Dedekind's construction à la Baire

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Outline This is an attempt to give a summary of Baire's (1905) elementary work *"Théorie des Nombres Irrationnels, des limites et de la continuité"*. The ultimate goal is to give connections to Conway's (1970's) theory of surreal numbers which completes ordinals via Dedekind-like cuts, in a way analogous to the way that rationals are completed by Dedekind cuts. This classical construction is beautifully explained in Baire's work and is summarised, in slightly more modern terms, below. Ideally, we should start from nothing (the empty set $\varnothing$) but we won't: We will assume knowledge of sets, of natural ($\mathbb{N}$) and rational numbers ($\mathbb{Q}$), as well as algebraic operations on them. Briefly, $\mathbb{Q}$ is defined as a set of equivalence classes of pairs (α, β), with $\beta \neq 0$, of integers ($\mathbb{Z} := \mathbb{N} \cup \{0\} \cup (-\mathbb{N})$), under the equivalence relation $\alpha \sim \beta$ if there exists integer n such that n divides both α and β. We write α/β for the equivalence class of (α, β). In particular, we make use of the theorem that between any two rationals q_1, q_2, there are infinitely many other rationals.

Cuts A CUT in $\mathbb{Q}$ is a pair (A, B) of disjoint subsets of $\mathbb{Q}$ such that $\mathbb{Q} = A \cup B$ and such that

$$\forall a \in A \quad \forall b \in B \quad a < b.$$

We refer to A as the LEFT CUT and to B as the RIGHT CUT. Such cuts exist: for example, (I) let $q \in \mathbb{Q}$, let $A := \{r \in \mathbb{Q} : \ r \leq q\}$, $B = \mathbb{Q} - A$. Or, (II) let $A := \{r \in \mathbb{Q} : \ r < q\}$, $B = \mathbb{Q} - A$. In case (I), A has a maximum. In case (II), B has a minimum.

Existence of irrational cuts But there are cuts such that neither A has a maximum nor B a minimum. For example, let $A^* := \{q \in \mathbb{Q} : \ q^2 < 2\}$. Since there is no rational whose square is 2, we have $B^* = \{q \in \mathbb{Q} : \ q^2 > 2\}$. So A^* has no maximum, and B^* has no minimum. Each cut of this form is called irrational number.

We define the REAL NUMBERS $\mathbb{R}$ to be the set of irrational and rational numbers.

Given $q_1, q_2 \in \mathbb{Q}$, with $q_1 < q_2$, we can find an irrational number (A, B) such that $q_1 \in A$, $q_2 \in B$. To do this, consider the function $f : \mathbb{Q} \to \mathbb{Q}$, $f(x) := 2(x - q_1)/(q_2 - q_1)$, and then consider the cut $(f^{-1}(A^*), f^{-1}(B^*))$. We have $f(q_1) = 0$, $f(q_2) = 2$. Since $0^2 < 2 < 2^2$, we have $f(q_1) = 0 \in A^*$, $f(q_2) = 2 \in B^*$. So $q_1 \in f^{-1}(A^*)$, $q_2 \in f^{-1}(B^*)$. So there are plenty of irrational numbers.

Order Now we define an order on $\mathbb{R}$. First, to compare two rational numbers, we use the standard comparison. Second, if $\lambda = (A, B)$ is irrational, we say, *by definition*, that $a < \lambda < b$, for all $a \in A, b \in B$. So we can compare a rational with an irrational.

To compare two distinct irrationals $\lambda = (A, B)$, $\lambda = (A', B')$ we first observe that either A is a proper subset of A' or vice versa. Indeed, since the two irrationals are distinct, so are the sets A, A'. Without loss of generality, assume $A' \setminus A \neq \varnothing$. Hence

$$\exists q \in A' \setminus A = A' \cap B = B \setminus B'.$$

Then A must be a proper subset of A'. Because, if this is not the case, then

$$\exists r \in A \setminus A' = A \cap B' = B' \setminus B.$$

From the above we see that $q \in A'$, $r \in B'$, so $q < r$; but, also, $q \in B$, $r \in A$, so $r < q$, and this is a contradiction.

We can now *define* $\lambda < \lambda'$ to mean A is a proper subset of A' (equivalently, B' is a proper subset of B).

The argument above actually shows more: between any two distinct irrationals there is a rational number.

So any two distinct real numbers can be compared. Indeed, if they are both rationals, we can do it. If one is rational and the other irrational then the rational either belongs to the left or the right cut, so it is, by definition, either smaller or larger than the irrational. And λ, λ' are both irrational, then their left cuts are ordered (one is a subset of the other), so either $\lambda < \lambda'$ or $\lambda' < \lambda$.

To see that this is a total order, we check transitivity. Transitivity is obvious on rationals. If we have three irrationals such that $\lambda < \mu$ and $\mu < \nu$, then the left cut of λ is a proper subset of the left cut of μ which is a proper subset of the left cut of ν, and so $\lambda < \nu$. If $\lambda < q$, $q < \mu$, where q is rational, then q belongs to the right cut of λ and so every member of the left cut of λ is smaller than q; but since q also belongs to the left cut of μ, every rational smaller than q belongs to the left cut of μ, and so every member of the left cut of λ belongs to the left cut of μ. Hence $\lambda < \mu$. If $q < \lambda$, $\lambda < r$ then $q < r$ because q belongs to the left cut of λ, while r belongs to the right cut.

Hence $<$ is a total order on $\mathbb{R}$. We also define $\lambda \leq \mu$ to mean $\lambda < \mu$ or $\lambda = \mu$.

Define POSITIVE REAL NUMBERS by $\mathbb{R}_{++} = \{x \in \mathbb{R} : \ 0 < x\}$, and the negative by $\mathbb{R}_{--} = \{x \in \mathbb{R} : \ x < 0\}$. To every $q \in \mathbb{Q}$ there corresponds a negative $-q$. So to every irrational $\lambda = (A, B)$ there corresponds a negative $-\lambda := (-B, -A)$, where $-A = \{-a : \ a \in A\}$, $-B = \{-b : \ b \in B\}$. We can see that $-(-\lambda) = \lambda$, and We can see that $\lambda = \mu$ implies $-\lambda = -\mu$. Also, if $\lambda < \mu$ then $-\mu < -\lambda$. Given λ, either λ or $-\lambda$ is positive. We define $|\lambda|$ to be the positive of the two. By definition, $|0| = 0$. Hence, $|\lambda| = |-\lambda|$ for all $\lambda \in \mathbb{R}$.

Completeness If E is a nonempty subset of $\mathbb{R}$, we let

$$B_E := \{q \in \mathbb{Q} : \ \forall x \in E \ q > x\}.$$

If B_E is nonempty we say that E is UPPER BOUNDED. Assume that this is the case and define the SUPREMUM of E by

$$\sup E := (A_E, B_E),$$

where $A_E := \mathbb{Q} - B_E$. We claim that

(i) For all $x \in E$, $x \leq \sup E$.

(ii) For all $x \in E$ with $x < \sup E$, there exists $y \in E$ such that $x < y < \sup E$.

Indeed, suppose $x = (A, B) \in E$, but that $x > \sup E$. Then B is a proper subset of B_E. So there is $q \in B_E$, but $q \notin B$, i.e. $q \in A$. Hence $q > x$, and also $q < x$, which is a contradiction.

To show (ii), suppose that $x = (A, B) < \sup_E = (A_E, B_E)$. Then there is a rational q between x and $\sup E$, that is, $q \in A_E$, and so $q < \sup E$.

If define the set

$$U_E := \{y \in \mathbb{R} : \ \forall x \in E \ y \geq x\}$$

of upper bounds to E, then (i) says that $\sup E \in U_E$, i.e. $\sup E = \min U_E$. This is the COMPLETENESS PROPERTY of $\mathbb{R}$.

Similarly, we define the INFIMUM $\inf E = (A'_E, B'_E)$ by letting $A'_E = \{q \in \mathbb{Q} : \forall x \in E\ q < x\}$, assuming that the latter is nonempty, i.e. that, by definition, the set E is LOWER BOUNDED. We have similar properties for $\inf E$.

So far, we have established that $\mathbb{R}$ is a totally ordered set with the least upper bound (and greatest lower bound) property.

If E is not upper bounded we let $\sup E := +\infty$. If E is not lower bounded we let $\inf E := -\infty$.

Note that if $E \subset E'$ then $B_E \subset B_{E'}$, and so $\sup E \leq \sup E'$. Similarly, $\inf E \geq \inf E'$.

Limits A nondecreasing sequence of real numbers is such that

$$x_1 \leq x_2 \leq \cdots$$

Its limit is defined to be equal to $M = \sup x_n$. If the sequence is not upper bounded then $M = +\infty$. Otherwise, according to what proved earlier, for all $x < M$ there is an element x_n of the sequence such that $x < x_n \leq M$. Since the sequence is nondecreasing, we also have $x < x_k \leq M$ for all $k \geq n$.

Similarly, we define the limit of nonincreasing sequence

$$x_1 \geq x_2 \geq \cdots$$

and have similar remarks.

For an arbitrary sequence $x_1, x_2, \ldots$, we let

$$\begin{aligned} M_p &= \sup\{x_p, x_{p+1}, \ldots\} \\ m_p &= \inf\{x_p, x_{p+1}, \ldots\}. \end{aligned}$$

Hence

$$\begin{aligned} M_1 &\geq M_2 \geq \cdots \\ m_1 &\leq m_2 \leq \cdots \end{aligned}$$

We define the UPPER and LOWER LIMIT by, respectively,

$$\begin{aligned} M &= \overline{\lim}\, x_n := \inf M_p \\ m &= \underline{\lim}\, x_n := \sup m_p \end{aligned}$$

If $b > M$ then, eventually, all terms of the sequence are below b. If $a < m$ then, eventually, all terms of the sequence are above a. Notice that $m \leq M$. Because, otherwise, there is $M < a < m$ and so, eventually, all terms of the sequence would simultaneously be below and above a, which is impossible.

If $a < M$ then, eventually, all M_p are above a. But $M_p > a$ implies that $x_n > a$ for some $n > p$. Similarly, if $m < b$ then for all p there is $n > p$ such that $x_n < b$.

We see that the converse is also true, namely we have: $M = \overline{\lim}\, x_n$ *if and only if, for all* $a < M < b$ *we have that (i) infinitely many of the terms of the sequence are above* a *and (ii) eventually all the terms of the sequence are below* b.

Similarly, $m = \underline{\lim}\, x_n$ *if and only if, for all* $a < m < b$ *we have that (i) infinitely many of the terms of the sequence are below* b *and (ii) eventually all the terms of the sequence are above* a.

If $M = m$ we define the LIMIT of x_n by $\lim x_n = M = m$. We also use $x_n \to M$ to denote the same thing. If M is not $\pm\infty$, from the above we have that if $a < M < b$, eventually all terms of the sequence are between a and b. This is a characterising property.

If $\lim x_n = M$ then $\lim(-x_n) = -M$ and $\lim |x_n| = |M|$.

Rational approximation Let $\lambda = (A, B) \in \mathbb{R}$, $\alpha \in \mathbb{Q}, \alpha > 0$. Fix $a \in A$ consider integers p with $p < a/\alpha$. Fix $b \in B$ consider integers q with $q < b/\alpha$.

$$p\alpha < a < \lambda < b < q\alpha.$$

Since there are finitely many integers n with $p \le n \le q$, we let n to be the largest n such that $n\alpha \le \lambda$. Then

$$n\alpha \le \lambda < (n+1)\alpha.$$

In other words, given $\lambda \in \mathbb{R}$ and α a positive rational there is a unique $n \in \mathbb{Z}$ such that the above holds. We define

$$n\alpha =: \lfloor \lambda \rfloor_\alpha, \quad (n+1)\alpha =: \lceil \lambda \rceil_\alpha.$$

We may refer to the pair $\lfloor \lambda \rfloor_\alpha$, $\lceil \lambda \rceil_\alpha$ as the BEST α-RATIONAL APPROXIMATION to the number λ. If $\alpha > \beta > 0$ are rationals then

$$\lfloor \lambda \rfloor_\alpha < \lfloor \lambda \rfloor_\beta \le \lambda < \lceil \lambda \rceil_\beta < \lceil \lambda \rceil_\alpha.$$

By decreasing α we can obtain a better approximation. Consider then $\alpha_1, \alpha_2, \ldots$, such that $\alpha_m/\alpha_{m+1} = k_m \in \mathbb{N}$, $k_m \ge 2$. Then $\lim \alpha_m = 0$ and

$$\lfloor \lambda \rfloor_{\alpha_1} < \lfloor \lambda \rfloor_{\alpha_2} < \cdots \le \lambda < \cdots < \lceil \lambda \rceil_{\alpha_2} < \lceil \lambda \rceil_{\alpha_1}$$

Let $\mu = \sup \lfloor \lambda \rfloor_{\alpha_m}$, $\nu = \inf \lceil \lambda \rceil_{\alpha_m}$. Then $\mu \le \lambda \le \nu$, but it is impossible to have $\mu < \nu$ because, if it were so, we would be able to find $a, b \in \mathbb{Q}$, such that $\mu < a < b < \nu$. Then, for all m, $\lfloor \lambda \rfloor_{\alpha_m} < a < b < \lceil \lambda \rceil_{\alpha_m}$, so $\lceil \lambda \rceil_{\alpha_m} - \lfloor \lambda \rfloor_{\alpha_m} > b - a$. Since $\lim \alpha_m = 0$, by taking m large we would have $\alpha_m < b - a$, i.e. $\alpha_m < \lceil \lambda \rceil_{\alpha_m} - \lfloor \lambda \rfloor_{\alpha_m}$, which is impossible since the latter difference equals α_m.

In particular, we have proved that, for all $x \in \mathbb{R}$ and all $\varepsilon > 0$, we can find $a, b \in \mathbb{Q}$ such that $a < x < b$ and $b - a < \varepsilon$.

Subtraction We define the first algebraic operation, namely SUBTRACTION. Let $x < y$ be reals, not both rational. Define

$$y - x := \sup\{b - a : \ x \le a < b \le y, \ a, b \in \mathbb{Q}\}.$$

The set on the right is upper bounded because there are rationals $\alpha < x$, $\beta > y$ and for any rationals $a < b$ between x and y we have $b - a < \beta - \alpha$. Also, if x, y are rational, then the definition reduces to the usual difference of rationals, because we can take $a = x$, $b = y$. We also let $x - y = -(y - x)$, and if $x = y$ we let $x - y = y - x = 0$.

Note that if $x' \le x \le y \le y'$ then $y - x \le y' - x'$ because the first set is contained in the second.

Having defined difference we can talk more intelligently about limits

Limit theorems

Theorem I. A sequence u_n converges to a finite limit if and only if for all $\varepsilon > 0$ there is an integer p such that for all integers $m, n > p$ we have $|u_m - u_n| < \varepsilon$.

To prove this, we note that, if u_n is a sequence, there are three mutually disjoint cases: (i) the sequence has a finite limit, (ii) the sequence has an infinite ($\pm\infty$) limit, (iii) the sequence has no limit.

In case (i), let λ be the limit. By rational approximation, if $\varepsilon > 0$, we can find $\alpha, \beta \in \mathbb{Q}$, $\beta - \alpha < \varepsilon$, $\alpha < x < \beta$. By the definition of λ, we can find $p \in \mathbb{N}$ such that

$$\alpha < u_n < \beta,$$

for all $n > p$. So if $\mu, \nu > p$, since both u_μ, u_ν are between α and β we certainly have $|u_\mu - u_\nu| \leq \beta - \alpha < \varepsilon$.

In case (ii), suppose $+\infty$ is the limit. Then for all A there is $p \in \mathbb{N}$ such that $u_n > A$ if $n > p$. Suppose that $A \in \mathbb{Q}$, $A > 0$. Fix $\mu \in \mathbb{N}$. Take $B \in \mathbb{Q}$, with $u_m < B$. Then $B + A > B$. And we can find $\nu \in \mathbb{N}$ such that $u_\nu > B + A$. Thus, no matter what the rational A is, we can find $\mu, \nu \in \mathbb{N}$ such that

$$u_\mu < B < B + A < u_\nu,$$

whence $|u_\mu - u_\nu| \geq A$ (from the definition of the difference).

In case (iii), we have $-\infty \leq m = \underline{\lim}\, u_n < \overline{\lim}\, u_n = M \leq +\infty$. Choose rationals α, β, such that $m < \alpha < \beta < M$. Then for all $p \in \mathbb{N}$ there exists $\mu, \nu > p$ such that

$$u_\mu < \alpha < \beta < u_\nu.$$

This implies that $|u_\mu - u_\nu| \geq \beta - \alpha$.

The theorem is actually proved by exhaustion of all cases. $\square$

Theorem II. If u_n, v_n are two sequence and if λ is the limit of the first then the second has limit λ if and only if $|u_n - v_n|$ has limit 0.

Suppose first that $v_n \to \lambda$. Then, for all $\varepsilon > 0$ we can find rationals α, β, $\beta - \alpha < \varepsilon$, $\alpha < \lambda < \beta$, such that $\alpha < u_n, v_n < \beta$, for all $n > p$, whence $|u_n - v_n| \leq \beta - \alpha < \varepsilon$. Hence $|u_n - v_n|$ converges to 0.

Suppose next that $|u_n - v_n|$ converges to 0. Suppose that v_n does not converge to λ. Let $M = \overline{\lim}\, v_n$, $m = \underline{\lim}\, v_n$. Then $\lambda < M$ or $\lambda > m$. Suppose, for example, $\lambda < M$. Pick rational α, β such that $\lambda < \alpha < \beta < M$. Eventually, all terms of the first sequence are below α, while infinitely many of the terms of the second are above β. So $|u_n - v_n| \geq \beta - \alpha$ for infinitely many n and this is impossible. Similarly, if $\lambda > m$, we can also obtain a contradiction. $\square$

Theorem III. If u_n converges to λ and v_n to μ then $u_n - v_n$ converges to $\lambda - \mu$.

We only have to deal with the case $\lambda < \mu$. For any $\varepsilon > 0$, chose rationals a, b, c, d, $b - a < \varepsilon$, $d - c < \varepsilon$, $\alpha < \lambda < b < c < \mu < d$, and deduce that, eventually, $\alpha < u_n < b < c < v_n < d$. Hence $c - b \leq \mu - \lambda \leq d - a$, and $c - b \leq v_n - u_n \leq d - a$. And so $|(v_n - u_n) - (\mu - \lambda)| \leq (d - a) - (c - b) = (d - c) + (b - a) < 2\varepsilon$. $\square$

Continuity Consider a function $f(x^1, \ldots, x^d)$ of d real variables. We say that it is CONTINUOUS at $(a^1, \ldots, a^d)$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x^1, \ldots, x^d) - f(a^1, \ldots, a^d)| < \varepsilon$$

whenever $(x^1, \ldots, x^d)$ satisfy

$$|x^1 - a^1| < \delta, \ldots, |x^d - a^d| < \delta.$$

This is equivalent to the following statement:

$$f(x_n^1, \ldots, x_n^d) \to (a^1, \ldots, a^d)$$

whenever $x_n^1, \ldots, x_n^d$ are sequences such that

$$x_n^1 \to a^1, \ldots x_n^d \to a^d.$$

Indeed, if f is continuous at $(a^1, \ldots, a^d)$ then for arbitrary ε, let δ be as in the definition, and consider sequences $x_n^1, \ldots, x_n^d$ converging to $a^1, \ldots, a^d$, respectively. Then, eventually, $|x_n^1 - a^1| < \delta, \ldots, |x_n^d - a^d| < \delta$. By definition of continuity again, this implies that, eventually, $|f(x^1, \ldots, x^d) - f(a^1, \ldots, a^d)| < \varepsilon$, which means that $f(x_n^1, \ldots, x_n^d) \to (a^1, \ldots, a^d)$. Conversely, if f is not continuous at $(a^1, \ldots, a^d)$ then *there exists* an $\varepsilon > 0$, such that, no matter what $\delta > 0$ is, we can find $x^1, \ldots, x^d$ such that

$$|x^1 - a^1| < \delta, \ldots, |x^d - a^d| < \delta, \text{ but } |f(x^1, \ldots, x^d) - f(a^1, \ldots, a^d)| > \varepsilon.$$

So, for example, if $\delta = 1/n$, let $x_n^1, \ldots, x_n^d$ be the numbers satisfying the conditions above, namely,

$$|x_n^1 - a^1| < 1/n, \ldots, |x_n^d - a^d| < 1/n, \text{ but } |f(x_n^1, \ldots, x_n^d) - f(a^1, \ldots, a^d)| > \varepsilon,$$

no matter what $n \in \mathbb{N}$ is. Hence

$$x_n^1 \to a^1, \ldots, x_n^d \to a^d \text{ but } f(x_n^1, \ldots, x_n^d) \not\to f(a^1, \ldots, a^d).$$

Consider now a CLOSED RATIONAL RECTANGLE, i.e.

$$C = \{(x^1, \ldots, x^d) \in \mathbb{Q}^d : \ a^1 \le x^1 \le b^1, \ldots, a^d \le x^1 \le b^d\}$$

We say that $f : C \to \mathbb{R}$ is UNIFORMLY CONTINUOUS if any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x^1, \ldots, x^d) - f(y^1, \ldots, y^d)| < \varepsilon$$

whenever $(x^1, \ldots, x^d) \in C$, $(y^1, \ldots, y^d) \in C$ satisfy

$$|x^1 - y^1| < \delta, \ldots, |x^d - y^d| < \delta.$$

Theorem (extension principle): Suppose that $f(x^1, \ldots, x^d)$ is a function of d rational arguments such that it is uniformly continuous on every closed rational rectangle C. Then there exists a unique continuous function $F(X^1, \ldots, X^d)$ of d real arguments, which coincides with f whenever its arguments are rational.

Let $X_0 = (X_0^1, \ldots, X_0^d)$ be d real numbers. We will define F at X_0. First we observe that is a sequence $x_n = (x_n^1, \ldots, x_n^d)$ of rational numbers, such that

$$x_n \to X_0, \quad \text{i.e. } x_n^1 \to X_0^1, \ldots, x_n^d \to X_0^d.$$

(For example, we may take $x_n^i = \lceil X_0^i \rceil_{1/n}$.) We claim that $f(x_n)$ has a limit. Indeed, we first choose a closed rectangle C containing X_0 and all the x_n. Then we use the uniform continuity of f on C, ensuring, for every $\varepsilon > 0$, the existence of a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $|x^1 - y^1| < \delta, \ldots, |x^d - y^d| < \delta$. Since $x_n^1, \ldots, x_n^d$ all converge, we have that there is integer p such that, for all $m, n > p$, $|x_n^1 - x_m^1| < \delta, \ldots, |x_n^d - x_n^d| < \delta$. This implies that $|f(x_n) - f(x_m)| < \varepsilon$ for all $m, n > p$, which implies that $f(x_n)$ has a (finite) limit, say λ.

If we choose another sequence, say $y_n = (y_n^1, \ldots, y_n^d)$ such that $y_n \to X_0$, we also have that $f(y_n)$ has a limit. We show that this limit is λ. Indeed, since both $x_n \to X_0$ and $y_n \to X_0$, we have that, eventually,

$$|x_n^1 - y_n^1| < \delta, \ldots, |x_n^d - y_n^d| < \delta.$$

By uniform continuity, eventually,

$$|f(x_n) - f(y_n)| < \varepsilon.$$

This means that the limit of $f(x_n)$ is the same as the limit of $f(y_n)$.

We therefore *define*

$$F(X_0) = \lambda.$$

If $X_0 = (X_0^1, \ldots, X_0^d)$ is rational (i.e. all the X_0^i are rational) we claim that $F(X_0) = f(X_0)$. Indeed, f is continuous at X_0. Therefore $f(x_n) \to f(X_0)$. So $f(X_0) = \lambda = F(X_0)$. Hence F is an extension of f, and so F has been defined.

We show that F is continuous. Fix ε and let δ be chosen by the uniform continuity of f. Let $X_0 = (X_0^1, \ldots, X_0^d)$ be d real numbers and $Y_0 = (Y_0^1, \ldots, Y_0^d)$ be d real numbers such that $|X_0^1 - Y_0^1| < \delta, \ldots |X_0^d - Y_0^d| < \delta$. Pick $x_n \to X_0$, $y_n \to Y_0$. Then, eventually,

$$|x_n^1 - y_n^1| < \delta, \ldots, |x_n^d - y_n^d| < \delta.$$

This implies that, eventually,

$$|f(x_n) - f(y_n)| < \varepsilon.$$

Since $f(x_n) \to F(X_0)$, $f(y_n) \to F(Y_0)$, we conclude that

$$|F(X_0) - F(Y_0)| < \varepsilon.$$

□

Algebraic operations Let x, y be rationals. We know what $x + y$ is. We show that the function $f(x, y) = x + y$ is uniformly continuous. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/2$. Suppose $|x - x'| < \delta$, $|y - y'| < \delta$, Then $|(x + y) - (x' + y')| = |(x - x') + (y - y')| \le |x - x'| + |y - y'| < 2\delta = \varepsilon$. We can then *define* $X + Y = \lim(x_n + y_n)$, where x_n, y_n are rational, and $x_n \to X$, $y_n \to Y$.

We also know what xy is, when x, y are rational. Let $A > 0$ be rational, and let C be all rational numbers x, y such that $|x| < A$, $|y| < A$. We show that $f(x, y) = xy$ is uniformly continuous on C. Fix $\varepsilon > 0$ and let $\delta = \varepsilon/2A$. Suppose $|x - x'| < \delta$, $|y - y'| < \delta$. Then

$$\begin{aligned} |xy - x'y'| &= |(x - x')y + (y - y')x| \\ &\le |x - x'|\ |y| + |y - y'|\ |x| < 2\delta A = \varepsilon. \end{aligned}$$

Hence we can *define* $XY = \lim(x_n y_n)$, where x_n, y_n are rational, and $x_n \to X$, $y_n \to Y$.

We also know what x/y is when x, y are rational, $y \neq 0$. Let $0 < a < A$ be rational and let C be all rational x, y such that $|x| < A$, $a < |y| < A$. Fix $\varepsilon > 0$ and let $\delta = \varepsilon a^2/2A$. Suppose $|x - x'| < \delta$, $|y - y'| < \delta$. Then

$$\begin{aligned}\left|\frac{x}{y} - \frac{x'}{y'}\right| &= \frac{|(x-x')y - (y-y')x|}{|y|\,|y'|} \\ &\leq \frac{|x-x'|\,|y| + |y-y'|\,|x|}{|y|\,|y'|} < \frac{2\delta A}{a^2} = \varepsilon.\end{aligned}$$

We know that Q is a field. Therefore, the following are true for all $x, y, z \in \mathbb{Q}$:

$$\begin{aligned} x + y &= y + x \\ (x+y) + z &= x + (y+z) \\ x + 0 &= x \\ x - x &= 0 \\ x + (-y) &= x - y \\ xy &= yx \\ x(yz) &= (xy)z \\ x1 &= x \\ x(1/x) &= 1, \quad x \neq 0 \\ x(1/y) &= x/y, \quad y \neq 0 \\ (x+y)z &= xz + yz \\ x0 &= 0. \end{aligned}$$

By the extension theorem, the same properties are true for all $x, y, z \in \mathbb{R}$.

Archimedean property *Theorem: Let $0 < x < y$ be real numbers. There exists an integer n such that $nx > y$.*

Pick a rational number α between 0 and x. Then we know (see rational approximation) that there is an integer n such that $n\alpha > y$. Hence $nx > y$. □

Summary The set of real numbers has been proved to have the following properties:
1. It is totally ordered.
2. It is complete.
3. It has the Archimedean property.
4. It is a field.

Theorem: Any two fields F, G satisfying 1+2+3 are isomorphic, i.e. there is a bijection $\varphi : F \to G$ that preserves addition and multiplication.

Some (counter)examples: $\mathbb{Z}$ satisfies 1+2+3. $\mathbb{Q}$ satisfies 1+3+4. $\mathbb{C}$ satisfies 2+4. Ordinals satisfy 1+2. Surreal numbers satisfy 1+2+4. (The last two statements are not, strictly speaking, mathematically correct because neither the ordinals nor the surreal numbers form sets.)

REFERENCES

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