Option pricing made EZ

There is something out there called stock. It's an invention of capitalism and is exactly like the stake in gambling. When you play a game of chance you gamble an amount of money and hope to win more. Let us say that the game is simple: heads or tails. You are being told that if you pay \$1 and heads come up you will get back your dollar plus \$0.75 extra, or, if tails come up, you will lose your dollar. The payment S is either \$1.75 or \$0, and is per dollar you pay. So if you pay \$3 then you will get $3S$ (meaning \$5.25 or \$0).

In the capitalist language, you can think of S as the price of a unit of stock. You started by owing 1 unit of stock priced \$1. Something happened and the unit price changed to S. You still own a unit, so the cash value of what you own is S dollars. If, instead, you started with $u_0 = 3$ units of stock then, after the coin is tossed, you still own 3 units of stock whose value is $3S$ dollars.

The first and second paragraphs above say exactly the same thing but in different language. The first paragraph uses the language of gambling. The second uses the language of capitalism.

Let's see what will happen if you decide to continue gambling (or "investing", in capitalist language). You now own u_0 units of stock. You may decide to play them all. The unit stock price changes (another gambling takes place) from S_0 to S_1. So the cash value of you have is $u_0 S_1$. However, you, being cautious, may decide to put some money away and only gamble a part of what you own. So you put away c_1 dollars and only have $u_0 S_0 - c_1$ dollars to play. You immediately transform this money into u_1 units of stock, where u_1 is such that

$$u_0 S_0 = u_1 S_0 + c_1.$$

That is, just before the next gambling, you take the decision to put some money away and play the rest. The unit stock price changes to S_1 and thus you own

$$X_1 = u_1 S_1 + c_1 \text{ dollars}.$$

Again, you put some money away and transform the rest into stock:

$$X_1 = u_2 S_1 + c_2 \text{ dollars}.$$

This equation is supposed to be read as $u_1 S_1 + c_1 = u_2 S_1 + (c_2 - c_1) + c_1$, or $u_1 S_1 - (c_2 - c_1) = u_2 S_2$, meaning that if, from our winnings we put away $c_2 - c_1$ dollars and transform the remaining into stock, we own $u_2 = (u_1 S_1 - (c_2 - c_1))/S_1$ units of stock.

Think of $c_1, c_2, \ldots$ or, equivalently, $u_1, u_2, \ldots$, as the "strategy" you follow.

Since the game is "random", just like the so-called stock market, S_n evolves as a random process. Usually, people are interested in the change of unit stock price as a proportion of its previous value, i.e., in the ratio $(S_{n+1} - S_n)/S_n$. And they think that it is this ratio that is "random". So they say

$$S_n - S_{n-1} = R_n S_{n-1},$$

where $-1 \le R_n < \infty$ is the "random interest rate". Think of S_n as the state of a random dynamical system.

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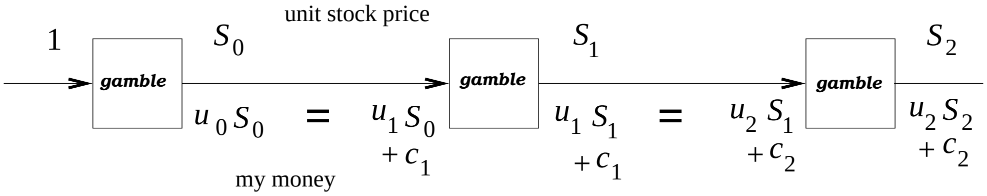
Pictorially, here is what happens:



The top line indicates the price of the unit of stock, starting from 1, to S_0, to S_1, etc. The bottom line indicates the amount of my money at each stage.

Just after the n-th gamble we have

$$X_n = u_n S_n + c_n \text{ dollars.}$$

Let us then summarize our discrete-time dynamical system as follows:

$$\text{Unit stock price evolution: } S_n - S_{n-1} = R_n S_{n-1}, \quad n \geq 1$$
$$\text{Money balance equation: } X_m = u_m S_m + c_m = u_{m+1} S_m + c_{m+1}, \quad n \geq 0$$

The dynamics can be simplified if we write

$$X_n - X_{n-1} = (u_n S_n + c_n) - (u_n S_{n-1} + c_n),$$

where we used the money balance equation for $n = m$ and $n = m - 1$. Therefore,

$$\text{Unit stock price evolution: } S_n - S_{n-1} = R_n S_{n-1}, \quad n \geq 1$$
$$\text{Money balance equation: } X_n - X_{n-1} = u_n(S_n - S_{n-1}), \quad n \geq 1.$$

So we have dynamics without the $\{c_n\}$ at all, which is natural since the $\{u_n\}$ and $\{c_n\}$ are functions of one another.

We will take X_0 as an initial condition (in dollars). We also let S_0 be the unit stock price initially. We can easily solve and find

$$S_n = S_0(1 + R_0) \cdots (1 + R_n),$$

and

$$X_n - X_{n-1} = u_n R_n S_{n-1} = S_0(1 + R_0) \cdots (1 + R_{n-1}) R_n u_n,$$

or

$$X_n = X_0 + S_0 \sum_{j=1}^{n} (1 + R_0) \cdots (1 + R_{j-1}) R_j u_j.$$

An “option” is a contract with the casino owner (the bank, say). They tell us: if at time N, the unit stock price is S_N then we’re going to pay you $\psi(S_N)$ amount of money, regardless of whether you own any stock ar have any money in your bank then. “Great”, we say. All we have to do is wait. But, of course, this is not the way things are in capitalism. They say, “well, to play this game, pay us a bit of money now”. And they quote a price. We then think, and decide. But what is the price they tell us, and how do they figure it out? Of course, they can set up an exorbitantly high price. But then they’ll have no customers.

So, here is how they think. Surely, they won't risk losing money. If they quote a price π, say, and you pay them this price then they should be able to invest it themselves in stock and be able to make them at least the amount of money $\psi(S_N)$ that they are going to give you.

So π should be such that, if $X_0 = \pi$, then there exist controls $u_0, \ldots, u_{N-1}$ such that $X_N = \psi(S_N)$, regardless of the behaviour of the market (which means regardless of the values of R_n).

In control language, π is that initial condition for which there is a control which will steer the system into the region $X_N = \psi(S_N)$.

We will call a control u which achieves $X_N = \psi(S_N)$ a "hedging-strategy", or, more precisely, a $[N, \psi]$–hedging strategy.

We define Ω to be the set of values of the sequence $\omega := (R_0, R_1, \ldots)$. Clearly, all quantities above are functions on Ω. For example, $S_n : \Omega \to \mathbb{R}$ is the function $S_n(\omega) = S_0 \prod_{j=0}^{n-1}(1+R_j)$. To denote the dependence of this on the initial price π and the control u we write $S_n(\pi, u, \omega)$.

The fundamental theorem here states that, a $[N, \psi]$–hedging strategy exists if and only if there is a probability measure P on Ω, such that the equation

$$\pi = \int_\Omega \psi(S_N(\pi, u, \omega)) P(d\omega)$$

has a solution. Questions that arise are the following:
1) What is this probability measure P?
2) How do we compute π?
3) Is the solution unique?
4) How do we find u?
5) Is the u found nonnegative?

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Let us consider the simplest nontrivial case. Stock market dynamics:

$$S_n = (1 + a\varepsilon_n) S_{n-1}.$$

Here, $0 < a < 1$ is a fixed number. We assume that

$$\varepsilon_1, \varepsilon_2, \ldots \in \{-1, +1\}.$$

We can easily solve:

$$S_n = S_0(1 + a\varepsilon_1)(1 + a\varepsilon_2) \cdots (1 + a\varepsilon_n).$$

Consider $N \geq 1$ and a function $\psi : \mathbb{R}_+ \to \mathbb{R}$ which is increasing. Define the probability space $(\Omega, \mathscr{F}, P)$, where $\Omega = \{-1, +1\}^{\mathbb{N}}$, $\mathscr{F}$ the cylinder σ-field and P the infinite product of uniform distributions on $\{-1 + 1\}$. Let also $\mathscr{F}_n := \sigma(\varepsilon_1, \ldots, \varepsilon_n)$. Since $S_{n-1} \in \mathscr{F}_{n-1}$, we have $E[S_n \mid \mathscr{F}_{n-1}] = S_{n-1} E(1 + a\varepsilon_n) = S_{n-1}$. So $\{S_n\}$ is a $\{\mathscr{F}_n\}$–martingale. Define now

$$X_n := E[\psi(S_N) \mid \mathscr{F}_n].$$

Then $\{X_n\}$ is another $\{\mathscr{F}_n\}$–martingale. Therefore, there exist random variables $H_n \in \mathscr{F}_{n-1}$ such that

$$X_n - X_{n-1} = H_n \varepsilon_n.$$

To see this directly, let

$$H_n := \varepsilon_n(X_n - X_{n-1}).$$

We have

$$H_n\varepsilon_n = (X_n - X_{n-1})\varepsilon_n^2 = X_n - X_{n-1}.$$

We claim that $H_n \in \mathscr{F}_{n-1}$. To see this, notice that the martingale property for $\{X_n\}$ says

$$E[X_n - X_{n-1} \mid \mathscr{F}_{n-1}] = 0.$$

Since

$$X_n = f_n(\varepsilon_1, \ldots, \varepsilon_{n-1}, \varepsilon_n) \equiv f_n(\boldsymbol{\varepsilon}, \varepsilon_n)$$

the martingale property is written as

$$\frac{1}{2}[f_n(\boldsymbol{\varepsilon}, 1) - f_{n-1}(\boldsymbol{\varepsilon})] + \frac{1}{2}[f_n(\boldsymbol{\varepsilon}, -1) - f_{n-1}(\boldsymbol{\varepsilon})] = 0.$$

That is,

$$f_n(\boldsymbol{\varepsilon}, 1) - f_{n-1}(\boldsymbol{\varepsilon}) = f_{n-1}(\boldsymbol{\varepsilon}) - f_n(\boldsymbol{\varepsilon}, -1).$$

Therefore,

$$\begin{aligned} H_n = \varepsilon_n(X_n - X_{n-1}) &= \varepsilon_n(f_n(\boldsymbol{\varepsilon}, \varepsilon_n) - f_{n-1}(\boldsymbol{\varepsilon})) \\ &= \begin{cases} f_n(\boldsymbol{\varepsilon}, 1) - f_{n-1}(\boldsymbol{\varepsilon}), & \text{if } \varepsilon_n = 1 \\ -(f_n(\boldsymbol{\varepsilon}, -1) - f_{n-1}(\boldsymbol{\varepsilon})), & \text{if } \varepsilon_n = -1 \end{cases}, \end{aligned}$$

and the two cases are identical. So, despite appearances, H_n is only a function of $(\varepsilon_1, \ldots, \varepsilon_{n-1})$.
Having written X_n in the form

$$X_n - X_{n-1} = H_n\varepsilon_n$$

with $H_n \in \mathscr{F}_{n-1}$, we let u_n be defined by

$$u_n := H_n/aS_{n-1}.$$

Thus $u_n \in \mathscr{F}_{n-1}$ and

$$X_n - X_{n-1} = au_nS_{n-1}\varepsilon_n = u_n(S_n - S_{n-1}),$$

and so $\{X_n\}$ satisfies money dynamics.
The only catch is that we want $u_n \geq 0$ for all n. This is equivalent to $H_n \geq 0$ for all n. And, since $H_n = \varepsilon_n(X_n - X_{n-1})$, this is further equivalent to

$$\operatorname{sgn}(\varepsilon_n) = \operatorname{sgn}(X_n - X_{n-1}) = \operatorname{sgn}(E[\psi(S_N) \mid \mathscr{F}_n] - E[\psi(S_N) \mid \mathscr{F}_{n-1}]).$$

Since

$$S_N = S_{n-1}(1 + a\varepsilon_n)(1 + a\varepsilon_{n+1}) \cdots (1 + a\varepsilon_N),$$

we have

$$\begin{aligned} E[\psi(S_N) \mid \mathscr{F}_{n-1}] &= E[\psi(S_N) \mid S_{n-1}], \\ E[\psi(S_N) \mid \mathscr{F}_n] &= E[\psi(S_N) \mid S_{n-1}, \varepsilon_n]. \end{aligned}$$

With $\Pi := (1 + a\varepsilon_{n+1}) \cdots (1 + a\varepsilon_N)$,

$$\begin{aligned}\Phi_1(x) &:= E[\psi(x(1 + a\varepsilon_n)\Pi],\\ \Phi_2(x) &:= E[\psi(x\Pi)].\end{aligned}$$

Then

$$\begin{aligned}E[\psi(S_N) \mid \mathscr{F}_{n-1}] &= \Phi_1(S_{n-1})\\ E[\psi(S_N) \mid \mathscr{F}_n] &= \Phi_2(S_{n-1}(1 + a\varepsilon_n))\end{aligned}$$

We thus need to check if

$$\mathrm{sgn}(\varepsilon_n) = \mathrm{sgn}(\Phi_2(S_{n-1}(1 + a\varepsilon_n)) - \Phi_1(S_{n-1})),$$

or, equivalently,

$$\Phi_2(s(1 + a)) \geq \Phi_1(s), \quad \Phi_2(s(1 - a)) \leq \Phi_2(s).$$

But

$$\Phi_1(x) = \frac{1}{2}\Phi_2(x(1 + a)) + \frac{1}{2}\Phi_2(x(1 - a)).$$

So the last two inequalities are equivalent to

$$\Phi_1(x(1 + a)) \geq \Phi_1(x), \quad \Phi_1(x(1 - a)) \leq \Phi_1(x).$$

Since ψ is an increasing function, it follows that Φ_1 is increasing and from this the last two inequalities are obvious.

Having realized the hedging strategy $\{u_n\}$, and having proved that $u_n \geq 0$, we can now define $\{c_n\}$ too, simply from the equation

$$X_n = u_n S_n + c_n.$$

Here, there is no guarrantee that $c_n \geq 0$. If $c_n < 0$ for some n, this means that, instead of putting money in a bank account, we borrow money from the bank. So, if as in the diagram, we start with $X_0 = u_0 S_0$, then define u_1 as above, and redistribute money according to $u_0 S_0 = u_1 S_0 + c_1$, then $c_1 < 0$ means that it is necessary to borrow money from the bank so that $u + 0S_0 + |c_1| = u_1 S_0$.

In all this business, there is no probability measure at all. The probability measure P was only introduced as an artifact for finding a strategy. Since

$$X_0 = E[\psi(S_N) \mid S_0],$$

it follows that this equation is the equation which we can use in order to compute the price for the option $[N, \psi]$. Using $S_N = S_0(1 + a\varepsilon_1) \cdots (1 + a\varepsilon_N)$, we find that if, initially, the unit stock price is S_0 then we pay

$$X_0 = \sum_{r=0}^{N} \psi(S_0(1 + a)^r (1 - a)^{N-r}) \binom{N}{r} 2^{-N}.$$

If the bank asks us to pay X_0 then this is a fair price. Anything above is not.

For the option $[N, \text{identity}]$, the last formula gives

$$X_0 = S_0,$$

which seems to be natural.

For the option $[N, \psi]$ with $\psi(x) = (x - K)^+$, the price is

$$X_0 = \sum_{r > r^*} (S_0(1+a)^r (1-a)^{N-r} - K) \binom{N}{r} 2^{-N},$$

where

$$r^* = \frac{\log(K/S_0) - N \log(1-a)}{\log(1+a) - \log(1-a)}.$$

This is the popular European option.

In all of the above we assumed that we put money in the bank (or borrow from the bank) without interest rate. We can easily modify everything to accomodate the situation of fixed interest rate.

Also, we can accommodate the situation where R_n takes two values, not necessarily equal in magnitude.