BGG-resolution for α-stratified modules over simply-laced finite-dimensional Lie algebras

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Abstract

We construct the strong BGG-resolution for irreducible α-stratified modules over finite-dimensional simple Lie algebras with simply-laced diagrams.

1 Introduction

This paper is a sequel of [6] where the submodule structure of α-stratified (i.e. torsion free with respect to the subalgebra corresponding to a root α) generalized Verma modules was studied. The results obtained there generalize the classical theorem of Bernstein-Gelfand-Gelfand on Verma modules inclusions. The crucial role in the study is played by the generalized Weyl group W_α that acts on the space of parameters of generalized Verma modules.

Let $\mathfrak{G}$ be a simple finite-dimensional Lie algebra over the complex with a simply-laced Coxeter-Dynkin diagram (i.e. there is no multiple arrows). In the present paper for any such algebra we construct a strong BGG-resolution for α-stratified irreducible modules in the spirit of [1,10].

The structure of the paper is the following. In the section 2 we collect the notation and preliminary results. A weak generalized BGG-resolution is costructed in section 3. Here we follow closely [1]. Section 4 contains an extension lemma for α-stratified modules which generalizes a well-known result of Rocha-Caridi for Verma modules [10]. Our proof is analogous to the one of Humphreys for Verma modules [8]. In section 5 we study the maximal submodule of the generalized Verma module and construct a strong generalized BGG-resolution for α-stratified irreducible modules in section 6. Finally, in section 7 we give a character formulae for certain α-stratified irreducible modules.

2 Notation and preliminary results

Let $\mathbb{C}$ denotes the complex numbers, $\mathbb{Z}$ denotes all integers, $\mathbb{N}$ denotes all positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Let π be a basis of Δ containing α, $\Delta_\pm = \Delta_\pm(\pi)$ be the set of positive (negative) roots with respect to π. For any $S \subset \pi$ let $\Delta_\pm(S)$ be a closed subset in $\Delta_\pm$ generated by S.

Also let $\rho = \frac{1}{2}\sum_{\gamma\in\Delta_+}\gamma$. For λ, $\mu \in \mathfrak{H}^*$ we will say that $\lambda \geq \mu$ if $\lambda - \mu = \sum_{\beta\in\pi} k_\beta \beta$, $k_\beta \in \mathbb{Z}_+$. Further $(\cdot,\cdot)$ will denote the standard form on $\mathfrak{H}^*$. If $\beta \in \Delta_+$ then $s_\beta \in W$ will denote a corresponding reflection in $\mathfrak{H}^*$: $s_\beta(\lambda) = \lambda - \frac{2(\lambda,\beta)}{(\beta,\beta)}\beta$.

Fix a basis $\{H_\beta, \beta \in \pi\}$ of $\mathfrak{H}$ normalized by the condition $\beta(H_\beta) = 2$ and a non-zero element X_γ in each subspace $\mathfrak{G}_\gamma$, $\gamma \in \Delta$ such that $[X_\beta, X_{-\beta}] = H_\beta, \beta \in \pi$.

Denote $\mathfrak{N}_\pm = \sum_{\gamma\in\Delta_+}\mathfrak{G}_{\pm\gamma}$, $\mathfrak{N}^\alpha_\pm = \sum_{\gamma\in\Delta_+\setminus\{\alpha\}}\mathfrak{G}_{\pm\gamma}$, $\mathfrak{H}^\alpha = \{h \in \mathfrak{H} | \alpha(h) = 0\}$. Then we have

$$\mathfrak{G} = \mathfrak{N}_- \oplus \mathfrak{H} \oplus \mathfrak{N}_+ = \mathfrak{G}^\alpha \oplus \mathfrak{N}^\alpha_- \oplus \mathfrak{H}^\alpha \oplus \mathfrak{N}^\alpha_+$$

Where $\mathfrak{G}^\alpha$ is generated by $\mathfrak{G}_{\pm\alpha}$. Also let $\mathfrak{H}_\alpha = \mathfrak{G}^\alpha \cap \mathfrak{H}$ and thus $\mathfrak{G}^\alpha = \mathfrak{G}_\alpha \oplus \mathfrak{H}_\alpha \oplus \mathfrak{G}_{-\alpha}$.

For $m \in \mathbb{Z}_+$ denote by $U(\mathfrak{G})^{(m)}$ the subspace in $U(\mathfrak{G})$ spanned by the elements of degree m (with respect to the fixed PBW-basis above).

For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ the universal enveloping algebra of $\mathfrak{A}$ and by $Z(\mathfrak{A})$ the centre of $U(\mathfrak{A})$.

Consider a linear space $\Omega = \mathfrak{H}^* \times \mathbb{C}$. For (λ, p) and (μ, q) in Ω we say that $(\lambda, p) > (\mu, q)$ if $\lambda - \mu = \sum_{\beta\in\pi\setminus\{\alpha\}} n_\beta\beta$, $n_\beta \in \mathbb{Z}_+$ and $\lambda \neq \mu$.

Let $r \in \mathbb{C}$. Consider a linear space $B_r = \sum_{\beta\in\pi\setminus\{\alpha\}}\mathbb{C}\beta + r\alpha$ with a fixed point $r\alpha$, a $\mathbb{Z}$-module $\tilde{B}_r = B_r \oplus \mathbb{Z}\alpha$ and let $\Omega_r = B_r \times \mathbb{C}$, $\tilde{\Omega}_r = \tilde{B}_r \times \mathbb{C}$.

In [6] we introduced the generalized Weyl group W_α acting on the space Ω_r in the following way.

Consider a partition of π: $\pi = \pi_1 \cup \pi_2$ where $\pi_1 = \{\gamma \in \pi | \alpha + \gamma \in \Delta, \ \}, \pi_2 = \{\gamma \in \pi | \alpha + \gamma \notin \Delta\}$. For $(\lambda, p) \in \Omega$ and $\beta \in \pi_1$ denote

$$n^\pm_\beta(\lambda, p) = \frac{1}{2}(\lambda(H_\alpha + 2H_\beta) \pm p)$$

and define $(\lambda_\beta, p_\beta) \in \Omega$, where $\lambda_\beta = \lambda - n^-_\beta(\lambda, p)\beta$, $p_\beta = n^+_\beta(\lambda, p)$.

For each $\beta \in \pi$ consider $\ell_\beta \in GL(\Omega)$ such that

$$\ell_\beta(\lambda, p) = \begin{cases} (\lambda, -p), & \beta = \alpha \\ (s_\beta\lambda, p), & \beta \in \pi_2 \setminus \{\alpha\} \\ (\lambda_\beta, p_\beta), & \beta \in \pi_1. \end{cases} \qquad (*)$$

Then $W_\alpha = < \ell_\beta, \beta \in \pi >$.

It is easy to see that W_α is isomorphic to the Weyl group W. Moreover, there exists a root system $\Delta_{\alpha,r}$ in Ω_r with respect to which W_α is the Weyl group [6]. We denote by σ_β the reflection in Ω_r corresponding to a root $\beta \in \Delta_{\alpha,r}$. Also let $(\cdot,\cdot)_r$ denotes a corresponding non-degenerated form on Ω_r and $\zeta = \zeta_{\alpha,r} : \Delta \to \Delta_{\alpha,r}$ be a natural bijection.

Let pr_i, $i = 1, 2$ be a natural projection on the i-th component of Ω_r.

For a $\mathfrak{G}$-module V with a Jordan-Hölder series let $\mathcal{JH}(V)$ denotes the set of all irreducible subquotients of V. A $\mathfrak{G}$-module V is called weight if

$$V = \bigoplus_{\lambda \in \mathfrak{H}^*} V_\lambda$$

where $V_\lambda = \{v \in V | hv = \lambda(h)v \text{ for all } h \in \mathfrak{H}\}$. If $V_\lambda \neq 0$ then λ is called a weight of V. Denote by $\operatorname{supp} V$ the set of all weights of V. A weight λ is called highest weight if $V_{\lambda+\beta} = 0$ for all $\beta \in \Delta_+$. A weight $\mathfrak{G}$-module V is called α-stratified if X_α and $X_{-\alpha}$ act injectively on V.

Let V be a weight $\mathfrak{G}$-module. A non-zero element $v \in V$ is called α-primitive (with respect to $\mathfrak{G}$) if $v \in V_\lambda$ for some $\lambda \in \mathfrak{H}^*$ and $\mathfrak{N}_+^\alpha v = 0$.

It is known that $c = (H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha$ generates $Z(\mathfrak{G}^\alpha)$. Let $a, b \in \mathbb{C}$. Any such pair defines a unique indecomposable weight $\mathfrak{G}^\alpha$-module $N(a,b)$ on which $X_{-\alpha}$ acts injectively and where a is an eigenvalue of H_α and b is an eigenvalue of c. The module $N(a,b)$ has a $\mathbb{Z}$-basis $\{v_i\}$ such that $X_{-\alpha}v_i = v_{i-1}$, $H_\alpha v_i = (a+2i)v_i$ and $X_\alpha v_i = \frac{1}{4}(b - (a+2i+1)^2)v_{i+1}$.

One can easily check (see [6, Lemma 2.2]) that the module $N(a,b)$ is torsion free if and only if $b \neq (a + 2\ell + 1)^2$ for all $\ell \in \mathbb{Z}$.

Set $\Omega^s = \{(\lambda, p) \in \Omega | p \neq \pm(\lambda(H_\alpha) + 2\ell) \text{ for all } \ell \in \mathbb{Z}\}$, $\Omega_r^s = \Omega_r \cap \Omega^s$, $\tilde{\Omega}_r^s = \tilde{\Omega}_r \cap \Omega^s$. Hence, if $(\lambda, p) \in \Omega^s$ then $N((\lambda - \rho)(H_\alpha), p^2)$ is irreducible and torsion free.

Since $\mathfrak{H} = \mathfrak{H}_\alpha \oplus \mathfrak{H}^\alpha$, any element $\lambda \in \mathfrak{H}^*$ can be written uniquely as $\lambda = \lambda_\alpha + \lambda^\alpha$ where $\lambda_\alpha \in \mathfrak{H}_\alpha^*$ and $\lambda^\alpha \in (\mathfrak{H}^\alpha)^*$. Let $a, b \in \mathbb{C}$ and $\lambda \in \mathfrak{H}^*$ such that $(\lambda - \rho)(H_\alpha) = (\lambda_\alpha - \rho)(H_\alpha) = a$. Define a $\mathfrak{H}$-module structure on $N(a,b)$ by letting $hv = \lambda^\alpha(h)v$ for any $h \in \mathfrak{H}^\alpha$ and any $v \in N(a,b)$. Thus $N(a,b)$ becomes a $\mathfrak{H}^\alpha + \mathfrak{H}$-module. Moreover, we can consider $N(a,b)$ as $D = \mathfrak{H} + \mathfrak{G}^\alpha + \mathfrak{N}_+^\alpha$-module with a trivial action of $\mathfrak{N}_+^\alpha$.

The generalized Verma module associated with α, λ, b is defined as follows:

$$M_\alpha(\lambda, b) = U(\mathfrak{G}) \bigotimes_{U(D)} N(a,b).$$

Set $M(\lambda, b) = M_\alpha(\lambda, b)$.

It will be more convenient to use a slightly different parametrization of generalized Verma modules replacing $M(\lambda, b)$ by $M(\lambda, p)$ where $p^2 = b$. Thus we always have $M(\lambda, p) = M(\lambda, -p)$.

Note that module $M(\lambda, p)$ has a unique maximal submodule and it is α-stratified if and only if $(\lambda, p) \in \Omega^s$.

It follows from [3, Corollary 1.11] that module $M(\lambda, p)$ admits a central character $\theta_{(\lambda,p)} \in Z^*(\mathfrak{G})$, i.e. $zv = \theta_{(\lambda,p)}(z)v$ for any $z \in Z(\mathfrak{G})$ and $v \in M(\lambda, p)$.

Denote by $L(\lambda, p)$ the unique irreducible quotient of $M(\lambda, p)$.

Lemma 1. *$L(\lambda, p) \simeq L(\lambda + k\alpha, p)$ for all $k \in \mathbb{Z}$.*

The following order on Ω_r was introduced in [6]: Let (λ, p), $(\mu, q) \in \Omega_r$ and $\beta \in \Delta_{\alpha,r}$. We will write $(\lambda, p) \xrightarrow{\beta} (\mu, q)$ if $(\mu, q) = \sigma_\beta(\lambda, p)$ and $(\beta, (\lambda, p))_r \in \mathbb{N}$ for $\beta \neq \zeta(\alpha)$. Then

$(\mu, q) \ll (\lambda, p)$ will describe the fact that there exists a sequence $\beta_1, \beta_2, \ldots, \beta_k$ in $\Delta_{\alpha,r}$ such that $(\mu, q) \xrightarrow{\beta_1} \sigma_{\beta_1}(\mu, q) \xrightarrow{\beta_2} \ldots \sigma_{\beta_{k-1}} \ldots \sigma_{\beta_1}(\mu, q) \xrightarrow{\beta_k} (\lambda, p)$.

The main result of [6, Theorem 7.6] is the following theorem which describes the structure of α-stratified generalized Verma module with respect to the order on Ω_r.

Theorem 1. *Let (λ, p) and $(\mu, q) \in \tilde{\Omega}_r^s$. The following statements are equivalent:*

1. *$M(\mu, q) \subset M(\lambda, p)$;*
2. *$L(\mu, q) \in \mathcal{J}H(M(\lambda, p))$;*
3. *There exists $k \in \mathbb{Z}$ such that $(\mu + k\alpha, q) \ll (\lambda, p)$.*

Let

$$P^{++} = \{(\lambda, p) \in \Omega_r^s \mid w(\lambda, p) \ll (\lambda, p) \text{ for all } w \in W_\alpha\}.$$

For $\beta \in \pi$ denote by Δ_β a root subsystem of rank 2 generated by α and β.

In this paper we discuss the construction of analogues of the weak and the strong BGG-resolutions for irreducible modules $L(\lambda, p)$ with $(\lambda, p) \in P^{++}$.

3 Cohomological part of the weak BGG-resolution

Let $P = \Delta_+(\pi \setminus \{\alpha\})$ and let $\mathfrak{P}$ be a subalgebra of $\mathfrak{G}$ generated by all root subspaces

$$\mathfrak{G}_{-\beta}, \; \beta \in P.$$

An element (λ, p) will be called minimal if

$$\mathrm{pr}_1((\lambda, p) - \sigma_\beta(\lambda, p)) = \beta$$

holds for every $\beta \in \pi \setminus \{\alpha\}$. In this section we fix a minimal element (λ, p).

Consider the subalgebra $\mathfrak{P}$ as a module over a subalgebra $\mathfrak{a} = \mathfrak{N}_+^\alpha + \mathfrak{H}$ under the following action:

$$h \cdot a = [h, a] + \lambda(h)(a)$$

for any $h \in \mathfrak{H}$ and $a \in \mathfrak{P}$, and

$$b \cdot a = \begin{cases} [b, a], & [b, a] \in \mathfrak{P}; \\ 0, & [b, a] \notin \mathfrak{P}. \end{cases}$$

for all $b \in \mathfrak{N}^\alpha$ and $a \in \mathfrak{P}$. Clearly, this action can be naturally extended to the action on the external powers $\bigwedge^k \mathfrak{P}$ for all $k \in \mathbb{N}$.

Let ε be a unique eigenvalue on $M(\lambda, p)$ of a quadratic Casimir operator

$$C = H + \sum_{\alpha\in\Delta_+} X_{-\alpha} X_\alpha,$$

where H is a certain fixed element in $\mathfrak{H}$. Note that this eigenvalue is determined uniquely by (λ, p) via generalized Harish-Chandra homomorphism [5].

Define $U_\varepsilon = U(\mathfrak{G})/(C-\varepsilon)$ and consider the following $\mathfrak{G}$-modules:

$$D_k = U_\varepsilon \bigotimes_{U(\mathfrak{a})} \bigwedge^k \mathfrak{P},$$

where $k \in \mathbb{Z}_+$.

Following [1], for $k \in \mathbb{N}$ define the homomorphisms $d_k : D_k \to D_{k-1}$ as follows

$$d_k(X \otimes X_1 \wedge X_2 \wedge \cdots \wedge X_k) = \sum_{i=1}^{k} (-1)^{i+1} X X_i \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k + \sum_{1\leqslant i<j\leqslant k} (-1)^{i-j} X \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k.$$

Since $d_k \circ d_{k+1} = 0$ we immediately obtain that the sequence

$$0 \leftarrow D_0/\operatorname{Im} d_1 \overset{\eta}{\leftarrow} D_0 \overset{d_1}{\leftarrow} D_1 \overset{d_2}{\leftarrow} D_2 \overset{d_3}{\leftarrow} \dots$$

is a complex. Here η is a natural projection. We will denote this complex by $V_\alpha(\lambda, \varepsilon)$.

Theorem 2. *The complex $V_\alpha(\lambda, \varepsilon)$ is exact.*

Proof. The algebra U_ε inherites the natural gradation on $U(\mathfrak{G})$ by the degree of the monomials. Using that we can define a gradation on D_k. For $l \geq k$ let $D_k^{(l)}$ be a subspace spanned by the elements $x \otimes y$ where x is an element in U_ε of degree less or equal $l-k$ and $y \in \bigwedge^k \mathfrak{P}$. It is clear that $d_k(D_k^{(l)}) \subset D_{k-1}^{(l)}$ and thus d_k induces a homomorphism

$$d_k^{(l)} : D_k^{(l)}/D_k^{(l-1)} \to D_{k-1}^{(l)}/D_{k-1}^{(l-1)}.$$

Also set $M^{(l)} = D_0^{(l)}/\operatorname{Im} d_1^{(l)}$ and let $\eta^{(l)}$ be a corresponding induced homomorphism.

It is sufficient to show for every l the exactness of the complex

$$0 \leftarrow M^{(l)} \overset{\eta^{(l)}}{\leftarrow} \hat{D}_0^{(l)} \overset{d_1^{(l)}}{\leftarrow} \hat{D}_1^{(l)} \overset{d_2^{(l)}}{\leftarrow} \hat{D}_2^{(l)} \overset{d_3^{(l)}}{\leftarrow} \dots \qquad (1)$$

with $\hat{D}_k^{(l)} = D_k^{(l)}/D_k^{(l-1)}$.

By the PBW theorem for every $k \in \mathbb{Z}_+$ one can write:

$$D_k = \left(U(\mathfrak{N}_-) \bigotimes \bigwedge^k \mathfrak{P} \right) \bigoplus \left(\sum_{m \geq 1} X_\alpha^m U(\mathfrak{N}_-^\alpha) \bigotimes \bigwedge^k \mathfrak{P} \right)$$

and hence

$$\hat{D}_k^{(l)} \simeq \left(U(\mathfrak{N}_-)^{(l-k)} \bigotimes \bigwedge^k \mathfrak{P} \right) \bigoplus \left(\sum_{m=1}^{l-k} X_\alpha^m U(\mathfrak{N}_-^\alpha)^{(l-k-m)} \bigotimes \bigwedge^k \mathfrak{P} \right).$$

We will denote by $s_\alpha \mathfrak{N}_-$ a subalgebra generated by $\mathfrak{N}_-^\alpha$ and X_α. Let $\mathfrak{N}_-^{\mathfrak{P}}$ ($s_\alpha \mathfrak{N}_-^{\mathfrak{P}}$ resp.) be a subalgebra generated by $X_{-\beta}$, $\beta \in \Delta_+$, $\beta \notin \Delta_+(\pi \backslash \{\alpha\})$ ($\beta \in s_\alpha \Delta_+$, $\beta \notin s_a \Delta_+(\pi \backslash \{\alpha\})$ resp.) and let $S_j(\mathfrak{P})$ be a set of all homogeneous elements of degree j in the symmetric algebra of $\mathfrak{P}$. Then

$$\hat{D}_k^{(l)} \simeq \left(\sum_{j=0}^{l-k} U(\mathfrak{N}_-^{\mathfrak{P}})^{(l-j-k)} S_j(\mathfrak{P}) \bigotimes \bigwedge^k \mathfrak{P} \right) \bigoplus \left(\sum_{j=0}^{l-k} U(s_\alpha \mathfrak{N}_-^{\mathfrak{P}})^{(l-j-k)} S_j(\mathfrak{P}) \bigotimes \bigwedge^k \mathfrak{P} \right).$$

For any homogeneous element $u \in U(\mathfrak{N}_-^{\mathfrak{P}})$ ($u \in U(s_\alpha \mathfrak{N}_-^{\mathfrak{P}})$ resp.) of degree $l - j - k$ we have that $d_k^{(l)}(u S_j(\mathfrak{P}) \otimes \bigwedge^k \mathfrak{P}) \subset u S_{j+1}(\mathfrak{P}) \otimes \bigwedge^{k-1} \mathfrak{P}$. Therefore the element u generates a complex which is in fact the Koszul complex [2] and hence is exact. Using the PBW theorem we conclude that the complex (1) decomposes into a direct sum of exact complexes and therefore is exact. The theorem is proved. $\square$

For a weight $\mathfrak{G}$-module V consider a formal character

$$\operatorname{ch} V = \sum_{\mu \in \mathfrak{H}^*} (\dim V_\mu)\, e^\mu.$$

Corollary 1.

$$\operatorname{ch} D_0 / \operatorname{Im} d_1 = \sum_{i \geq 1} (-1)^{i+1} \operatorname{ch} D_i.$$

4 Extension lemma

In this section we prove an analogue of the Extension lemma ([8,10]) for α-stratified generalized Verma modules.

Recall that α-stratified generalized Verma modules are the important objects in the category $\mathcal{O}^\alpha$ wich was studied in [3,7]. This category has properties similar to those of the classical category $\mathcal{O}$. It was shown, in particular, that $\mathcal{O}^\alpha$ has enough projective objects. Let $P(\lambda, p)$ be the projective cover of $L(\lambda, p)$.

Theorem 3. *Let (λ, p), $(\mu, q) \in \Omega_r^s$. If*

$$\mathrm{Ext}_{\mathcal{O}^\alpha}(M(\mu, q), M(\lambda, p)) \neq 0$$

then $(\mu, q) \ll (\lambda, p)$.

Proof. The proof is based on the properties of the category $\mathcal{O}^\alpha$ [7] and is analogous to the proof of the extension lemma in [8]. □

Consider a subgroup $W_\alpha^+ \subset W_\alpha$ generated by all l_β, $\beta \in \pi \setminus \{\alpha\}$. Since W_α^+ is a reflection group we have a well-defined notion of the length $l(w)$ for any $w \in W_\alpha^+$.

Corollary 2. *For $(\lambda, p) \in P^{++}$ and $w_1, w_2 \in W_\alpha^+$ with $l(w_1) = l(w_2)$ holds*

$$\mathrm{Ext}_{\mathcal{O}^\alpha}(M(w_1(\lambda, p)), M(w_2(\lambda, p))) = 0.$$

5 The structure of the maximal submodule of $M(\lambda, p)$

The main result of this section is the following

Theorem 4. *The module $D_0/\operatorname{Im} d_1$ is irreducible.*

Corollary 3. *If $(\lambda, p) \in P^{++}$ and $\mathcal{N}$ is the maximal submodule of $M(\lambda, p)$ then*

$$\mathcal{N} = \sum_{\gamma \in \pi \setminus \{\alpha\}} M(\sigma_\gamma(\lambda, p)).$$

Proof. Follows immediately from theorem 2 and theorem 4. □

To prove the theorem 4 we will need several lemmas.

Let $K = \Delta_-(\pi) \setminus P$ and $K(\mathfrak{G})$ be a subalgebra generated by X_β, $\beta \in K$.

Lemma 2. *Let $(\mu, q) \in \Omega_r^s$. If $\beta \in K$ and $(\beta, \alpha) \neq 0$ then X_β acts injectively on $L(\mu, q)$.*

Proof. Suppose that there exists a non-zero $v \in L(\mu, q)$ such that $X_\beta v = 0$. Since $(\alpha, \beta) \neq 0$ then either $\alpha + \beta \notin \Delta$ or $\alpha - \beta \notin \Delta$. Thus, either $X_\beta X_\alpha v = 0$ or $X_\beta X_{-\alpha} v = 0$. Viewing α-stratified module $L(\mu, q)$ as a module over Lie algebra $< X_\beta, X_{-\beta} > \simeq sl(2, \mathbb{C})$ and using the fact that $L(\mu, q)$ is $X_{-\beta}$-finite we obtain that $L(\mu, q)$ contains irreducible finite-dimensional $sl(2, \mathbb{C})$-submodules of the same dimension and with different highest weights which is impossible. Lemma is proved. □

Lemma 3. *Let $(\mu, q) \in \Omega_r^s$ and $0 \neq v \in M(\mu, q)_{\mu - \rho}$. Then for $\beta \in K$ and $k \geq 1$ an element $X_\beta^k v$ is not α-primitive.*

Proof. If $(\alpha,\beta)\neq 0$ then the statement follows from lemma 2.

Suppose now that $(\alpha,\beta)=0$ and consider the maximal (with respect to the height of roots) $\gamma\in\Delta_+$ such that $\gamma\neq\alpha$, $\gamma+\beta\in K$ and $(\beta+\gamma,\alpha)\neq 0$. The existence of such γ is obvious.

Let k be the minimal positive integer for which $X_\beta^k v$ is α-primitive. Then

$$0 = X_\gamma X_\beta^k v = a X_{\beta+\gamma} X_\beta^{k-1} v + \dots$$

with $a\neq 0$. It follows from PBW theorem that $X_\gamma X_\beta^k v = 0$ which contradicts lemma 2. □

Let M be a $\mathfrak{G}$-module. A non-zero weight element $v\in M$ will be called quasi-primitive if there exists a submodule $N\subset M$ such that v becomes α-primitive in the quotient M/N.

Lemma 4. *Let $(\mu,q)\in\Omega_r^s$, $N\subset M(\mu,q)$, $0\neq v\in M(\mu,q)_{\mu-\rho}$ and $K(\mathfrak{G})v\cap N\neq 0$. Then $K(g)v$ contains a quasi-primitive element.*

Proof. Since module N is α-stratified and finitely generated one can choose a set of generators $w_1,\dots,w_l$ (which are not necessary α-primitive) of N such that $w_i\in U(\mathfrak{N}_-)v$ for all i. Let $0\neq v'\in K(\mathfrak{G})v\cap N$. There exists $k>0$ for which

$$X_{-\alpha}^k v' \in \sum_i U(\mathfrak{N}_-) w_i.$$

We obtain a contradiction now from the PBW theorem since $v'\in K(\mathfrak{G})v$. This completes the proof of lemma. □

Lemma 5. *Let $(\mu,q)\in\Omega_r^s$ and $0\neq v\in M(\mu,q)_{\mu-\rho}$. Then $K(\mathfrak{G})v$ has no quasi-primitive elements except $\mathbb{C}X_{-\alpha}^k v$, $k\geq 0$.*

Proof. It follows from theorem 1 that if $0\neq v'\in M(\mu,q)_\nu$, $\nu\leqslant\mu-\rho$ is α-primitive for all q then $v'\notin K(\mathfrak{G})v$. On the other hand, a direct calculation shows that for any $\tau\in\mathfrak{H}^*$ the existence of a non-zero α-primitive element in $K(\mathfrak{G})v$ of weight $\mu-\tau$ is equivalent to the system of linear equations on μ. This implies that the only α-primitive elements in $K(\mathfrak{G})v$ are $\mathbb{C}X_{-\alpha}^k v$, $k\geq 0$.

Now suppose that $v'\in(K(\mathfrak{G})v)_\nu$ is quasi-primitive and $(K(\mathfrak{G})v)_\xi$ has no quasi-primitive elements if $\xi>\tau$. Consider the minimal generating system G in $\Delta_+\setminus\{\alpha\}$ containing $\gamma\in\pi\setminus\{\alpha\}$. Then obviously $X_\gamma v'=0$ for all $\gamma\in\pi\setminus\{\alpha\}$. If $\gamma\in G\setminus\pi$ then $(\gamma,\alpha)\neq 0$. Let $\mathfrak{b}\simeq sl(2,\mathbb{C})$ be a subalgebra generated by $X_{\pm\gamma}$ and N be a $\mathfrak{b}$-module generated by v'.

Suppose that $X_\gamma v'\neq 0$. Since v' is quasi-primitive it implies that $v'\notin X_{-\gamma}N$ and thus N has a finite-dimensional subfactor. Using the fact that our module is α-stratified and the fact that $(\gamma,\alpha)\neq 0$ we easily obtain a contradiction from $sl(2)$-theory. Hence v' is α-primitive and thus belongs to $\mathbb{C}X_{-\alpha}^k v$ for some $k\geq 0$. □

Lemma 6. *Let V be a quotient of $M(\mu,q)$, $0\neq v\in M(\mu,q)_{\mu-\rho}$ and $\nu\in\mathfrak{H}^*$ be a weight of V. Then $\dim V_\nu\geq\dim(K(\mathfrak{G})v)_\nu$ where $(K(\mathfrak{G})v)_\nu=K(\mathfrak{G})v\cap M(\mu,q)_\nu$. Moreover, if $\dim V_\nu=\dim(K(\mathfrak{G})v)_\nu$ for infinitely many weights ν_i of V, where $\nu_i-\nu_j\notin\mathbb{Z}\alpha$ for all $i\neq j$, then module V is irreducible.*

Proof. Follows immediately from lemmas above. ☐

of theorem 4. Let $0 \neq v \in M(\lambda,p)_{\lambda-\rho}$. It follows from corollary 1 that $\dim(D_0/\operatorname{Im} d_1)_\nu = \dim(K(\mathfrak{G})v) \cap M(\lambda,p)_\nu$ for infinitely many weights $\nu \in \mathfrak{H}^*$ satisfying the conditions of lemma 6. Using lemma 6 we conclude that $D_0/\operatorname{Im} d_1$ is irreducible which completes the proof. ☐

6 Strong BGG-resolution

In this section we follow [1,10] to construct the strong BGG-resolution for irreducible α-stratified module $L(\lambda,p)$ with $(\lambda,p) \in P^{++}$.

Let $(\lambda,p) \in P^{++}$. For $k \geq 0$ denote

$$(W_\alpha^+)^k = \{w \in W_\alpha^+ \mid l(w) = k\}$$

and set

$$C_k = \sum_{w \in (W_\alpha^+)^k} M(w(\lambda,p)).$$

Define a map $\mathcal{D}_i : C_i \to C_{i-1}$ using the matrix $(d^i_{w_1 w_2})$, $w_1 \in (W_\alpha^+)^i$, $w_2 \in (W_\alpha^+)^{i-1}$ where $d^i_{w_1 w_2} = s(w_1,w_2)$ if $w_1 > w_2$ (with respect to Bruhat order) and zero otherwise. Here the numbers $s(w_1,w_2)$ are defined as in [1, Lemma 10.4]. Set $m = |\Delta_+(\pi \setminus \{\alpha\})|$.

Theorem 5. *Let $\eta : M(\lambda,p) \to L(\lambda,p)$ be a natural projection. Then the sequence*

$$0 \leftarrow L(\lambda,p) \overset{\eta}{\leftarrow} C_0 \overset{\mathcal{D}_1}{\leftarrow} C_1 \overset{\mathcal{D}_2}{\leftarrow} \dots \overset{\mathcal{D}_m}{\leftarrow} C_m \leftarrow 0$$

is exact.

Proof. It follows from the construction that this sequence is a complex.

To show the exactness in each term we will follow the proof of [10, Corollary 10.6].

Let $\mathcal{K}$ be a category of all weight $\mathfrak{G}$-modules having central character. Clearly every module $V \in \mathcal{K}$ has a decomposition

$$V = \sum_{\chi \in Z^*(\mathfrak{G})} V(\chi),$$

where $V(\chi)$ is a component with central character χ. Let $\theta \in Z^*(\mathfrak{G})$ be a central character of $M(\lambda,p)$ and let $F_\theta : \mathcal{K} \to \mathcal{K}$ be a functor such that $F_\theta(V) = V(\theta)$ for all $V \in \mathcal{K}$.

Obviously, there exists a minimal element $(\mu,q) \in P^{++}$ and a finite-dimensional $\mathfrak{G}$-module U such that $Y = F_\theta(L(\mu,q) \otimes U)$ contains an α-primitive element with parameters (λ,p). Moreover, the dimension of $Y_{\lambda-\rho}$ equals 1.

We will show that in fact $Y \simeq L(\lambda,p)$. Suppose that Y is not irreducible and N is some non-trivial submodule of Y. Then it follows from lemma 6 that the dimension growth of

Y/N is strictly less then the dimension growth of any irreducible module $L(\lambda', p')$ in $\mathcal{K}$. The obtained contradiction implies that $Y \simeq L(\lambda, p)$.

Let ε be an eigenvalue of C on $L(\mu, q)$. Consider an exact complex $V_\alpha(\mu, \varepsilon)$. Applying the functor $F_\theta(\,\cdot \otimes U)$ to $V_\alpha(\mu, \varepsilon)$ we obtain the following exact complex:

$$0 \leftarrow L(\lambda, p) \stackrel{\eta}{\leftarrow} B_0 \stackrel{d_1}{\leftarrow} B_1 \stackrel{d_2}{\leftarrow} B_2 \stackrel{d_3}{\leftarrow} \dots$$

where $B_i = F_\theta(D_i \otimes U)$, $i \geq 0$.

Using [1, Proposition 9.6] and theorem 3 we conclude that

$$B_i \simeq C_i, i \geq 0.$$

Following [10, Lemmas 10.2,10.5] there exists a sequence of isomorphisms $\nu^i : B_i \to C_i$ which makes the following diagram commutative:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & B_2(\lambda,p) & \stackrel{d_2}{\longrightarrow} & B_1(\lambda,p) & \stackrel{d_1}{\longrightarrow} & B_0(\lambda,p) & \stackrel{\eta}{\longrightarrow} & L(\lambda,p) & \longrightarrow & 0 \\
 & & \nu^2\big\downarrow & & \nu^1\big\downarrow & & \nu^0\big\downarrow & & 1\big\downarrow & & \\
\dots & \longrightarrow & C_2(\lambda,p) & \stackrel{\mathcal{D}_2}{\longrightarrow} & C_1(\lambda,p) & \stackrel{\mathcal{D}_1}{\longrightarrow} & C_0(\lambda,p) & \stackrel{\eta}{\longrightarrow} & L(\lambda,p) & \longrightarrow & 0.
\end{array}$$

This completes the proof of the theorem. □

7 Character formulae

In this section we use the strong BGG-resolution to obtain a character formulae for a $\mathfrak{G}$-module $L(\lambda, p)$ with $(\lambda, p) \in P^{++}$.

For $\nu \in \mathfrak{H}^*$ let

$$\mathfrak{H}_\nu = \nu + \sum_{\beta \in \pi \setminus \{\alpha\}} \mathbb{Z}\beta.$$

Set for any $\nu \in \operatorname{supp} V$

$$\operatorname{ch}^{\alpha,\nu}(V) = \sum_{\mu \in \mathfrak{H}_\nu} (\dim V_\mu)\, e^\mu.$$

Lemma 7. *Let V be an α-stratified $\mathfrak{G}$-module and $\nu \in \operatorname{supp} V$ then*

$$\operatorname{ch}(V) = \left(\sum_{i=-\infty}^{+\infty} e^{i\alpha} \right) \operatorname{ch}^{\alpha,\nu}(V).$$

Proof. Follows from the fact that $X_{\pm\alpha}$ act injectively on V. □

Let $\varphi : \mathfrak{H}^* \to \mathfrak{H}_0$ be a natural projection along the root α. Set $\Delta' = \{\varphi(\beta) \mid \beta \in \Delta_+\}$. It is easy to see (see for example [9]) that for any $(\mu, q) \in \Omega$

$$\mathrm{ch}^{\alpha,\mu-\rho}(M(\mu,q)) = e^{\mu-\rho} \prod_{\beta\in\Delta'} (1 - e^{-\beta})$$

and thus

$$\mathrm{ch}(M(\mu,q)) = e^{\mu-\rho} \prod_{\beta\in\Delta_+\setminus\{\alpha\}} (1 - e^{-\beta}) \left(\sum_{i=-\infty}^{+\infty} e^{i\alpha}\right)$$

by lemma 7.

Set $\rho' = \dfrac{1}{2}\sum_{\beta\in P} \beta$.

Theorem 6. *Let $(\lambda, p) \in P^{++}$. Then there exists an element $a(\lambda,p) \in \mathfrak{H}^*$ such that*

$$\mathrm{ch}(L(\lambda,p)) = \left(\sum_{i=-\infty}^{+\infty} e^{i\alpha}\right)\left(\prod_{\beta\in -K\setminus\{\alpha\}} (1-e^{-\beta})^{-1}\right) \times \\ \times \left(\sum_{w\in W_\alpha^+} (-1)^{l(w)} e^{w(\lambda+a(\lambda,p)+\rho')-a(\lambda,p)}\right)\left(\sum_{w\in W_\alpha^+} (-1)^{l(w)} e^{w(\rho')}\right)^{-1}$$

Proof. It follows from theorem 5, that the character $\mathrm{ch}\, L(\lambda,p)$ satisfies the following alternating formulae:

$$\mathrm{ch}\, L(\lambda,p) = \sum_{i\geq 0}(-1)^i \sum_{w\in (W_\alpha^+)^{(i)}} \mathrm{ch}\, M(w(\lambda,p)).$$

Thus using the character formulae for $M(\mu,q)$ above we obtain

$$\mathrm{ch}\, L(\lambda,p) = \left(\sum_{i=-\infty}^{+\infty} e^{i\alpha}\right)\left(\prod_{\beta\in -K\setminus\{\alpha\}} (1-e^{-\beta})^{-1}\right) \times \\ \times \sum_{i\geq 0}(-1)^i \sum_{w\in (W_\alpha^+)^{(i)}} e^{\mathrm{pr}_1(w(\lambda,p))-\rho} \prod_{\beta\in P}(1-e^{\beta})^{-1}.$$

Since the group W_α^+ is an affine reflection group in every Ω_r the result follows from the classical Weyl character formulae for finite-dimensional modules [4, Theorem 7.5.9]. $\square$

Note that the element $a(\lambda,p)$ in theorem 6 is determined uniquely by the element in Ω_r with respect to which the group W_α^+ is linear.

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