PROBLEMS IN MARKOV CHAINS FOR A CLASS IN MARKOV AT THE MSC LEVEL IN THE DEPARTMENT OF MATHEMATICS AT UPPSALA UNIVERSITY

**1.** The transition probability matrix of a Markov chain is given by $P = \begin{pmatrix} 0 & 2/3 & 0 & 1/3 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. (1) Which states are inessential? (2) Find all invariant probability distributions. (3) What are the periods of inessential states? (4) Show that $\lim_{n\to\infty} P^n$ exists and compute it.

**2.** There are $d$ labeled boxes and $N$ identical balls. Show that there are $\binom{N+d-1}{d-1}$ ways to put the balls into the boxes. Let $x = (x_1, \ldots, x_d)$ be one such way (configuration), where $x_i$ is the number of balls in box $i$. Pick a box at random. If the box is nonempty pick a ball and move it to a different box chosen at random. If the box is empty do nothing. Write down the probability $p(x, y)$ that the configuration changes from $x$ to $y$. Let $X_n$, $n = 0, 1, 2, \ldots$, be a Markov chain with transition probabilities $p(x, y)$. (1) Is it irreducible? (2) Show that there is a unique stationary distribution. (3) Is the chain time-reversible? (4) Find the stationary distribution. (5) Find the periods of all states. (6) Show that, for any distribution of the initial configuration $X_0$, $P(X_n = x)$ converges; where to?

**3.** Consider an tree with countably many vertices such that each vertex has exactly $d$ neighbors and a Markov chain on this tree that starts from a vertex and moves by selecting one of the $d$ neighbors with probability $1/d$ at each step. Examine for which values of $d$ the chain is transient.

**4.** Your money in a stock changes according to the rule $x_{n+1} = x_n + \xi_{n+1}$, where $\xi_1, \xi_2, \ldots$, are i.i.d. Bernoulli random variables with $P(\xi_1 = +1) = p$, $P(\xi_1 = -1) = 1 - p$. Suppose that money is discounted according to some $\alpha < 1$. Let $T$ be the first $n$ such that $x_n = b$. Compute the expected present value of $x_T$ when $x_0 = 0$, that is, $E_0 \alpha^{x_T}$.

**5.** Let $T$ be a positive random variable. Given $a \geq 0$, define the random function $X_a(t), t \geq 0$, as follows. If $a = 0$ let $X_a(t) := 0$ for all $t$. If $a > 0$ let $X_a(t) := 0$, $t \leq T$, while $X_a(t) = t - T$, $t \geq T$. (1) Under what condition does $X_a(t)$, $t \geq 0$, have the Markov property? (2) If $X_a(t)$, $t \geq 0$, has the Markov property and $S$ is a finite stopping time, is it true that $(X_a(S + t), t \geq 0)$ has the same distribution as $(X_a(t), t \geq 0)$, conditional on $X_a(S)$?

**6.** Someone has $N$ umbrellas in his possession, some at home and some in his office. In the morning, he leaves home. If it rains, he picks an umbrella (if any) and walks to his office. In the evening, he leaves his office to go home. If it rains, he picks an umbrella (if any) and walks home. When he steps out of his home or office there is a chance $p$ that it will rain, independently of everything prior to that instant. (Assume, of course, that he picks no umbrell if it doesn't rain.) Let $w_k^H$ (respectively $w_k^O$) be the probability that he gets wet when he leaves home (respectively office) for the $k$-th time. Show that $\lim_{k\to\infty} w_k^H = \lim_{k\to\infty} w_k^O$ and find this limit.
Hint: Let $U_n$ be the number of umbrellas he leaves behind when he steps on the street for the $n$-th time (leaving home or office). Show that $(U_n)$ is a Markov chain, find its transition probabilities and analyze it.

**7.** Consider a Markov chain with two states, 1 and 2. Let $p_{1,2} = \alpha$, $p_{2,1} = \beta$. (1) Give necessary and sufficient conditions on $\alpha$ and $\beta$ so that there is a unique stationary distribution. (2) Under these conditions, and if $\alpha \max \beta > 0$, find the rate of convergence of $p_{1,2}^{(n)}$ as $n \to \infty$. (3) Compute $p_{x,y}^{(n)}$ for all $x, y$ and $n$.

**8.** If $T$, $S$ are totally ordered countable sets and $X_t$, $t \in T$, a collection of random variables such that (i) has the Markov property (ii) $t_1 < t_2 \Rightarrow X_{t_1} < X_{t_2}$, a.s., define $Y_s$, $s \in S$ such that $Y_{X_t} = t$, $t \in T$ (the inverse function) and show that it also has the Markov propery.

**9.** Let $X_n$, $n \in \mathbb{Z}$, be a sequence of random variables. Show that it has the Markov property if and only if for all $m < n$, $(X_k, m \le k \le n)$ and $(X_k, k < m \text{ or } k > n)$ are independent conditional on $(X_m, X_n)$.

**10.** Show that for any Markov chain $X_n$, $n = 0, 1, \ldots$, in a discrete state space $S$ with transition probabilities $p_{x,y}$ there is a set $U$, there is a function $f : S \times U \to S$ and there are i.i.d. random variables $\xi_1, \xi_2, \ldots$ with values in $U$ such that

$$X_{n+1} = f(X_n, \xi_{n+1}), \quad n = 0, 1, \ldots$$

**11.** Let $X_n$, $n \in \mathbb{Z}$, be a Markov chain in a discrete state space $S$. Let $\mu$ be the distribution of $X_0$, that is, $\mu(x) = P(X_0 = x)$, $x \in S$. Let $p_{x,y}$ be the transition probabilities, that is, $p_{x,y} = P(X_{n+1} = y | X_n = x)$. In other words, we assume that the transition probabilities do not depend on $n$. Define $Y_n := X_{-n}$, $n \in \mathbb{Z}$. Show that the transition probabilites $P(Y_{n+1} = y | Y_n = x)$ may depend on $n$. Give conditions so that this does not happen.

**12.** Let $X_n$, $n \in \mathbb{Z}$, be a simple symmetric random walk and let $T := \inf\{n \ge 0 : X_n = 0 \text{ or } X_n = N\}$. Find the distribution of the random variable $T$ when $N = 3$.

**13.** A permutation (i.e., a bijection) $\sigma$ on $\{1, \ldots, N\}$ is called a transposition if there are distinct $i$ and $j$ such that $\sigma(i) = j$, $\sigma(j) = i$ and $\sigma(k) = k$ for all $k \ne i, j$. Let $S$ be the set of permutations and $T$ the subset of transpositions. Let $\xi_1, \xi_2, \ldots$ be an i.i.d. sequence of uniformly distributed random variables with values in $T$ and let

$$X_n = \xi_n \circ \cdots \xi_2 \circ \xi_1(X_0),$$

where $\circ$ denotes composition. In other words, this is a mathematical formulation of shuffling a deck of $N$ cards by using the (silly) method of picking two cards at random and interchanging them and then repeating the process. (1) Is $(X_n)$ a random walk on a graph? (3) Show that there is a unique stationary distribution that is the uniform distributiion on $S$. (4) Draw the transitition diagram for $N = 3$.

**14.** Consider a Markov chain $(X_t, t \ge 0)$ in $S = \mathbb{Z}_+ \times \mathbb{Z}_+$ with continuous time and transition rates

$$q(x, x + e_1) = \lambda, \quad q(x + e_1, x + e_2) = \mu_1, \quad q(x + e_2, x) = \mu_2,$$

for all $x = (x_1, x_2) \in S$, and $q(x, y) = 0$ for all other cases of $x$ and $y$ as long as $x \neq y$. By $e_1, e_2$ we mean $e_1 = (1, 0)$, $e_2 = (0, 1)$. (0) Describe a physical system modeled by this chain. (1) Show that $\pi(x) = C(\lambda/\mu_1)^{x_1}(\lambda/\mu_2)^{x_2}$, $x_1, x_2 \in \mathbb{Z}_+$ is the unique stationary distribution if $\lambda < \mu_1 \wedge \mu_2$. (2) Why is there no stationary distribution if this condition is violated?

**15.** Let $(X_t, t \geq 0)$ be a Markov chain in continuous time and values in a finite set $S$. Let $q_{ij}$ be the transition rate from $i$ to $j \neq i$, $i, j \in S$. By convention, assume that the paths are right-continuous. Define

$$Y_0 := X_0, \quad Y_n := X_{T_n}, \, n = 1, 2, \ldots$$

Show that $(Y_n)$ is a Markov chain in discrete time and compute the one-step transition probabilities.

**16.** (1) If $T, S$ are independent positive random variables such that $T$ is expontial, show that the distribution of $T - S$, conditional on $\{T > S\}$, is the distribution of $T$. (2) Let $T_1, T_2, \ldots, T_n$ be independent exponential random variables with mean 1. Show that $T_1 + \cdots + T_n$ has the same distribution as $T_1 + \frac{T_2}{2} + \cdots + \frac{T_n}{n}$. (3) If $T_1, T_2$ are independent exponential random variables compute the conditional expectation of $E(T_1 \vee T_2 \mid T_1 \wedge T_2)$. (And by this I mean, find a function $f$ such that $E(T_1 \vee T_2 \mid T_1 \wedge T_2) = f(T_1 \wedge T_2)$.)

**17.** Let $(U_n, n = 0, 1, \ldots)$ be a Markov chain in $\mathbb{Z}_+$ and transition probabilities

$$p_{i,i+1} = p_{i+1,i} = \alpha, \text{ if } i \text{ is even}$$
$$p_{i,i+1} = p_{i+1,i} = 1 - \alpha, \text{ if } i \text{ is odd.}$$

All other $p_{i,j}$ are zero. Assume $0 < \alpha < 1$. The Markov chain is obviously irreducible. (1) Show that it has no stationary distribution. (2) Find the unique solution of the system of equations $\nu(i) = \sum_j \nu(j)p_{j,i}$, $\nu(i) \geq 0$, for all $i$. (3) What is the limit of $P(U_n = j \mid U_0 = i)$ as $n \to \infty$?

**18.** Let $\xi_1, \xi_2, \ldots$ be an i.i.d. sequence of strictly positive geometric random variables with $P(\xi_1 > k) = p^k$, $k = 0, 1, \ldots$. Let $T_k := \xi_1 + \cdots + \xi_k$ and

$$X_n := \inf\{k \geq n : T_k \geq n\} - n.$$

Show that $(X_n, n \geq 0)$ has the Markov property and find the limit of $P(X_n = x)$ as $n \to \infty$ for all $x$.

**19.** Consider two independent simple symmetrix random walks $(X_n, n \geq 0)$ and $(Y_n, n \geq 0)$ starting from arbitrary deterministic $X_0, Y_0$. Compute the probability of the event that $X_n = Y_n$ for infinitely many values of $n$. Put it otherwise, what's the chance that two drunkards moving at random on the integers will meet infinitely many times?

**20.** Are there values of the constant $\alpha \in (0, 1)$ so that the Markov chain in $S = \{1, 2, \ldots\}$ with $p_{i,2i} = \alpha$, and $p_{i,i-1} = 1 - \alpha$, $i \geq 1$, $p_{1,1} = 1 - \alpha$, is positive recurrent?

**21.** For a simple symmetric random walk on the set $S = \{0, 1, \ldots, N\}$ (the transition probabilities are $p_{i,i+1} = pi+1, i = 1/2$, $0 \leq i \leq N-1$, $p_{0,0} = p_{N,N} = 1$) define $T := \inf\{n \geq 0 : \{X_0, \ldots, X_n\} = S\}$. (1) Is $T$ a stopping time? (2) Compute $E_i(T)$, for $0 \leq i \leq N$.

**22.** When flipping a fair coin independently, calculate the expected number $N_k$ of coin flips required until $k$ consecutive heads are obtained for the first time. For example, if the outcome is 0000011010100111 then $N_1 = N_2 = 6$, $N_3 = 16$. Hint: Play with the idea that there is a Markov chain that, at each flip $n$, keeps track of the number of consecutive heads (if any) have been seen just prior to $n$. Once you find this Markov chain, the problem is easy to solve.

**23.** A monkey is sitting in front of a typewriter whose keys are the 26 capital letters of the english alphabet. The monkey presses one key at random again and again. Find the expended number of steps until the monkey types the 6-letter word HELLO. How about the 6-letter word CUBIC? Are the two expectations the same?

**24.** Let $X_1, X_2, \ldots$ be i.i.d. random variables in $\mathbb{R}$ with $P(X_1 = x) = 0$ for all $x \in \mathbb{R}$. (Such random variables are continuous because their distribution function $x \mapsto P(X_1 \leq x)$ is continuous. For example, they may possess density, but not necessarily.) Let $\xi_n := \mathbf{1}_{X_n > X_{n-1}, \ldots, X_1}$. Show that the random variables $\xi_n$, $n = 1, 2, \ldots$ are independent but not identically distributed. Show that $P(\xi_n = 1) = 1/n$.

**25.** Let $G_1, G_2, \ldots$ be i.i.d. nonnegative geometric random variables with $P(G_1 \geq k) = p^k$, $k = 0, 1, \ldots$. Find a deterministic sequence $u_n$ of real numbers such that $P(\overline{\lim}_{n\to\infty}(G_n/u_n) = 1) = 1$. Hint: For $\varepsilon > 0$, compute the probabilities $P_n^{\pm} := P(G_n \geq (1 \pm \varepsilon)u_n)$ and choose $u_n$ such that $\sum_n P_n^+ < \infty$ but $\sum_n P_n^- = \infty$. (Subhint: $\sum_n \frac{1}{n^{1\pm\varepsilon}} < \infty$ if the plus sign is used and infinite if the minus sign is used. Also recall that if the sequence $A_n$ of events satisfies $\sum_n P(A_n) < \infty$ then the probability that infinitely many of them will occur is zero and that is because the number $\sum_n \mathbf{1}_{A_n}$ of occurrences of the events has, under the given condition, finite expectation. And recall that if the sequence $A_n$ of *independent* events satisfies $\sum_n P(A_n) = \infty$ then the probability that infinitel many of them will occur is zero, simply because we can use independece to compute this probability.)

**26.** Let $X_n, n = 0, 1, \ldots$, be simple symmetric random walk in $\mathbb{Z}$. Let $A_n := \min(X_0, \ldots, X_n)$, $B_n := \max(X_0, \ldots, X_n)$. Examine whether the sequence $(A_n, B_n)$, $n = 0, 1, \ldots$, has the Markov property.

**27.** Let $(X_n, Y_n)$, $n = 0, 1, \ldots$, be a simple symmetric random walk in $\mathbb{Z}^2$. Show that $X_n$, $n = 0, 1, \ldots$, and $Y_n$, $n = 0, 1, \ldots$ are both Markov chains and find their transition probabilities. Show that $X_n$, $n = 0, 1, \ldots$, and $Y_n$, $n = 0, 1, \ldots$ are not independent.

**28.** Let $(X_t, Y_t)$, $t \geq 0$, be a simple symmetric random walk in $\mathbb{Z}^2$ with continuous time. Show that $X_t$, $t \geq 0$, and $Y_t$, $t \geq 0$, are both Markov chains and find their transition probabilities. Show that $X_t$, $t \geq 0$, and $Y_t$, $t \geq 0$, are independent. (The process $(X_t, Y_t)$,

$t \geq 0$, is defined as a Markov chain in $\mathbb{Z}^2$ with transition rates $q_{(x,y),(x\pm 1,y)} = q_{(x,y),(x,y\pm 1)} = \lambda$ for some $\lambda > 0$.)

**29.** Let $N_1(t)$, $t \geq 0$, and $N_2(t)$, $t \geq 0$, be two independent Poisson processes. Is $N_1(N_2(t))$, $t \geq 0$, a Markov chain in continuous time?

**30.** Let $\xi_0, \xi_1, \ldots$ be i.i.d. Bernoulli random variables: $P(\xi_0 = 1) = p$, $P(\xi_0 = 0) = 1 - p$, where $0 < p < 1$. With arbitrary $X_0$ (e.g., $X_0 = 1$), define

$$X_{n+1} = \frac{1}{1 + \xi_n X_n}, \quad n = 0, 1, \ldots$$

(1) What is the state space of the Markov process $X_n$, $n = 0, 1, \ldots$? (2) Describe the asymptotic behavior of the process.

**31.** Let $X_n$, $n = 0, 1, \ldots$, be a symple symmetric random walk starting from $X_0 = 0$. (1) Show that $\lim_{n\to\infty} P(X_n/\sqrt{n} \leq x) = \int_0^x f(y)\, dy$, where $f$ is the standard normal density. (2) Let $0 < a < b$ be real numbers and compute the limit

$$\lim_{n\to\infty} P(X_{[na]}/\sqrt{n} \leq x, X_{[nb]}/\sqrt{n} \leq y),$$

where $[t]$ is the integer part of $t$, i.e., the largest integer $m$ such that $m \geq t$.

**32.** If $f : \mathbb{R} \to \mathbb{R}$ is a function such that its derivative at 0 exists (that is, $\lim_{h\to 0} \frac{1}{h}(f(h) - f(0))$ exists), is it true that $f$ is continuous in an open interval containing 0? If yes, prove it. If no, give an example.

**33.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose derivative exists everywhere. Then it is elementary that $f$ is continuous. Let $f'$ be its derivative. Is it true that $f'$ is continuous? If yes, prove it. If no, give an example.

**34.** Let $N(t)$, $t \geq 0$, be a Poisson process with rate $\lambda$. Let $z$ be a complex number. (1) Is $z^{N(t)}$, $t \geq 0$, Markov? (2) If yes, for which values of $z$ does it have a non-trivial stationary distribution; compute that stationary distribution. (By non-trivial, I mean a distribution not supported on a single point.)