Notes for course "Markov processes"

Canonical scale and escape probabilities (DY) Escape probability for a Markov process is the probability of a question of the form "hit a set of states before hitting another". Clearly, such probabilities only depend on the successive values attained by the process and not on the sojourn times at each state. So we may as well consider the embedded chain in order to find these probabilities.

Consider a chain x_n with values in $\mathbb{Z}_+ = \{0, 1, \ldots\}$ such that it only moves to neighboring states and it is irreducible. So the only nonzero transition probabilities are $p(i, i \pm 1)$. We set

$$p_i = p(i, i+1), \quad q_i = 1 - p_i, \quad i \in \mathbb{Z}_+.$$

For irreducibility on the whole $\mathbb{Z}_+$ we need

$$p_0 = 1, \quad 0 < p_i < 1, \quad i \geq 1.$$

This becomes a chain with reflection at 0. A continuous-time chain with x_n as embedded chain has transition rates

$$\begin{aligned} \lambda_{i,i+1} &= a_i p_i, \quad i \geq 0, \\ \lambda_{i,i-1} &= a_i q_i, \quad i \geq 1, \end{aligned}$$

where the α_i are arbitrary strictly positive numbers.

Here is another chain y_n with values in some countable set of the form

$$U = \{0 = u_0 < 1 = u_1 < u_2 < u_3 < \cdots\}$$

and transition probabilities according to the rule: the one-step probability from i to $i-1$ (resp., $i+1$) is the chance that a Brownian motion started at u_i exits $[u_{i-1}, u_i]$ from u_{i-1} (resp., from u_{i+1}). This entails that

$$p(i, i-1) = \frac{u_{i+1} - u_i}{u_{i+1} - u_{i-1}}, \quad p(i, i+1) = \frac{u_i - u_{i-1}}{u_{i+1} - u_{i-1}},$$

and

$$p(0, 1) = 1.$$

To make the algebra a bit easier, let

$$\delta_i := u_{i+1} - u_i, \quad i \geq 0.$$

In particular, $\delta_0 = 1$. By convention, set $\delta_{-1} := 0$. Then

$$p(i, i-1) = \frac{\delta_i}{\delta_{i-1} + \delta_i}, \quad p(i, i+1) = \frac{\delta_{i-1}}{\delta_{i-1} + \delta_i}$$

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work for all $i \geq 0$. The chain y_n is an irreducible chain in U. We can identify x_n and y_n by letting

$$q_i = \frac{\delta_i}{\delta_{i-1} + \delta_i}, \quad p_i = \frac{\delta_{i-1}}{\delta_{i-1} + \delta_i}.$$

That is, to pass from y_n to x_n we define the p_i as above. Conversely, to pass from x_n to y_n we let

$$\delta_i = \frac{q_i}{p_i} \delta_{i-1}, \quad i \geq 1, \quad \delta_0 = 1,$$

and so

$$\begin{aligned}
u_0 &= 0, \\
u_1 &= 1, \\
u_2 &= 1 + \frac{q_1}{p_1}, \\
u_3 &= 1 + \frac{q_1}{p_1} + \frac{q_1 q_2}{p_1 p_2}, \\
u_i &= 1 + \frac{q_1}{p_1} + \cdots + \frac{q_1 \cdots q_{i-1}}{p_1 \cdots q_{i-1}}, \quad i \geq 2.
\end{aligned}$$

We take the latter chain, y_n, and find escape probabilities for the interval

$$I := U \cap (\alpha, \beta),$$

where $\alpha < \beta$ are real numbers. We define the boundary ∂I of I by the formula

$$\partial I = \{\sup([0, \alpha] \cap \overline{U})\} \cup \{\inf([\beta, \infty] \cap \overline{U})\}.$$

where $\overline{U} := U \cap \{r := \lim_{n \to \infty} u_m\}$. The set ∂I contains at 1 or 2 elements.

1° If $I = \{u_M, \ldots, u_N\}$ with $M \geq 1$ then $\partial I = \{u_{M-1}, u_{N+1}\}$.

2° If $I = \{u_0, \ldots, u_N\}$ then $\partial I = \{u_{N+1}\}$.

3° If $I = \{u_M, u_{M+1}, \ldots\}$ with $M \geq 1$ then $\partial I = \{u_{M-1}, r\}$.

4° If $I = U$ then $\partial I = \{r\}$.

Observe that ∂I is a singleton if and only if $u_0 \in I$.

> Let $p(u)$ be the probability that the process y_n, started at $u \in I$, escapes from I at the largest point of ∂I.

We will see that this is consistent with the case when ∂I is a singleton. Patience! We have

$$p(u_i) = q_i\, p(u_{i-1}) + p_i\, p(u_{i+1}). \tag{1}$$ hh

This gives

$$\frac{p(u_{i+1}) - p(u_i)}{\delta_i} = \frac{p(u_i) - p(u_{i-1})}{\delta_{i-1}}, \quad i \geq 1,$$

For $i = 0$ we have

$$p(u_0) = p_0\, p(u_1),$$

which is written as

$$\frac{p(u_1) - p(u_0)}{\delta_0} = 0.$$

If $u_0 \in I$ (that is, if ∂I is a singleton), these equations yield that $p(u)$ is a constant function. It can easily be shown that regardless of whether we are in case 2° or 4° and regardless of whether $r < \infty$ or not, the escape probability is 1. So

$$p(u) = 1, \text{ if } u_0 \in I.$$

If $u_0 \notin T$ (cases 1° or 3°) then, letting $\partial I = \{a, b\}$ we have

$$p(u) = \frac{u - a}{b - a}.$$

Again, this works in all cases. For example, if $\partial I = \{a, \infty\}$ then $p(u) = 0$.

One thing that seems puzzling is that $p(u) = 1$ of $0 = u_0 \in I$, even when $r = \infty$. But recall that r depends in the canonical scale, and $r = 1 + \sum_{n=1}^{\infty} \frac{q_1 \cdots q_n}{p_1 \cdots p_n}$. For example, if we take x_n to be a simple random walk in $\mathbb{Z}_+$ with probability $p < 1/2$ of moving one step to the right then $r = 1 + \sum_{n=1}^{\infty} (q/p)^n = \infty$. In this case, $p(u) = 1$ means that the random walk "will reach $+\infty$", i.e., that $\lim_{n \to \infty} x_n = \infty$. If $p > 1/2$ then $r = \frac{p}{p-q}$ and then $p(u) = \frac{u-a}{b-a}$, in the canonical scale (this should be translated to the original scale to give the usual geometric formula). Again, the walk exits from $+\infty$ because it keeps reflecting at 0. Similarly, when $p = 1/2$, as long as $0 \in I$, we have $p(u) = 1$.

Kind of weird, but one should pay close attention to the definitions.

Considering the last results we have that the chain y_n is recurrent if and only if $r = \infty$ (and transient if $r < \infty$). This result can be tranferred to x_n as well as to its continuous-time version $x(t)$. Thus, if, for ℓ positive integer,

$$\lambda_{i,i+1} = a_i(i^\ell - (i-1)^\ell), \quad \lambda_{i,i-1} = a_i((i+1)^\ell - i)^\ell)$$

we can see that $u_i \to \infty$ and so the chain is recurrent. But if, for $\ell > 0$, $\ell \neq 1$,

$$\lambda_{i,i+1} = a_i(\ell^i - \ell^{i-1}), \quad \lambda_{i,i-1} = a_i(\ell^{i+1} - \ell^i)$$

the nature of the chain depends on whether $\ell < 1$ or > 1.

Canonical scale and exit times (DY) We keep the same model as above, but in continuous time. Let $x(t)$ be the CTMC, a birth-death process in U. Let $m(u)$ be the mean exit time from I, starting from $u \in I$. If I is bounded, $m(u)$ is finite, by a geometric argument. Assume $m(u) < \infty$. Then,

$$m(u_i) = \frac{1}{a_i} + p_i m(u_{i+1}) + q_i m(u_{i-1}). \tag{2}$$

$\boxed{\texttt{mm}}$

This is an equation that holds inside the interval I. There are boundary conditions, so the actual solution depends on I. Let us write $m_I(u)$ and $m_J(u)$ for two solutions. Observe that $\Delta(u) := m_I(u) - m_J(u)$ satisfies the homogeneous harmonic equation–see (1)–that is,

$$\Delta(u_i) = p_i \Delta(u_{i+1}) + q_i \Delta(u_{i-1}).$$

Solving this equation is done as in the previous section. We know that $\Delta(u)$ is either constant or linear. The general solution of (2) is obtained by the homogeneous solution (with boundary conditions) and a special solution. To find the special solution, we consider $I = U$. Write $S_i = S(u_i)$ for the special solution, assuming, arbitrarily, $S(u_0) = 0$:

$$S(u_i) = \frac{1}{a_i} + p_i S(u_{i+1}) + q_i S(u_{i-1}).$$

Rewrite as follows

$$\underbrace{\frac{S(u_{i+1}) - S(u_i)}{\delta_i}}_{:=-V(u_i)} = \underbrace{\frac{S(u_i) - S(u_{i-1})}{\delta_{i-1}}}_{:=-V(u_{i-1})} - 2\mu_i, \qquad 2\mu_i := \frac{1}{a_i} \frac{\delta_{i-1} + \delta_i}{\delta_{i-1}\delta_i}.$$

Solving,

$$\begin{aligned} 2\mu_i &= \frac{1}{a_i} \frac{p_1 \cdots p_{i-1}}{q_1 \cdots q_{i-1} q_i}, \\ V(u_i) &= \frac{1}{a_0} + \sum_{k=1}^{i} \frac{1}{a_k} \frac{p_1 \cdots p_{k-1}}{q_1 \cdots q_{k-1} q_k}, \\ S(u_i) &= - \sum_{0 \le k \le m \le i-1} \frac{1}{a_k} \frac{q_{k+1} \cdots q_m}{p_k p_{k+1} \cdots p_m}. \end{aligned}$$

Now consider a continuous function $S(u)$, $u \ge 0$, that coincides with $S(u_i)$ at all $u = u_i$ and is obtained by linear interpolation in-between these points. We have $S(0) = 0$, S is decreasing and concave. If $r = \lim u_i = \infty$ then $S(\infty) = -\infty$ but if $r < \infty$ then $S(r)$ could be finite or infinite. By convention, let us extend $S(u)$ for $u > r$ by setting it to $-\infty$. Actually, this is the only choice that will preserve concavity. Dynkin and Yushkevich give a beautiful geometric way of finding the actual $m(u)$.

If we are in case 1° the set ∂I contains two finite points a and b. We draw a straight line joining $(a, S(a))$ and $(b, S(b))$ on the plane. The difference between $S(u)$ and this straight line is $m(u)$:

$$m(u) = S(u) - \frac{(b-u)S(u) + (u-a)S(b)}{b-a}.$$

If we are in case 2° the set ∂I contains only one point, b. Then $m(u)$ is interpreted as the mean time for the chain to hit b started from u (recall that when it hits $u_0 = 0$ it reflects and keeps going). This is a finite quantity that is obtained by the same method as above.

$$m(u) = S(u) - S(b).$$

If I has infinitely many points (the rest two cases) then we have to worry whether r is finite or infinite. Suppose that $r < \infty$.

Case 3° is the case where $\partial I = \{a, r\}$, and we assume $\boxed{r < \infty}$. We obtain $m(u)$ by the letting $b \uparrow r$ in case 1° .

$$m(u) = S(u) - \frac{(r-u)S(u) + (u-a)S(r)}{r-a}.$$

This works even when $S(r) = -\infty$ because, in this case, $m(u) = \infty$.

Case 4° is the case where $\partial I = \{r\}$, and we assume $r < \infty$. Taking the limit as $b \uparrow r$ in case 2° we obtain

$$m(u) = S(u) - S(r).$$

Again, this works even when $S(r) = -\infty$ because, in this case, $m(u) = \infty$.

It remains to see what happens when $\boxed{r = \infty}$.

Consider case 3° with $\partial I = \{a, r = \infty\}$. Necessarily, $S(\infty) = -\infty$. Here we must take limit of our result for case 3° when $r \to \infty$. This gives

$$m(u) = -(u-a) \lim_{r\to\infty} \frac{S(r)}{r} = -(u-a) \lim_{r\to\infty} S'(r) = (u-a) \lim_{i\to\infty} V(u_i).$$

As the first limit is a negative quantity, we have $m(u) \geq 0$, as it should.

Consider finally case 4° with $\partial I = \{a, r = \infty\}$. Taking limit as $r \to \infty$ in $m(u) = S(u) - S(r)$ we find

$$m(u) = \infty.$$

This is obvious: The time to hit ∞ for a reflected random walk is infinity.

Boundary classification If $r < \infty$ then the probability $p(u)$ of the event that the canonical chain y_n exits from r before visiting a state $a < u$ is $(u-a)/(r-a) > 0$. Therefore $\lim_{n\to\infty} y_n = \infty$ and so the chain is transient. In this case, we say that the boundary r is *attracting*.

If $r = \infty$ then $p(u) = 0$. Here the chain is recurrent. We say that the boundary r is *repelling*.

These results and classification can be transferred to the original chain x_n and its continuous-time version $x(t)$. We are, however, also interested in the total time T that the boundary r is reached. In the repelling case, obviously $T = \infty$. But in the attracting case, we may have $T < \infty$. It can be shown that the event $\{T < \infty\}$ has probability zero or one. A priori then, in the repelling case, there are the following possibilities: (i) $T < \infty$, $E_u T < \infty$, (ii) $T < \infty$, $E_u T = \infty$, (iii) $T = \infty$, $E_u T = \infty$. It turns out that the quantity that decides these things is $S(r)$.

If $S(r) = -\infty$ then $T = \infty$ (and so $E_u T = \infty$). Here, we say that the boundary r is *inaccessible.*

If $S(r) > -\infty$ then $E_u T < \infty$ (and so $T < \infty$). Here, we say that the boundary r is *accessible.*

Combining the above we have

$\boxed{r < \infty,\ |S(r)| < \infty}$ Attracting and accessible boundary. The chain is transient and reaches the boundary in finite time.

$\boxed{r < \infty,\ |S(r)| = \infty}$ Attracting but inaccessible boundary. The chain is transient and the boundary is never reached.

$\boxed{r = \infty,\ |S(r)| = \infty}$ Repelling but inaccessible boundary. The chain is recurrent and (therefore) the boundary is never reached.

$\boxed{r = \infty,\ |S(r)| < \infty}$: Impossible case.

Note that r depends only on the jump chain, but $S(r)$ depends also on the actual rates.

Let x_n be a random walk reflected at 0. For $x(t)$, let a_i be the rate out of i. Suppose that $p > q$. Then $r = \sum_{n\geq}(q/p)^n < \infty$. So the chain is transient. Recall that

$$|S(r)| = \sum_{0\leq k\leq m<\infty} \frac{1}{a_k} \frac{q_{k+1}\cdots q_m}{p_k p_{k+1}\cdots p_m}.$$

In our case at hand,

$$|S(r)| = \sum_{k\geq 0} \frac{1}{a_k p} \sum_{m\geq k} (q/p)^{m-k} = \sum_{k\geq 0} \frac{1}{a_k p} \frac{1}{1-(q/p)} = \frac{1}{p-q} \sum_{k\geq 0} \frac{1}{a_k}.$$

Hence, if $p > q$ and $\sum_k (1/a_k) < \infty$ we have an attracting accessible boundary. And so

$$\lim_{t\uparrow T} x(t) = \infty, \quad T < \infty.$$

But if $p > q$ and $\sum_k (1/a_k) = \infty$ we have an attracting inaccessible boundary. And so

$$\lim_{t\to\infty} x(t) = \infty, \quad T = \infty.$$

On the other hand, if $p \leq q$ then $r = \infty$ and thus the boundary is repelling. Necessarily, $|S(r)| = \infty$ because, by convexity, $S(\infty) = -\infty$. So if $p \leq q$ we have a repelling inaccessible boundary. And so

$$0 = \underline{\lim}_{t\to\infty} x(t) < \overline{\lim}_{t\to\infty} x(t) = \infty, \quad T = \infty.$$