4. Problem session

Exercise 4.5.

b) Prove that the tangent bundle TM of M is defined as an n dimensional bundle over all of M (precisely the same definition as before - with our extended notion of what a smooth chart is) and that canonically $T_x\partial M$ is a sub-bundle of $TM_{|\partial M}$.

Proof. The following uses some of the conclusions from part a). Let

$$\psi\colon\ H^k \overset{\circ}{\supset} U \cong U' \overset{\circ}{\subset} M$$

be any smooth chart. This defines a local trivialization of TM by

$$\Psi\colon\ TM_{|U'} \to U' \times \mathbb{R}^k$$

by

$$\Psi(x,v) = (x, D_x\psi^{-1}(v)).$$

Indeed, the differentials $D_x\psi^{-1} = D_x(\psi^{-1})$ are invertible also at boundary points $x \in \partial M$. The transition maps between two such trivializations Ψ_1 and Ψ_2 given two charts $\psi_1\colon\ U_1 \cong U_1' \overset{\circ}{\subset} M$ and $\psi_2\colon\ U_2 \cong U_2' \overset{\circ}{\subset} M$ are

$$(\Psi_2 \circ \Psi_1^{-1})(x,v) = (x, (D_x\psi_2^{-1})(D_x\psi_1^{-1})^{-1}(v)), \qquad x \in U_1' \cap U_2', v \in \mathbb{R}^k$$

and satisfies the cocycle condition (by chain rule) - hence defines a vector bundle. Note that the equivalence relation given by these transition maps is fiber-wise the same as the one we originally used (for a manifold without boundary) to define the fiber-wise tangent spaces T_xM.

Going back to the one chart $\psi\colon\ H^k \overset{\circ}{\supset} U \cong \mathrm{U}' \overset{\circ}{\subset} M$ we can also restrict to the boundary $V = U \cap \partial H^k$ and get a chart on ∂M. We denote $V' = U' \cap \partial M = \psi(V)$. Note that we may restrict TM to V' and thus we have a bundle $TM_{|V'}$ which is one dimensionen higher than the manifold it is defined on. As above we have a trivialization

$$\Phi\colon\ T(\partial M)_{|V'} \to V' \times \mathbb{R}^{k-1}$$

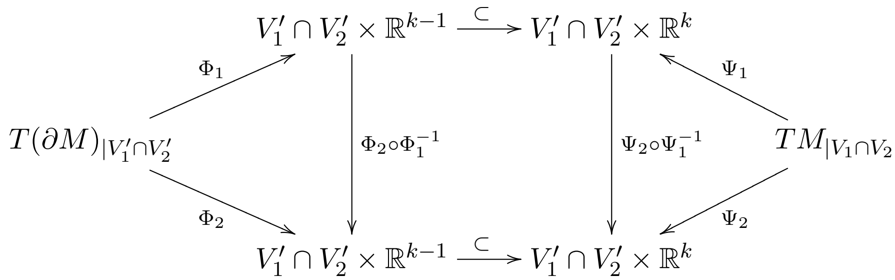
by

$$\Phi(x,v) = (x, D_x(\psi^{-1}_{|V'})(v))$$

Using the standard inclusion $\mathbb{R}^{k-1} \subset \mathbb{R}^k$ (which is [the differential of] the inclusion of the boundary of H^k into $\mathbb{R}^k$) we locally define a sub bundle:

$$\begin{array}{ccc} T(\partial M)_{|V'} & \xrightarrow{\quad\Phi\quad} & V' \times \mathbb{R}^{k-1} \\ & & \downarrow \\ TM_{|V'} & \xrightarrow{\quad\Psi_{|V'}\quad} & V' \times \mathbb{R}^k \end{array}$$

This does not depend on the choice of chart. Indeed, for two charts as above - also with notation $V_i' = \partial M \cap U_i'$ and Φ_i for the induced trivializations of $T\partial M$ - we see that



commutes. Indeed, the middle square commutes because; for $(x, v) \in (V_1' \cap V_2') \times \mathbb{R}^{k-1}$ we have

$$\begin{aligned}(\Psi_2 \circ \Psi_1^{-1})(x, v) &= (x, (D_x \psi_2^{-1})(D_x \psi_1^{-1})^{-1}(v)) = \\ &= (x, (D_{\psi_1^{-1}(x)}(\psi_2^{-1} \circ \psi_1))(v)) = \\ &= (x, (D_{\psi_{1|V_1}^{-1}(x)}(\psi_{2|V_2}^{-1} \circ \psi_{1|V_1}))(v)) = \\ &= (x, (D_x \psi_{2|V_2}^{-1})(D_x \psi_{1|V_1}^{-1})^{-1}(v)) = (\Phi_2 \circ \Phi_1^{-1})\end{aligned}$$

Here we are using that the restriction of the diffeomorphism

$$\psi_2^{-1} \circ \psi_1 \colon\ R^k \mathring{\supset} U_1 \to U_2 \mathring{\subset} \mathbb{R}^k$$

in both source and target to the diffeomorphism

$$\psi_{2|V_2}^{-1} \circ \psi_{1|V_1} \colon\ R^{k-1} \mathring{\supset} V_1 \to V_2 \mathring{\subset} \mathbb{R}^{k-1}$$

has differential equal to the similar restriction of the differential. This simply relies on the fact that the diffeomorphism takes $\mathbb{R}^{k-1}$ to $\mathbb{R}^{k-1}$. $\square$