ON AN ANALOGUE OF BGG-RECIPROCITY

Xavier Gomez

Centre de Physique Theorique, CNRS, Luminy case 907, 13288 Marseille cedex 9, France, e-mail: gomez@cpt.univ-mrs.fr

Volodymyr Mazorchuk

Department of Mechanics and Mathematics, Kyiv Taras Shevchenko University, 64, Volodymyrska st., 01033, Kyiv, Ukraine, e-mail: mazor@mechmat.univ.kiev.ua,

Abstract

We prove an analogue of BGG-reciprocity for certain categories of modules associated with a parabolic subalgebra of a simple finite-dimensional Lie algebra.

1 Introduction and Set-up

The aim of this note is to present a generalization of the celebrated BGG-reciprocity principle for the category $\mathcal{O}$ associated with a triangular decomposition of a simple comlex finite-dimensional Lie algebra, $\mathfrak{G}$ ([BGG]). In particular, we cover another generalization of this result, obtained in [FKM] for a special category, $\mathcal{O}(\mathcal{P},\Lambda)$, of modules, associated with a parabolic subalgebra, $\mathcal{P}$, of $\mathfrak{G}$. The machinery worked out in [FKM] applies to the situation where a block of $\mathcal{O}(\mathcal{P},\Lambda)$ corresponds to a projectively stratified finite-dimensional algebra. Our goal in this paper is to obtain a more general result than [FKM, Theorem 4] (see also [ADL, Theorem 2.5]). The situation we consider can not be described by projectively stratified algebras in general.

For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ the universal enveloping algebra of $\mathfrak{A}$ and by $Z(\mathfrak{A})$ the center of $U(\mathfrak{A})$.

We start with a simple complex finite-dimensional Lie algebra $\mathfrak{G}$ with a fixed Cartan subalgebra $\mathfrak{H}$, a fixed triangular decomposition $\mathfrak{G}=\mathfrak{N}_-\oplus\mathfrak{H}\oplus\mathfrak{N}_+$ and a parabolic subalgebra $\mathcal{P}\supset\mathfrak{H}\oplus\mathfrak{N}_+$. Consider the Levi decomposition $\mathcal{P}=(\mathfrak{A}\oplus\mathfrak{H}_{\mathfrak{A}})\oplus\mathfrak{N}$, where $\mathfrak{A}$ is semisimple, $\mathfrak{H}_{\mathfrak{A}}\subset\mathfrak{H}$, $[\mathfrak{A},\mathfrak{H}_{\mathfrak{A}}]=0$ and $\mathfrak{N}$ is nilpotent. Denote by $\mathfrak{A}'=\mathfrak{A}\oplus\mathfrak{H}_{\mathfrak{A}}$ the reductive Levi factor of $\mathcal{P}$.

Let Λ be a full subcategory of the category of finitely generated $\mathfrak{A}'$-modules, satisfying the following conditions:

1. any module from Λ is locally $Z(\mathfrak{A})$-finite and $\mathfrak{H}_{\mathfrak{A}}$-diagonilizable;

2. The full subcategory Λ_χ of Λ, corresponding to a central character $\chi \in Z(\mathfrak{A}')^*$, is the module category of a finite-dimensional self-injective associative algebra;

3. Any simple finite-dimensional $\mathfrak{A}'$-module F defines an exact functor $F \otimes \_ : \Lambda \to \Lambda$.

We will call such Λ *admissible*. For instance, the category of $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable finite-dimensional $\mathfrak{A}'$-modules is admissible (for other examples see [FKM]).

Fix an admissible category, Λ. Denote by $\mathcal{O}(\mathcal{P}, \Lambda)$ the full subcategory of the category of $\mathfrak{G}$-modules, whose objects are finitely generated, $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable, locally $\mathfrak{N}$-finite $\mathfrak{G}$-modules, which decompose into a direct sum of modules from Λ, when viewed as $\mathfrak{A}'$-modules. We assume that the abelian structure on Λ admits a natural extension to an abelian structure on $\mathcal{O}(\mathcal{P}, \Lambda)$.

Let M be an $\mathfrak{A}'$-module. Setting $\mathfrak{N}M = 0$, we turn M into a $\mathcal{P}$-module and now we can consider the induced module $\Delta(M) = U(\mathfrak{G}) \otimes_{U(\mathcal{P})} M$. As in [FKM, Proposition 2], one has that for any $M \in \Lambda$ the module $\Delta(M)$ is an object of $\mathcal{O}(\mathcal{P}, \Lambda)$, and any simple object in $\mathcal{O}(\mathcal{P}, \Lambda)$ is a quotient of $\Delta(S)$ for a simple $S \in \Lambda$. We will denote the corresponding simple object in $\mathcal{O}(\mathcal{P}, \Lambda)$ by $L(S)$.

2 Projectives in $\mathcal{O}(\mathcal{P}, \Lambda)$ and a block decomposition

The construction of projective modules and a block decomposition in $\mathcal{O}(\mathcal{P}, \Lambda)$ is a routine procedure, analogous to the classical cases [BGG, RW] or [FKM, Sections 3,4], so we will omit all unnecessary details.

Proposition 1. *Any module $M \in \mathcal{O}(\mathcal{P}, \Lambda)$ is locally finite over $Z(\mathfrak{G})$ and each subcategory $\mathcal{O}(\mathcal{P}, \Lambda)_\chi$ of $\mathcal{O}(\mathcal{P}, \Lambda)$ corresponding to a central character, $\chi \in Z(\mathfrak{G})^*$, has only finitely many simple objects.*

Proof. As any module in $\mathcal{O}(\mathcal{P}, \Lambda)$ is finitely generated, to prove the first statement it is enough to show that $\Delta(S)$ is locally $Z(\mathfrak{G})$-finite for any simple $S \in \Lambda$. Consider the highest $\mathfrak{H}_{\mathfrak{A}}$-weight λ of $\Delta(S)$. Clearly, it is enough to prove that $Z(\mathfrak{G})$ acts locally finite on $\Delta(S)_\lambda \simeq S$. We can calculate this action using the generalized Harish-Chandra homomorphism ([DFO]), which reduces the action of $Z(\mathfrak{G})$ to the actions of $Z(\mathfrak{A})$ and $\mathfrak{H}_{\mathfrak{A}}$ on S, which are locally finite, since Λ is admissible. Now the second statment follows from finiteness properties of the generalized Harish-Chandra homomorphism [FKM, Section 4]. □

Proposition 2. *$\mathcal{O}(\mathcal{P}, \Lambda)$ has enough projective objects, in particular, each $\mathcal{O}(\mathcal{P}, \Lambda)_\chi$ is equivalent to the module category of a finite-dimensional associative algebra.*

Proof. Fix a central character, χ, and let $L(S)$ be a simple object in $\mathcal{O}(\mathcal{P}, \Lambda)_\chi$. Let λ be the highest $\mathfrak{H}_{\mathfrak{A}}$-weight of $L(S)$. As $\mathcal{O}(\mathcal{P}, \Lambda)_\chi$ has only finitely many simples, there exists

$k \in \mathbb{N}$ such that $\mathfrak{N}^k M_\lambda = 0$ for any $M \in \mathcal{O}(\mathcal{P}, \Lambda)_\chi$. Let $\hat{S}$ be the projective cover of S in Λ. Then the $\mathcal{O}(\mathcal{P}, \Lambda)_\chi$-projection of the module

$$P(L(S), k) = U(\mathfrak{G}) \bigotimes_{U(\mathcal{P})} \Big((U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)) \otimes \hat{S} \Big)$$

is a projective module in $\mathcal{O}(\mathcal{P}, \Lambda)_\chi$, which maps onto $L(S)$. Now the statement follows from the abstract nonsense. □

Corollary 1. *Each projective module in* $\mathcal{O}(\mathcal{P}, \Lambda)$ *has a* standard flag, *i.e. a filtration whose quotients are of the form* $\Delta(\hat{T})$, T *simple in* Λ.

Proof. Follows from the construction of $P(L(S), k)$ and exactness of $F \otimes \_$ on Λ by standard arguments (see, for example, [FKM, Proposition 3]). □

We denote by $P(S)$ the projective cover of $L(S)$ in $\mathcal{O}(\mathcal{P}, \Lambda)$ and by $[P(S) : \Delta(\hat{T})]$ the number of occurrences of $\Delta(\hat{T})$ as a quotient in a standard flag of $P(S)$. It is easy to see, that this number is well-defined (see [RW, Lemma 1]).

3 An analogue of BGG-reciprocity

This Section contains the main result of the paper, which generalizes the famous BGG-reciprocity principle.

Theorem 1. *Assume that the Chevalley involution,* σ, *on* $\mathfrak{A}$ *(resp.* $\mathfrak{G}$*) in a natural way defines a duality,* *, *on* Λ *(resp.* $\mathcal{O}(\mathcal{P}, \Lambda)$*). Then for any simples* $S, T \in \Lambda$ *holds*

$$[P(S) : \Delta(\hat{T})] = (\Delta(T) : L(S)).$$

Proof. First we note that, by definition, duality preserves simple modules, i.e. $T^* \simeq T$. Any module from Λ can be viewed as a $\sigma(\mathcal{P})$-module with the trivial action of $\sigma(\mathfrak{N})$ and we have a standard adjunction $\mathrm{Hom}_{\sigma(\mathcal{P})}(U(\sigma(\mathcal{P})) \otimes_{U(\mathfrak{A}')} V, W) \simeq \mathrm{Hom}_{\mathfrak{A}'}(V, W)$ for any V, $W \in \Lambda$. For simple W and projective indecomposable V this will mean, in particular, $\dim \mathrm{Hom}_{\sigma(\mathcal{P})}(U(\sigma(\mathcal{P})) \otimes_{U(\mathfrak{A}')} V, W) = \dim \mathrm{Hom}_{\sigma(\mathcal{P})}(\Delta(V), W)$ is zero if $V \not\simeq \hat{W}$ and one otherwise. From this we get $[P(S) : \Delta(\hat{T})] = \dim \mathrm{Hom}_{\sigma(\mathcal{P})}(P(S), T)$. Applying the duality, we have $\dim \mathrm{Hom}_{\sigma(\mathcal{P})}(P(S), T) = \dim \mathrm{Hom}_{\mathcal{P}}(T^*, P(S)^*) = \dim \mathrm{Hom}_{\mathcal{P}}(T, P(S)^*)$. Further, inducing the first module up to $\mathfrak{G}$, we get $\dim \mathrm{Hom}_{\mathcal{P}}(T, P(S)^*) = \dim \mathrm{Hom}_{\mathfrak{G}}(\Delta(T), P(S)^*)$. One more time applying the duality, $\dim \mathrm{Hom}_{\mathfrak{G}}(\Delta(T), P(S)^*) = \dim \mathrm{Hom}_{\mathfrak{G}}(P(S), \Delta(T)^*)$. Finally, we get $\dim \mathrm{Hom}_{\mathfrak{G}}(P(S), \Delta(T)^*) = (\Delta(T)^* : L(S)) = (\Delta(T) : L(S))$. □

Corollary 2. *Let* $L(S_i)$, $i \in I$, *be a complete list of simples in* $\mathcal{O}(\mathcal{P}, \Lambda)_\chi$. *Set* $a_{i,j} = (P(S_i) : L(S_j))$, $b_{i,j} = (\Delta(S_i) : L(S_j))$ *and* $c_{i,j} = (\hat{S}_i : S_j)$. *Let* $A = (a_{i,j})_{i,j \in I}$, $B = (b_{i,j})_{i,j \in I}$ *and* $C = (c_{i,j})_{i,j \in I}$. *Then* $A = B^t C B$.

In particular, Corollary 2 enables one to compute the Cartan matrix of $\mathcal{O}(\mathcal{P},\Lambda)$ if known are the Cartan matrix of Λ and the decomposition multiplicities of $\Delta(T)$, T simple in Λ. The last modules are usually called *generalized Verma modules* and have been intensively studied (see bibliography in [M]) and we note that the corresponding decomposition multiplicities are known in several cases ([M]).

Remark 1. *In case Λ is semi-simple, one has $\hat{T}\simeq T$ and we obtain the classical BGG-reciprocity ([BGG, FM, R]). In the case of projectively stratified algebras, considered in [ADL, FKM], each $\hat{T}$ has isomorphic simple subquotients (C is diagonal) and we have $(\hat{T}:T)[P(S):\Delta(\hat{T})]=[P(S):\Delta(T)]$, obtaining [FKM, Theorem 4].*

Remark 2. *This result can be easily extended to a truncated analogue of $\mathcal{O}(\mathcal{P},\Lambda)$ for a complex Lie algebra with triangular decomposition ([RW]), where Λ is admissible, $\mathcal{P}$ is a parabolic subalgebra containing a standard Borel subalgebra and $\mathfrak{A}$ is a semi-simple finite-dimensional algebra.*

4 An $sl(2,\mathbb{C})$-example

Classical category $\mathcal{O}$, as good as all examples from [FKM], embeds in the picture above. In this Section we give an example, in which Λ is not a sum of the module categories of projectively stratified algebras. This example definitely lies outside the area, where previously known results can be applied.

Let $\mathfrak{A}\simeq sl(2,\mathbb{C})$ with standard basis e,f,h. Fix $a,b\in\mathbb{C}$ and let $\hat{\Lambda}=\hat{\Lambda}(a,b)$ be the set of all simple weight $\mathfrak{A}$-modules, whose set of weights (*support*) is a subset in $a+\mathbb{Z}$ and such that b is the eigenvalue of the Casimir operator $\mathfrak{c}=(h+1)^2+4fe$ on these modules. The basic $sl(2,\mathbb{C})$-theory tells us that $\hat{L}$ contains two, four or six elements. Denote by Λ the full subcategory of the category of $\mathfrak{A}$-modules, generated by $F\otimes M$, F is simple finite-dimensional and M is indecomposable with non-isomorphic simple subquotients from $\hat{L}$. For instance, if $\hat{\Lambda}$ contains only two elements, Λ coincides with the category from [FKM, Section 10]. It is easy to check that Λ is an admissible category with abelian structure inherited from the category of $\mathfrak{A}$-modules, and hence $\mathcal{O}(\mathcal{P},\Lambda)$ satisfies all conditions of Theorem 1. However, for $|\hat{\Lambda}|>2$ any block of Λ does not correspond to a projectively stratified algebra.

For simplicity, here we consider in detail the case $|\hat{\Lambda}|=4$ (i.e. there is a unique $k\in\mathbb{Z}$ such that $b=(a+2k+1)^2$) and, in addition, we assume $b\neq 0$. Thus $\hat{\Lambda}$ contains a usual Verma module $M(a+2k)$ and a module $M'(a+2k+2)$, which is a Verma module with respect to the second choice for a basis of the root system (lowest weight module). Then the simple modules in Λ are exactly $M(a+i)$ and $M'(a+j)$, $i,j\in\mathbb{Z}$. $M(a+i)$ and $M'(a+i+2)$ can form (up to isomorphism) exactly two non-trivial extensions (both of length two), each of $M(a+i)$ and $M'(a+i+2)$ being a submodule exactly in one extension. Tensoring this with finite-dimensional modules, one can easily compute Λ. In particular, each block of Λ has exactly two simple modules ($M(a+i)$ and $M'(a+i+2)$ for some i) and each indecomposable projective module in Λ has length 2. In particular, the last means that each block of Λ is not equivalent to the module category of a projectively stratified algebra. The Cartan matrix of each block of Λ has the following form:

$$C = (c_{i,j}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Under such choice of Λ, Theorem 1 gives a non-trivial symmetry in $\mathcal{O}(\mathcal{P}, \Lambda)$ which can not be obtained from previously known results. In particular, the Cartan matrix of $\mathcal{O}(\mathcal{P}, \Lambda)_\chi$, computed by Corollary 2, has more complicated form than one in the classical situations (B^tB for category $\mathcal{O}$ or B^tCB, C diagonal, for projectively stratified algebras).

ACKNOWLEDGMENTS

The research was done during the visit of the second author to Bielefeld University as an Alexander von Humboldt fellow. Financial support of the Humboldt Foundation and hospitality of Bielefeld University are gratefully acknowledged.

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