An Introduction to Probability Theory and Its Applications, Wiley, New York (1957).
2. J. G. Kemeny and J. L. Snell, Finite Markov Chains, Van Nostrand, Princeton (1960).

The following monographs are devoted to new developments in this theory:

3. E. B. Dynkin, Theory of Markov Processes, Pergamon, New York (1960).
4. E. B. Dynkin, Markov Processes, Springer–Verlag, Berlin (1965).
5. G. A. Hunt, "Markov processes and potentials (I-III)," Illinois J. Math., 1:44-93 (1957); 1:316-369 (1957); 2:151-213 (1958).
6. K. L. Chung, Markov Chains with Stationary Transition Probabilities, 2nd ed., Springer–Verlag, Berlin (1967).
7. F. Spitzer, Principles of Random Walk, Van Nostrand, Princeton (1964).
8. K. Ito and H. P. McKean, Diffusion Processes and Their Sample Paths, Springer–Verlag, Berlin (1965).

(In this group, [6] represents a slight departure from the theme of the present book)

The following books also devote considerable attention to some related problems:

9. K. Ito, Probabilistic Processes, Nos. I and II [Russian translation], IL, Moscow (1960 and 1963).
10. M. M. Loève, Probability Theory, 3rd ed., Van Nostrand, Princeton (1963).
11. I. I. Gikhman and A. V. Skorokhod, Introduction to the Theory of Random Processes, Nauka, Moscow (1965).

The following publications are cited in the main text of the present work:

12. K. Ito and H. P. McKean, "Potentials and the random walk," Illinois J. Math., 4:119-132 (1960).
13. E. B. Dynkin, "The optimum choice of the instant for stopping a Markov process," Dokl. Akad. Nauk SSSR, 150:238-240 (1963).