On Arkhipov's and Enright's functors

Oleksandr Khomenko and Volodymyr Mazorchuk

Mathematical Subject Classification: Primary 16G10, 17B10; Secondary: 16G20, 17B35

Abstract

We give a description of Arkhipov's and (Joseph's and Deodhar-Mathieu's versions of) Enright's endofunctors on the category $\mathcal{O}$, associated with a fixed triangular decomposition of a complex finite-dimensional semi-simple Lie algebra, in terms of (co)approximation functors with respect to suitably chosen injective (resp. projective) modules. We establish some new connections between these functors, for example we show that Arkhipov's and Joseph's functors are adjoint to each other. We also give several proofs of braid relations for Arkhipov's and Enright's functors.

1 Introduction

The blocks of the the BGG category $\mathcal{O}$, associated with a fixed triangular decomposition of a complex finite-dimensional semi-simple Lie algebra $\mathfrak{g}$ (see [BGG]), are principal objects of study in the representation theory. They are equivalent to the categories of modules over certain quasi-hereditary algebras, which possess a lot of symmetries. In particular, the (basic) quasi-hereditary algebra A, describing the principal block of $\mathcal{O}$, is isomorphic to its opposite algebra, is isomorphic to its Koszul dual, and is isomorphic to its Ringel dual. The last isomorphism was first established by Soergel in [So1], using a special endofunctor on $\mathcal{O}$, which was inspired by the work [Ar1] of Arkhipov. Later on, in [Ar2], Arkhipov proposed a construction, which associates an analogous functor to every simple root of $\mathfrak{g}$. Basicly, every Arkhipov's functor is tensoring with a bimodule. Reading [Ar2] one gets a very strong impression that Arkhipov's functors must satisfy braid relation, especially as the statement of [Ar2, Lemma 2.1.10] says that two braid tensor products of Arkhipov's bimodules are isomorphic as left modules. However, we did not manage to derive the braid relations for functors (or tensor products of corresponding bimodules) from [Ar2]. Assuming braid relations for Arkhipov's functors one gets that in the case of finite-dimensional Lie algebra the functor, used by Soergel in [So1], is just the composition of Arkhipov's functors, constructed with respect to a reduced decomposition of the longest element in the Weyl group.

Another famous example of functors, associated with simple roots of $\mathfrak{g}$, is the family of the so-called *Enright's completions*, originally defined by Enright in [En]. Later on Deodhar, [De], proved that Enright's functors satisfy braid relations on a (rather big) subcategory of $\mathcal{O}$ (see also the paper [Bo] of Bouaziz). This was extended by Joseph, [Jo2], to the whole category $\mathcal{O}$ for a slight modification of Enright's functors.

In [KM] it is shown that Enright's functors (in Deodhar's or, more generally, in Mathieu's version, [Ma]) have a natural realization in terms of the approximation functor ([Au]) with respect to a suitably chosen (not necessarily indecomposable) injective module in $\mathcal{O}$. This realization is then used to get a short proof of the braid relations (on the subcategory of $\mathcal{O}$, considered by Deodhar), which is free from technicalities. Moreover, this realization also allows one to establish an equivalence between several categories of $\mathfrak{g}$-modules, arising in rather different contexts.

The aim of the present paper is to find analogous descriptions for Joseph's version of Enright's functors and also for Arkhipov's functors. To do this we "deform" in some sense the definition of the approximation functor and define a new functor, which we call partial approximation. The difference in the behavior of approximation and partial approximation is very similar to the difference in the behavior of Enright's and Joseph's functors, which suggests that the partial approximation can be a good candidate for a description of Joseph's functor. We show that the partial approximation really dercribes Joseph's functor. Finally, we also show that Arkhipov's functor can be described by a dual construction, called partial coapproximation.

Using our realization, we show that Joseph's and Arkhipov's functors are adjoint to each other, moreover, that one of them is a conjugation of the other by the natural duality on $\mathcal{O}$, implying that the studies of the braid relations for these two families of functors are equivalent. The results are quite surprising since the original definition of the functors is purely "Lie theoretic" and uses either the structure theory of the universal enveloping algebra or the theory of Harish-Chandra bimodules. Nevertheless, the description we get is categorical and thus it can be transfered to the case of any finite-dimensional associative algebra (provided that there exists a duality and a choice of injective modules, satisfying some special properties). In some sense our description establishes one more path connecting representation theory of Lie algebras and representation theory of finite-dimensional algebras.

Let α be a simple root of $\mathfrak{g}$, and let $\mathcal{O}_{int}^{\lambda}$ be the block of the category $\mathcal{O}$ for $\mathfrak{g}$ corresponding to an integral dominant weight λ. Denote by $\mathcal{A}^{\alpha}$ Arkhipov's functor, by $\mathcal{C}_J^{\alpha}$ Joseph's version of Enright's functor and by $\mathcal{C}_M^{\alpha}$ Deodhar-Mathieu's version of Enright's functor associated with α. Let Υ denote the set $\{w \cdot \lambda\}$, where w is the longest representative of a left coset of the Weyl group of $\mathfrak{g}$ modulo the subgroup, generated by s_{α}. Denote also by $\mathfrak{c}_{\Upsilon}$, $\mathfrak{d}_{\Upsilon}$ and $\tilde{\mathfrak{d}}_{\Upsilon}$ the functors of approximation, partial approximation and partial coapproximation with respect to Υ. The principal results of the paper can now be collected in the following three statements.

Theorem 1. *1. The functors $\mathcal{A}^{\alpha}$ and $\tilde{\mathfrak{d}}_{\Upsilon}$ are isomorphic;*

2. The functors $\mathcal{C}_J^{\alpha}$ and $\mathfrak{d}_{\Upsilon}$ are isomorphic;

3. The functors $\mathcal{C}_M^{\alpha}$ and $\mathfrak{c}_{\Upsilon}$ are isomorphic.

Theorem 2. *The functors $\mathcal{A}^\alpha$, α simple, satisfy braid relations.*

Theorem 3. *The functor $\mathcal{A}^\alpha$ is left adjoint to the functor $\mathcal{C}_J^\alpha$.*

The paper is organized as follows. In Section 2 we define our main objects: Arkhipov's and Enright's functors (the latter ones are defined both in the original and in the modified versions) on the category $\mathcal{O}$ and various approximations functors on module categories of finite dimensional algebras. In all cases the action of the functors is illustrated on the regular block of $\mathcal{O}$ for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. In Section 3, 4 and 5 we study the properties of and give a description for the original Enright's functor, the modified (Joseph's) version of Enright's functor and Arkhipov's functor respectively. In particular, we prove Theorem 3 and all statements combined in Theorem 1. In Section 6 we derive braid relations for Arkhipov's functors and transfer them to the case of Enright's functors. We finish the paper with an application of our results to some parabolic analogs of the category $\mathcal{O}$ in Section 7.

We have to say that during the preparation of the paper we learned that there is some overlap between our results and the recent results of Andersen and Stroppel, [AS]. We are indebted to Catharina Stroppel for this information. Moreover, as it was pointed out by Catharina, some of our original methods, especially those used for the study of Arkhipov's functors needed a serious revision. In particular, in the final form, presented in this paper, we use the fact, first completely proved in [AS], that Arkhipov's functors commute with translation functors.

2 Functors

We denote by $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of complex numbers, integers, non-negative integers and positive integers respectively.

We fix a triangular decomposition, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, for $\mathfrak{g}$ and denote by π the corresponding basis in the root system Δ for $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. We also fix some Weyl-Chevalley basis $\{X_\alpha : \alpha \in \Delta\} \cup \{H_\alpha : \alpha \in \pi\}$ in $\mathfrak{g}$. For a simple root α we denote by U_α the Ore localization of the universal enveloping algebra $U(\mathfrak{g})$ with respect to the multiplicative set $\{X_{-\alpha}^k : k \in \mathbb{N}\}$, see [Ma, Section 4] for details. Let W be the Weyl group of Δ. It is generated by simple reflections s_α, $\alpha \in \pi$. We denote by $l : W \to \mathbb{Z}_+$ the length function with respect to π. Let $(\cdot,\cdot)$ be the standard W-invariant form on $\mathfrak{h}^*$. We also denote by $\mathfrak{g}(\alpha)$ the $\mathfrak{sl}(2,\mathbb{C})$-subalgebra of $\mathfrak{g}$, generated by $X_{\pm\alpha}$.

From now on we restrict our consideration to the full subcategory $\mathcal{O}_{int}$ of the category $\mathcal{O}$, which consists of all modules having integral support. The general case can be easily reduced to the integral case via the equivalence of categories established in [So2]. Furthermore, with respect to the action of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ we can decompose $\mathcal{O}_{int}$ into a direct sum of full subcategories $\mathcal{O}_{int}^\lambda$, where λ is dominant and $\mathcal{O}_{int}^\lambda$ consists of all modules in $\mathcal{O}_{int}$, having the same (generalized) central character as the Verma module $M(\lambda)$. We will also widely use the natural duality on $\mathcal{O}$, denoted by $\star$, which is defined in terms of the Chevalley anti-involution on $\mathfrak{g}$.

All the categories, which we consider, have enough injective and projective objects. For every module M we fix some inclusion, $\mathfrak{z}_M$, of M into its injective envelope I_M, and some projection, $\mathfrak{y}_M$, from the projective cover P_M to M. To simplify notation we are going to omit the index M in $\mathfrak{z}_M$ and $\mathfrak{y}_M$ and write simply $\mathfrak{z}$ and $\mathfrak{y}$. For a (Lie) algebra A and a (Lie) subalgebra $B \subset A$ we denote by Res_B^A the restriction functor from A-mod to B-mod. By a *duality* on a category we mean an exact contravariant involutive self-equivalence, which preserves (isomorphism classes of) simple objects.

Later in the paper we will need the following general statement, which we call the *Comparison Lemma.*

Lemma 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and F, G, H be three additive functors from $\mathcal{A}$ to $\mathcal{B}$. Assume that*

1. *$\mathcal{A}$ has enough injective objects.*
2. *F and G are left exact.*
3. *For any injective $I \in \mathcal{A}$ the objects $F(I)$ and $G(I)$ are isomorphic.*
4. *There are natural transformations $\mathrm{nat}_F : H \to F$ and $\mathrm{nat}_G : H \to G$ such that for any injective I the maps $\mathrm{nat}_F(I)$ and $\mathrm{nat}_G(I)$ are epimorphisms.*
5. *For any injective object I we have $\mathrm{Ker}(\mathrm{nat}_F(I)) = \mathrm{Ker}(\mathrm{nat}_G(I))$, that is there exists an isomorphism $i : \mathrm{Ker}(\mathrm{nat}_F(I)) \to \mathrm{Ker}(\mathrm{nat}_G(I))$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Ker}(\mathrm{nat}_F(I)) & \longrightarrow & H(I) \\ {\scriptstyle i}\downarrow & & \| \\ \mathrm{Ker}(\mathrm{nat}_G(I)) & \longrightarrow & H(I) \end{array}$$

Then the functors F and G are isomorphic.

Proof. Since both functors are left exact and $\mathcal{A}$ has enough injective objects, taking the first two steps of the injective resolution it is sufficient to prove that F and G are isomorphic on the full subcategory $\mathcal{A}^{inj}$ of $\mathcal{A}$ consisting of injective objects. Let $I \in \mathcal{A}^{inj}$. Since both $\mathrm{nat}_F(I)$ and $\mathrm{nat}_G(I)$ are epimorphisms and $\mathrm{Ker}(\mathrm{nat}_F(I)) = \mathrm{Ker}(\mathrm{nat}_G(I))$, there exists an isomorphism $\mathfrak{p}(I) : F(I) \to G(I)$ such that the following digram commutes:

$$\begin{array}{ccc} F(I) & \xrightarrow{\mathfrak{p}(I)} & G(I) \\ {\scriptstyle \mathrm{nat}_F(I)}\uparrow & \nearrow{\scriptstyle \mathrm{nat}_G(I)} & \\ H(I) & & \end{array}$$

Let us show that $\{\mathfrak{p}(I) : I \in \mathcal{A}^{inj}\}$ defines a natural isomorphism from F to G. Since all $\mathfrak{p}(I)$ are isomorphisms, It is enough to show that $\{\mathfrak{p}(I) : I \in \mathcal{A}^{inj}\}$ defines a natural transformation from F to G.

Let I, J be injective objects in $\mathcal{A}$ and $f : I \to J$ be a morphism. As both nat_F and nat_G are natural, we have the following commutative diagrams:

$$\begin{CD} F(I) @>{F(f)}>> F(J) \\ @A{\mathrm{nat}_F(I)}AA @AA{\mathrm{nat}_F(J)}A \\ H(I) @>{H(f)}>> H(J) \end{CD} \qquad\qquad \begin{CD} G(I) @>{G(f)}>> G(J) \\ @A{\mathrm{nat}_G(I)}AA @AA{\mathrm{nat}_G(J)}A \\ H(I) @>{H(f)}>> H(J) \end{CD}$$

Hence all triangles and small quadrangles in the following diagram commute:

$$\begin{array}{ccccc} F(I) & & \xrightarrow{\quad F(f) \quad} & & F(J) \\ & \nwarrow^{\mathrm{nat}_F(I)} & & \nearrow^{\mathrm{nat}_F(J)} & \\ \downarrow{\scriptstyle \mathfrak{p}(I)} & & H(I) \xrightarrow{H(f)} H(J) & & \downarrow{\scriptstyle \mathfrak{p}(J)} \\ & \swarrow_{\mathrm{nat}_G(I)} & & \searrow_{\mathrm{nat}_G(J)} & \\ G(I) & & \xrightarrow[\quad G(f) \quad]{} & & G(J) \end{array}$$

Since the map $\mathrm{nat}_F(I)$ is an epimorphisms, we have the commutativity of the big square, which completes the proof. $\square$

2.1 Arkhipov's functors

In this subsection we follow [Ar1, Section 2]. For $\alpha \in \pi$ we define the functor $\mathcal{A} = \mathcal{A}^\alpha : \mathcal{O}_{int} \to \mathcal{O}_{int}$, which we will call *elementary Arkhipov's functor*, as follows:

$$M \mapsto \Theta_\alpha \left(\left(\mathrm{Res}^{U_\alpha}_{U(\mathfrak{g})} \left(U_\alpha \otimes_{U(\mathfrak{g})} M \right) \right) / \psi(M) \right),$$

where Θ_α denotes the twist with respect to the automorphism of $\mathfrak{g}$, corresponding to s_α, and $\psi(M)$ denotes the canonical image of M in the module $\mathrm{Res}^{U_\alpha}_{U(\mathfrak{g})} \left(U_\alpha \otimes_{U(\mathfrak{g})} M \right)$. The same functor can be realized as tensoring with a special "α-semi-regular" $U(\mathfrak{g})$-bimodule, namely the bimodule $\mathcal{S}_\alpha = U_\alpha / U$, followed by Θ_α-twist (we refer the reader to [Ar1, Section 2] for details). It is obvious that $\mathcal{A}$ is covariant and from the last realization it follows that it is right exact.

For any finite-dimensional $\mathfrak{g}$-module E the functor $\mathcal{A}$ commutes with the functor $E \otimes \_$. A complete proof of this can be found in [AS], where it is shown that the statement is closely connected to the fact that U_α has a comultiplication (over a completion). The isomorphism $\mathfrak{f}_E : \mathcal{A} \circ (E \otimes \_) \to (E \otimes \_) \circ \mathcal{A}$ (modulo the obvious Θ_α part) can be constructed as follows: If $M \in \mathcal{O}_{int}$ then the corresponding natural transformation $\mathfrak{f}_E : \mathcal{A} \circ (E \otimes \_) \to (E \otimes \_) \circ \mathcal{A}$ from $U_\alpha \otimes_{U(\mathfrak{g})} (M \otimes E)$ to $(U_\alpha \otimes_{U(\mathfrak{g})} M) \otimes E$ is given by

$$X_{-\alpha}^{-n} \otimes (m \otimes e) \mapsto \sum_{k \geqslant 0} (-1)^k \binom{n+k-1}{k} X_{-\alpha}^{-n-k} \otimes m \otimes X_{-\alpha}^{k} e.$$

The map in the opposite direction is

$$(X_{-\alpha}^{-n}\otimes m)\otimes e\mapsto X_{-\alpha}^{-ar}\otimes\sum_{k\geqslant 0}\binom{ar}{k}X_{-\alpha}^{ar-n-k}m\otimes X_{-\alpha}^{k}e,$$

where $r,a\in\mathbb{N}$ are chosen such that $X_{-\alpha}^{a}$ annihilates E and $(r-1)a\geqslant n-1$ (it is easy to see that the final map does not depend on the choice of a and r). Both formulae above are taken from [AS, Section 3]. Since $\mathcal{A}$ obviously preserves the (generalized) central character of the module, it follows that $\mathcal{A}$ commutes with all translation functors. An important property of the maps above is the following statements ([AS, Theorem 3.2]), which are checked by direct calculation (below for a finite-dimensional $\mathfrak{g}$-module E by E° we denote the usual dual module and we also remark that $E^{\circ}\not\cong E^{\star}$ in general):

Lemma 2. *Let $M\in\mathcal{O}$. For any finite-dimensional $\mathfrak{g}$-modules E and F the following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{A}(F\otimes E\otimes M) & \xrightarrow{\mathfrak{f}_F(M\otimes E)} & F\otimes\mathcal{A}(E\otimes M)\\
 & \searrow^{\mathfrak{f}_{F\otimes E}(M)} & \downarrow{\scriptstyle \mathfrak{f}_E(M)\otimes\mathrm{Id}}\\
 & & F\otimes E\otimes\mathcal{A}(M)
\end{array}$$

Moreover, if E is a finite-dimensional $\mathfrak{g}$-module, then the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{A}(E^{\circ}\otimes E\otimes M) & \xrightarrow{\mathfrak{f}_{E^{\circ}\otimes E}(M)} & E^{\circ}\otimes E\otimes\mathcal{A}(M)\\
{\scriptstyle \mathcal{A}(\overline{ev}\otimes\mathrm{Id})}\uparrow & & \uparrow{\scriptstyle \overline{ev}\otimes\mathrm{Id}}\\
\mathcal{A}(\mathbb{C}\otimes M) & \xrightarrow{\mathfrak{f}_{\mathbb{C}}(M)} & \mathbb{C}\otimes\mathcal{A}(M)
\end{array}$$

where $\overline{ev}:\mathbb{C}\to E\otimes E^{\circ}$ is defined by $1\mapsto\sum_{i=1}^{d}e_i^{*}\otimes e_i$ for a fixed basis $\{e_i\}$ in E and the corresponding dual basis $\{e_i^{*}\}$ in E°.

Dualizing the second part of Lemma 2 one easily gets the following statement:

Corollary 1. *Keep all the notation from Lemma 2, then the following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{A}(E^{\circ}\otimes E\otimes M) & \xleftarrow{\mathfrak{f}^{-1}_{E^{\circ}\otimes E}(M)} & E^{\circ}\otimes E\otimes\mathcal{A}(M)\\
{\scriptstyle \mathcal{A}(ev\otimes\mathrm{Id})}\downarrow & & \downarrow{\scriptstyle ev\otimes\mathrm{Id}}\\
\mathcal{A}(\mathbb{C}\otimes M) & \xleftarrow[\mathfrak{f}^{-1}_{\mathbb{C}}(M)]{} & \mathbb{C}\otimes\mathcal{A}(M)
\end{array}$$

where $ev:E\otimes E^{\circ}\to\mathbb{C}$ is the evaluation map.

A very important new property of $\mathcal{A}$ is given by the following statement.

Theorem 4. *There is a homomorphism of functors* $\mathfrak{r} : \mathcal{A} \to \mathrm{Id}$ *such that for every* $M \in \mathcal{O}_{int}$ *the cokernel of the map* $\mathfrak{r} : \mathcal{A}(M) \to M$ *is the maximal* $X_{-\alpha}$*-finite quotient of* M. *Moreover, for projective* $M \in \mathcal{O}_{int}$ *the map* $\mathfrak{r} : \mathcal{A}(M) \to M$ *is injective.*

We remark that by [AS, Proposition 5.4] the existence of a non-trivial homomorphism of functors $\mathfrak{r} : \mathcal{A} \to \mathrm{Id}$ implies that the cokernel of the map $\mathfrak{r} : \mathcal{A}(M) \to M$ is the maximal $X_{-\alpha}$-finite quotient of M. However, during the proof of Theorem 4 we derive this property from the construction.

Proof. Since $\mathcal{A}$ commutes with translation functors, it is enough to prove the statement for the block $\mathcal{O}_{int}^0$. Our aim is to reduce the statement to $\mathfrak{sl}(2,\mathbb{C})$-case, where we will check it by direct computation (using Lemma 2 and Corollary 1). Set $\mathfrak{a} = \mathfrak{g}(\alpha) + \mathfrak{h}$. Let us for a moment denote by $\mathcal{A}_{\mathfrak{g}}$ Arkhipov's functor for the algebra $\mathfrak{g}$ and by $\mathcal{A}_{\mathfrak{a}}$ Arkhipov's functor for the subalgebra $\mathfrak{a}$. We also denote by $\mathcal{S}_\alpha^{\mathfrak{g}}$ and $\mathcal{S}_\alpha^{\mathfrak{a}}$ the corresponding α-semi-regular bimodules.

Lemma 3. *The functors* $\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{g}} \circ \mathcal{A}_{\mathfrak{g}}$ *and* $\mathcal{A}_{\mathfrak{a}} \circ \mathrm{Res}_{\mathfrak{a}}^{\mathfrak{g}}$ *are isomorphic as functors from* $\mathcal{O}_{int}$ *to* $\mathfrak{a}$*-mod.*

Proof. The functor $\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{g}}$ can be realized as the functor ${}_{U(\mathfrak{a})}U(\mathfrak{g})_{U(\mathfrak{g})} \otimes_{U(\mathfrak{g})} -$. From the PBW Theorem it follows that the multiplication induces an isomorphism of $U(\mathfrak{a}) - U(\mathfrak{g})$-bimodules

$$ {}_{U(\mathfrak{a})}\mathcal{S}_\alpha^{\mathfrak{a}}{}_{U(\mathfrak{a})} \bigotimes_{U(\mathfrak{a})} {}_{U(\mathfrak{a})}U(\mathfrak{g})_{U(\mathfrak{g})} \cong {}_{U(\mathfrak{a})}\mathcal{S}_\alpha^{\mathfrak{g}}{}_{U(\mathfrak{g})}. $$

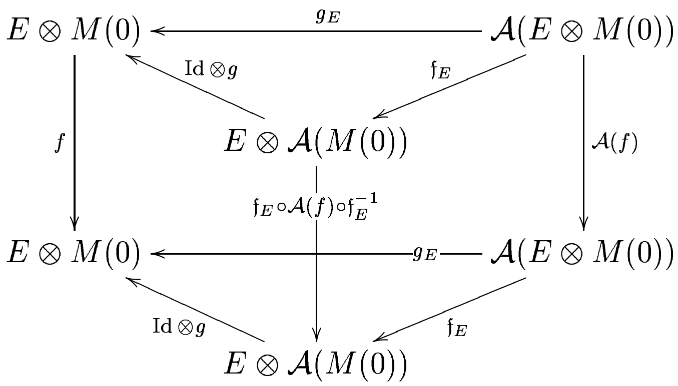
It is obvious that $\Theta_\alpha \circ \mathrm{Res}_{\mathfrak{a}}^{\mathfrak{g}} \circ \Theta_\alpha$ is isomorphic to $\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{g}}$ and the statement follows. □

We take the projective cover $M(0)$ in $\mathcal{O}$ of the trivial $\mathfrak{g}$-module. $M(0)$ is the Verma module with the highest weight $0 \in \mathfrak{h}^*$. First we want to show that $\mathcal{A}(M(0)) \cong M(s_\alpha \cdot 0)$. It is easy to see that the module $N = \mathcal{S}_\alpha^{\mathfrak{g}} \otimes_{U(\mathfrak{g})} M(0)$ is generated by the weight space N_α and hence the module $\mathcal{A}(M(0))$ is a highest weight module with the highest weight $s_\alpha \cdot 0$. Comparing the characters we derive the necessary statement. In particular, the module $\mathcal{A}(M(0))$ embeds into $M(0)$, this embedding is unique up to a scalar, and the cokernel of this embedding is the maximal $X_{-\alpha}$-finite quotient of $M(0)$. Let us fix some embedding of $\mathcal{A}(M(0))$ into $M(0)$, g say.

For every finite-dimensional $\mathfrak{g}$-module E the composition

$$ \mathcal{A}(E \otimes M(0)) \xrightarrow{\mathfrak{f}_E} E \otimes \mathcal{A}(M(0)) \xrightarrow{\mathrm{Id} \otimes g} E \otimes M(0) $$

gives a map, which we will denote by g_E. To complete the proof it is enough to show that for every homomorphism $f : E \otimes M(0) \to E \otimes M(0)$ the following diagram is commutative:



(1)

Indeed, choose E such that for every indecomposable projective module Q in $\mathcal{O}^0_{int}$ the projective module $E \otimes M(0)$ contains at least one direct summand isomorphic to Q. With this choice of E we have for any $M \in \mathcal{O}^0_{int}$ an exact sequence

$$E \otimes M(0) \xrightarrow{f} E \otimes M(0) \to M \to 0.$$

Applying $\mathcal{A}$ we get a unique morphism g_M, which makes the following commutative diagram:

$$\begin{array}{ccccccc}
E \otimes M(0) & \xrightarrow{f} & E \otimes M(0) & \longrightarrow & M & \longrightarrow & 0 \\
{\scriptstyle g_E}\uparrow & & {\scriptstyle g_E}\uparrow & & \vdots {\scriptstyle g_M} & & \\
\mathcal{A}(E \otimes M(0)) & \xrightarrow{\mathcal{A}(f)} & \mathcal{A}(E \otimes M(0)) & \longrightarrow & \mathcal{A}(M) & \longrightarrow & 0
\end{array}$$

It is easy to see that g_M does not depend on the choice of f.

Both triangles and right quadrangle in this diagram (1) commute by construction. So, to prove the commutativity it is enough to prove that the left quadrangle also commutes (this will imply the commutativity of the back quadrangle). Hence the question reduces to commutativity of the diagram

$$\begin{array}{ccc}
E \otimes M(0) & \xleftarrow{\mathrm{Id} \otimes g} & E \otimes \mathcal{A}(M(0)) \\
{\scriptstyle f}\downarrow & & \downarrow {\scriptstyle \mathfrak{f}_E \circ \mathcal{A}(f) \circ \mathfrak{f}_E^{-1}} \\
E \otimes M(0) & \xleftarrow{\mathrm{Id} \otimes g} & E \otimes \mathcal{A}(M(0))
\end{array}$$

But the last diagram commutes if and only if it commutes after restriction to $\mathfrak{a}$. This means that it is now left to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{a}}(E \otimes M(0)) & \xleftarrow{\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{a}}(\mathrm{Id} \otimes g)} & \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{a}}(E \otimes \mathcal{A}(M(0))) \\
{\scriptstyle \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{a}}(f)}\downarrow & & \downarrow {\scriptstyle \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{a}}(\mathfrak{f}_E \circ \mathcal{A}(f) \circ \mathfrak{f}_E^{-1})} \\
\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{a}}(E \otimes M(0)) & \xleftarrow{\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{a}}(\mathrm{Id} \otimes g)} & \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{a}}(E \otimes \mathcal{A}(M(0)))
\end{array}$$

Obviously, the functor $\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{g}}$ commutes with the functor $E\otimes\_$. Now observe that the orthogonal complement $\mathfrak{h}_\alpha^\perp$ to $\mathfrak{h}_\alpha = \mathbb{C}H_\alpha$ (this complement is taken in $\mathfrak{h}$ with respect to $(\cdot,\cdot)$) forms the center of $\mathfrak{a}$. Moreover, as all modules in $\mathcal{O}$ are $\mathfrak{h}$-diagonalizable by definition, for all $M \in \mathcal{O}$ we have the $\mathfrak{h}_\alpha^\perp$-weight decomposition $M = \oplus_{\nu\in(\mathfrak{h}_\alpha^\perp)^*} M_\nu$, where M_ν consists of all elements from M on which $\mathfrak{h}_\alpha^\perp$ acts via ν. It is clear that all $\mathfrak{g}$-morphisms respect this decomposition and thus to prove the commutativity of the last diagram it is sufficient to prove the commutativity of its restriction to every $\mathfrak{h}_\alpha^\perp$-weight space. However, for a fixed $\mathfrak{h}_\alpha^\perp$-weight space all components of the diagram become finitely-generated $\mathfrak{g}(\alpha)$-modules and all morphisms become $\mathfrak{g}(\alpha)$-morphisms.

Let $0_\alpha^\perp$ be the zero $\mathfrak{h}_\alpha^\perp$-weight (it is of course the restriction of 0 to $\mathfrak{h}_\alpha^\perp$). One has that the $\mathfrak{g}(\alpha)$-module $M(0)_{0_\alpha^\perp}$ is the Verma module with one-dimensional simple top. Thus, by Lemma 2, the restriction of the map g, which we fixed above, to $M(0)_{0_\alpha^\perp}$ fixes a map from the image of $M(0)_{0_\alpha^\perp}$ under the Arkhipov's functor $\mathcal{A}^\alpha$ for the algebra $\mathfrak{g}(\alpha)$ to $M(0)_{0_\alpha^\perp}$. Let us denote the last map by g' for a moment. The universal enveloping algebra $U(\mathfrak{g})$ is a direct sum of finite-dimensional $\mathfrak{g}(\alpha)$-modules under the adjoint action. This implies that for every $\mathfrak{h}_\alpha^\perp$-weight ν the $\mathfrak{g}(\alpha)$-module $M(0)_\nu$ is isomorphic to $E(\nu)\otimes M(0)_{0_\alpha^\perp}$, where $E(\nu)$ is some finite-dimensional $\mathfrak{g}(\alpha)$-module (in fact this module is spanned over $\mathbb{C}$ by all products of elements $X_{-\beta}$, where $\beta \neq \alpha$ is positive, which have $\mathfrak{h}_\alpha^\perp$-weight ν). Moreover, using the fact that g is a $\mathfrak{g}$-homomorphism, and thus commutes with all $X_{-\beta}$ above, it is easy to see that, under this identification, the restriction of g to $M(0)_\nu$ is exactly $\mathrm{Id}\otimes g'$. From this and the first statement of Lemma 2 we derive that all maps g_E are coordinated with the restriction to $\mathfrak{g}(\alpha)$, i.e.

$$\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{g}}(g_E|_{E\otimes M(0)_\nu}) = g_{E\otimes E(\nu)}.$$

This observation and Lemma 3 reduce our problem to the situation $\mathfrak{g} = \mathfrak{g}(\alpha)$, which we now assume to be the case till the end of the proof.

The advantage of the $\mathfrak{sl}(2,\mathbb{C})$-case is that we can do it by direct computation. Let λ be a dominant weight and denote by Pr_E^λ the projection of $\mathcal{O}_{int}$ onto $\mathcal{O}_{int}^\lambda$. It would certainly suffice to show that the following diagram is commutative (for all λ):

$$\begin{array}{ccc}
\mathrm{Pr}_E^\lambda(E\otimes M(0)) & \xleftarrow{\mathrm{Pr}_E^\lambda(\mathrm{Id}\otimes g)} & \mathrm{Pr}_E^\lambda(E\otimes \mathcal{A}(M(0))) \\
\Big\downarrow{\scriptstyle \mathrm{Pr}_E^\lambda(f)} & & \Big\downarrow{\scriptstyle \mathrm{Pr}_E^\lambda(\mathfrak{f}_E\circ\mathcal{A}(f)\circ\mathfrak{f}_E^{-1})} \\
\mathrm{Pr}_E^\lambda(E\otimes M(0)) & \xleftarrow{\mathrm{Pr}_E^\lambda(\mathrm{Id}\otimes g)} & \mathrm{Pr}_E^\lambda(E\otimes \mathcal{A}(M(0)))
\end{array} \qquad (2)$$

Now let us fix a decomposition of $E\otimes M(0)$ into a direct sum of indecomposable modules. Using the additivity of all our functors we get that it is enough to prove that for any two indecomposable direct summands P_1 and P_2 of $E\otimes M(0)$ the following diagram commutes:

$$\begin{array}{ccc}
P_1 & \xleftarrow{t_1} & \mathfrak{f}_E(\mathcal{A}(P_1)) \\
\Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \mathfrak{f}_E\circ\mathcal{A}(f)\circ\mathfrak{f}_E^{-1}} \\
P_2 & \xleftarrow{t_2} & \mathfrak{f}_E(\mathcal{A}(P_2))
\end{array}$$

where t_1 and t_2 denote the restriction of $\mathrm{Pr}^{\mu}_{E}(\mathrm{Id}\otimes g)$ to $\mathfrak{f}_E(\mathcal{A}(P_1))$ and $\mathfrak{f}_E(\mathcal{A}(P_2))$ respectively. By (2), we can assume that $P_1, P_2 \in \mathcal{O}^{\lambda}_{int}$

Now we remind that the projective modules in $\mathcal{O}^{\lambda}_{int}$ are well known. If λ is singular then $\mathcal{O}^{\lambda}_{int}$ contains the unique indecomposable projective module, which is simple and has trivial endomorphism ring. It this case the statement follows from the fact that $\mathcal{A}$ preserves the identity map. If λ is regular, we have (up to isomorphism) two projective modules: $P(\lambda)\cong M(\lambda)$, which is the projective Verma module in this block, and the big projective module $P(s_\alpha\cdot\lambda)$, which is the projective cover of the simple socle of $M(\lambda)$ (see also more detailed description in example at the end of this subsection). Let us check all the homomorphisms between them in more details.

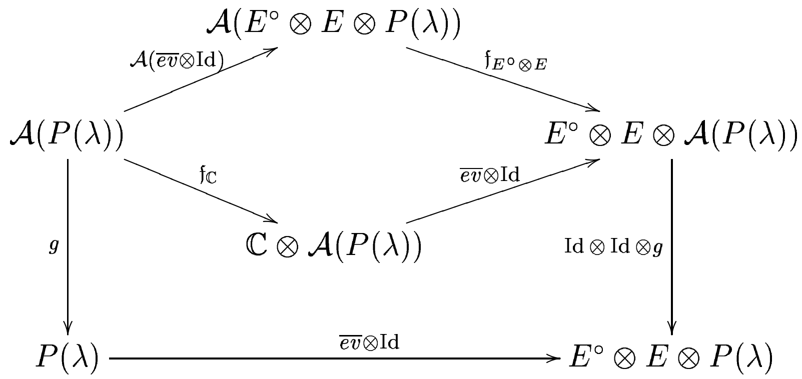
Every endomorphism of $P(\lambda)$ is scalar and hence in this case ($P_1\cong P_2\cong P(\lambda)$) the statement follows from the fact that $\mathcal{A}$ preserves the identity map.

The module $P(s_\alpha\cdot\lambda)$ has two-dimensional endomorphism ring. This module can be realized as the projection onto $\mathcal{O}^{0}_{int}$ of the module $E\otimes P(\lambda)$, where E is a simple $\mathfrak{g}(\alpha)$-module of dimension $2\lambda(H_\alpha)+3$. Under this translation the identity map on $P(\lambda)$ goes to the identity map on $P(s_\alpha\cdot\lambda)$ and the necessary statement for the identity map on $P(s_\alpha\cdot\lambda)$ follows easily. $P(s_\alpha\cdot\lambda)$ has also non-invertible and non-zero endomorphisms, but they all factor through $P(\lambda)$. Hence, checking the statement for homomorphisms from $P(\lambda)$ to $P(s_\alpha\cdot\lambda)$ and back we complete the proof.

To compute the homomorphisms between $P(\lambda)$ to $P(s_\alpha\cdot\lambda)$ we need some more notation. It is well known that the homomorphism spaces $\mathrm{Hom}_{\mathfrak{g}}(P(\lambda),P(s_\alpha\cdot\lambda))$ and $\mathrm{Hom}_{\mathfrak{g}}(P(s_\alpha\cdot\lambda),P(\lambda))$ are one-dimensional. Let θ_- and θ_+ denote translations to the α-wall and out of the α-wall and $\theta=\theta_+\circ\theta_-$ denotes the translation through the α-wall. As θ_- and θ_+ are left and right adjoint to each other, we have two natural adjunctions

$$F_1:\mathrm{Hom}(\theta_-(P(\lambda)),\theta_-(P(\lambda)))\to\mathrm{Hom}(P(\lambda),\theta(P(\lambda))),$$
$$F_2:\mathrm{Hom}(\theta_-(P(\lambda)),\theta_-(P(\lambda)))\to\mathrm{Hom}(\theta(P(\lambda)),P(\lambda)).$$

Observe that $\theta(P(\lambda))\cong P(s_\alpha\cdot\lambda)$ and we get the non-zero morphisms $F_1(\mathrm{Id}):P(\lambda)\to P(s_\alpha\cdot\lambda)$ and $F_2(\mathrm{Id}):P(s_\alpha\cdot\lambda)\to P(\lambda)$, which we choose as the basis elements in $\mathrm{Hom}_{\mathfrak{g}}(P(\lambda),P(s_\alpha\cdot\lambda))$ and $\mathrm{Hom}_{\mathfrak{g}}(P(s_\alpha\cdot\lambda),P(\lambda))$. So, to complete the proof it is enough to check the statement for $F_1(\mathrm{Id}):P(\lambda)\to P(s_\alpha\cdot\lambda)$ and $F_2(\mathrm{Id}):P(s_\alpha\cdot\lambda)\to P(\lambda)$. We start with $F_1(\mathrm{Id})$. Let now E be a simple $\mathfrak{g}(\alpha)$-module of dimension $\lambda(H_\alpha)+2$. Using induction in λ we can assume that the necessary statement is already proved for all smaller λ (starting with the singular block). Hence, we can substitute $P(s_\alpha\cdot\lambda)\cong\theta(P(\lambda))$ with $E^\circ\otimes E\otimes P(\lambda)$. Then the adjunction morphism $F_1(\mathrm{Id}):P(\lambda)\to E^\circ\otimes E\otimes P(\lambda)$ is exactly the morphism $\overline{ev}\otimes\mathrm{Id}$, which appeared in the second statement of Lemma 2. We have the following diagram:



The upper part of the diagram commutes by Lemma 2. As $\mathfrak{f}_{\mathbb{C}} = \mathrm{Id} \otimes \mathrm{Id}$ we get $(\overline{ev} \otimes \mathrm{Id}) \circ g = (\mathrm{Id} \otimes \mathrm{Id} \otimes g) \circ (\overline{ev} \otimes \mathrm{Id})$ for the lower part, which shows that this part commutes as well. Hence, the whole diagram commutes, giving us the necessary statement for the map $F_1(\mathrm{Id})$.

The statement for $F_2(\mathrm{Id})$ is proved by analogous arguments using Corollary 1. This completes the construction of $\mathfrak{r}$.

If E is finite dimensional and the top of M has only $X_{-\alpha}$-torsion-free simples, then it is clear that the top of $E \otimes M$ also has only $X_{-\alpha}$-torsion-free simples. Obviously, $E \otimes_{-}$ maps $X_{-\alpha}$-finite modules to $X_{-\alpha}$-finite modules. Hence, from the exactness of $E \otimes_{-}$ and definition of g we get that $M/\mathfrak{r}(M)$ is the maximal $X_{-\alpha}$-finite quotient of M. Using the right exactness of $\mathcal{A}$ one now easily extends this to all M.

As the map g_E is injective by construction and every indecomposable projective from $\mathcal{O}^0_{int}$ is a direct summand of $E \otimes M(0)$, we get that $\mathfrak{r}$ is injective on projective modules. This completes the proof. □

Let us consider as an example the action of $\mathcal{A}$ on the regular block of the category $\mathcal{O}$ for the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Let λ be dominant and integral. Then $\mathcal{O}^\lambda_{int}$ contains (up to isomorphism) only 5 indecomposable modules: the simple finite-dimensional module $L(\lambda)$, the simple Verma module $M(s_\alpha \cdot \lambda) = L(s_\alpha \cdot \lambda)$, the projective Verma module $M(\lambda) = P(\lambda)$, the injective envelope $I(\lambda)$ of $L(\lambda)$, and the projective-injective module $P(s_\alpha \cdot \lambda) = I(s_\alpha \cdot \lambda)$. All modules are rigid and uniserial and the radical filtrations of these modules (which at the same time are unique Loewy filtrations) can be depicted as follows:

Module N	$L(\lambda)$	$L(s_\alpha \cdot \lambda)$	$M(\lambda)$	$I(\lambda)$	$P(s_\alpha \cdot \lambda)$
$N/\mathrm{Rad}(N)$	$L(\lambda)$	$L(s_\alpha \cdot \lambda)$	$L(\lambda)$	$L(s_\alpha \cdot \lambda)$	$L(s_\alpha \cdot \lambda)$
$\mathrm{Rad}(N)/\mathrm{Rad}^2(N)$			$L(s_\alpha \cdot \lambda)$	$L(\lambda)$	$L(\lambda)$
$\mathrm{Rad}^2(N)/\mathrm{Rad}^3(N)$					$L(s_\alpha \cdot \lambda)$

The action of $\mathcal{A}$ on these modules is then described by the following statement.

Lemma 4. *1.* $\mathcal{A}(L(\lambda)) = 0$*;*

2. $\mathcal{A}(M(s_\alpha \cdot \lambda)) = I(\lambda)$*;*

3. $\mathcal{A}(M(\lambda)) = M(s_\alpha \cdot \lambda)$*;*

4. $\mathcal{A}(I(\lambda)) = I(\lambda)$*;*

5. $\mathcal{A}(P(s_\alpha \cdot \lambda)) = P(s_\alpha \cdot \lambda)$*.*

Proof. We prove this statement by direct elementary calculation and start with $L(\lambda)$. The module $L(\lambda)$ is locally $X_{-\alpha}$-finite and hence is annihilated by induction to U_α. Thus $\mathcal{A}(L(\lambda)) = 0$.

The element $X_{-\alpha}$ acts injectively on $M(s_\alpha \cdot \lambda)$. Hence, inducing $M(s_\alpha \cdot \lambda)$ to U_α, we get a dense completely pointed $\mathfrak{sl}(2,\mathbb{C})$-module, N say, of length 3 (see for example [FKM, Section 2], where these modules are denoted by $V(\lambda,\gamma)$). Factoring out the canonical image of the simple module $M(s_\alpha \cdot \lambda)$ in N gives an infinite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module of length 2, which has finite-dimensional socle. Twisting this module with respect to Θ_α we do not change the socle. But the only indecomposable module of length 2 with simple finite-dimensional socle in our list is $I(\lambda)$. Therefore $\mathcal{A}(M(s_\alpha \cdot \lambda)) = I(\lambda)$.

The element $X_{-\alpha}$ acts injectively on $M(\lambda)$. Hence, inducing $M(\lambda)$ up to U_α, we again get a dense completely pointed $\mathfrak{sl}(2,\mathbb{C})$-module of length 3, namely the same module N as before. Factoring out the canonical image of $M(\lambda)$ in N (which now has length 2) gives a simple infinite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module, which then obviously becomes the simple infinite-dimensional module in $\mathcal{O}$ after the twist with respect to Θ_α. But there is exactly one simple infinite-dimensional modules in the block, namely $M(s_\alpha \cdot \lambda)$.

The element $X_{-\alpha}$ acts locally nilpotent on the socle of $I(\lambda)$ and hence $\mathcal{A}(I(\lambda)) = \mathcal{A}(I(\lambda)/\operatorname{Soc}(I(\lambda))) = \mathcal{A}(M(s_\alpha \cdot \lambda)) = I(\lambda)$.

Finally, $X_{-\alpha}$ acts injectively on $P(s_\alpha \cdot \lambda)$. Hence, inducing $P(s_\alpha \cdot \lambda)$ up to U_α, we get a dense $\mathfrak{sl}(2,\mathbb{C})$-module with 2-dimensional weight spaces and of length 6. It is easy to see that the last module has simple top. Factoring $P(s_\alpha \cdot \lambda)$ out and twisting the module, we obviously get an indecomposable module with the same character. This one must coincide with $P(s_\alpha \cdot \lambda)$. □

2.2 Deodhar-Mathieu's version of Enright's functors

In this subsection we follow [KM, Sections 2 and 3]. For $\alpha \in \pi$ we define the functor $\mathcal{C}_M = \mathcal{C}_M^\alpha : \mathcal{O}_{int} \to \mathcal{O}_{int}$, which we call *elementary Enright's completion* (in Deodhar-Mathieu's version), in the following way:

$$M \mapsto \operatorname{Loc}_\alpha \left(\operatorname{Res}^{U_\alpha}_{U(\mathfrak{g})} \left(U_\alpha \otimes_{U(\mathfrak{g})} M \right) \right),$$

where $\operatorname{Loc}_\alpha$ denotes the functor of taking the locally X_α-finite part. It is obvious that $\mathcal{C}_M$ is covariant and that $\mathcal{C}_M \circ \mathcal{C}_M = \mathcal{C}_M$. Therefore the modules $N \in \mathcal{O}$ satisfying $\mathcal{C}_M(N) \cong N$ are called *complete*.

It follows from the definition that $\mathcal{C}_M$ is left exact. The essential part of $\mathcal{C}_M$ is the tensor induction to U_α, which is also the first part in the definition of the functor $\mathcal{A}$. It therefore follows from the corresponding property of $\mathcal{A}$ that $\mathcal{C}_M$ commutes with $E\otimes\_$ (with some efforts this can also be derived from Deodhar's paper [De]).

We would like now to illustrate the action of $\mathcal{C}_M$ on the same $\mathfrak{sl}(2,\mathbb{C})$-example as in the previous subsection.

Lemma 5. *In the notation of Lemma 4 we have:*

1. $\mathcal{C}_M(L(\lambda)) = 0$;
2. $\mathcal{C}_M(M(s_\alpha \cdot \lambda)) = M(\lambda)$;
3. $\mathcal{C}_M(M(\lambda)) = M(\lambda)$;
4. $\mathcal{C}_M(I(\lambda)) = M(\lambda)$;
5. $\mathcal{C}_M(P(s_\alpha \cdot \lambda)) = P(s_\alpha \cdot \lambda)$.

Proof. Case by case analysis analogous to that used in Lemma 4. □

2.3 Joseph's version of Enright's functors

In this subsection we follow [Jo1, Section 2]. For two $U(\mathfrak{g})$-modules M and N we denote by $\mathcal{L}(M,N)$ the space of all locally $\mathrm{ad}(g)$-finite linear maps from M to N (the so-called *maximal Harish-Chandra submodule* of $\mathrm{End}_{\mathbb{C}}(M,N)$). For a dominant regular λ we define the *elementary Enright's completion* $\mathcal{C}_J = \mathcal{C}_J^\alpha : \mathcal{O}_{int}^\lambda \to \mathcal{O}_{int}^\lambda$ (in Joseph's form) as the functor:

$$\mathcal{C}_J^\alpha = \mathcal{L}(M(s_\alpha \cdot \lambda), \_) \otimes_{U(\mathfrak{g})} M(\lambda).$$

It is the composition of the left exact functor $\mathcal{L}(M(s_\alpha \cdot \lambda), \_)$ (see [Ja, 6.8]) and the exact functor $\_ \otimes_{U(\mathfrak{g})} M(\lambda)$. Thus $\mathcal{C}_J : \mathcal{O}_{int}^\lambda \to \mathcal{O}_{int}^\lambda$ is left exact. The functor $\mathcal{C}_J$ commutes with $E\otimes\_$ (see [Jo1, 2.3]). The relation of $\mathcal{C}_J$ to $\mathcal{C}_M$ is given by the following well-known fact, [Jo1, 2.12]: if the action of $X_{-\alpha}$ on a module, $N \in \mathcal{O}_{int}^\lambda$, is injective, then $\mathcal{C}_M(N) \cong \mathcal{C}_J(N)$. In particular, the module N is $\mathcal{C}_J$-*complete* (i.e. $\mathcal{C}_J(N) \cong N$) if and only if N is complete.

The embedding $M(s_\alpha \cdot \lambda) \subset M(\lambda)$ gives a natural $U(\mathfrak{g})$-homomorphism $\mathcal{L}(M(\lambda), N) \to \mathcal{L}(M(s_\alpha \cdot \lambda), N)$ and hence a natural homomorphism $N \to \mathcal{C}_J(N)$. By [Jo1, 2.4], the kernel of this map is the largest submodule of N, the action of $X_{-\alpha}$ on which is locally finite. In particular, if the action of $X_{-\alpha}$ on N is injective, the natural homomorphism $N \to \mathcal{C}_J(N)$ is also injective.

Dualizing the arguments from [Ja, 6.9] one gets that the functor $\mathcal{L}(\_, N)$ is exact in case if N is injective. Applying this functor to the exact sequence

$$0 \to M(s_\alpha \cdot \lambda) \to M(\lambda) \to K \to 0$$

we get

$$0 \to \mathcal{L}(K, N) \to \mathcal{L}(M(\lambda), N) \to \mathcal{L}(M(s_\alpha \cdot \lambda), N) \to 0.$$

Applying $\_ \otimes_{\mathfrak{g}} M(\lambda)$ shows that in this case (when N is injective) the natural map $N \to \mathcal{C}_J(N)$ is surjective.

Using the properties of $\mathcal{C}_J$ obtained by Joseph, one can calculate the action of $\mathcal{C}_J$ for our $\mathfrak{sl}(2, \mathbb{C})$-example.

Lemma 6. *In the notation of Lemma 4 we have:*

1. $\mathcal{C}_J(L(\lambda)) = 0$;
2. $\mathcal{C}_J(M(s_\alpha \cdot \lambda)) = M(\lambda)$;
3. $\mathcal{C}_J(M(\lambda)) = M(\lambda)$;
4. $\mathcal{C}_J(I(\lambda)) = M(s_\alpha \cdot \lambda)$;
5. $\mathcal{C}_J(P(s_\alpha \cdot \lambda)) = P(s_\alpha \cdot \lambda)$.

Proof. The first statement follows from [Jo1, 3.5]. The fourth statement is [Jo1, Example 1] and everything else follows from Lemma 5 and [Jo1, 2.12]. □

2.4 Approximation with respect to an injective module

In this section we follow [Au] and [KM, Section 2]. Let A be a finite dimensional associative algebra and Λ the set of isomorphism classes of simple A-modules. The simple A-module $L(\lambda)$, $\lambda \in \Lambda$, has the projective cover $P(\lambda)$ and the injective envelope $I(\lambda)$.

Let Υ be a subset of Λ. An A-module M is Υ*-injectively cogenerated* if it is a submodule of a sum of indecomposable injective modules indexed by Υ (the latter will be called Υ-*injectives*). Further, M is called Υ*-injectively copresented* if it has a copresentation by Υ-injectives.

Denote by $P(\Upsilon)$ (resp. $I(\Upsilon)$) a direct sum of indecomposable projective (resp. injective) objects corresponding to the elements in Υ and by A_Υ the endomorphism ring of $P(\Upsilon)$. Then Υ-injectively copresented modules are exactly those A-modules M, for which the canonical morphism $M \to \mathrm{Hom}_{A_\Upsilon}(\mathrm{Hom}_A(P(\Upsilon), A), \mathrm{Hom}_A(P(\Upsilon), M))$ is an isomorphism or, equivalently, for which there exists an A_Υ-module, N say, such that M is isomorphic to the coinduced module $\mathrm{Hom}_{A_\Upsilon}(\mathrm{Hom}_A(P(\Upsilon), A), N)$. The full subcategory $\mathcal{C}(\Upsilon)$ of Υ-injectively copresented modules is equivalent to the category of A_Υ-modules, via coinduction and restriction. This gives $\mathcal{C}(\Upsilon)$ an abelian structure. With respect to this abelian structure, the inclusion $\mathcal{C}(\Upsilon) \subset A$-mod is left exact. (see [Au, 5.1, 5.4 and 5.6] for details).

Given an A-module M we can first map it to the category A_Υ-mod using the exact functor $\mathrm{Hom}_A(P(\Upsilon), \_)$ and then coinduce it to an injectively copresented module using $\mathrm{Hom}_{A_\Upsilon}(\mathrm{Hom}_A(P(\Upsilon), A), \_)$. By [Au, Section 3] the functor $\mathrm{Hom}_{A_\Upsilon}(\mathrm{Hom}_A(P(\Upsilon), A), \_)$ is right adjoint to the functor $\mathrm{Hom}_A(P(\Upsilon), \_)$.

For Subsection 2.5 we will need to know more explicitly the action of the above functors on the level of A-mod. This can be realized using the following two-step procedure.

On the first step we define the functor $\mathfrak{b}_\Upsilon : A\text{-mod} \to A\text{-mod}$ as follows: for an A-module M the module $\mathfrak{b}_\Upsilon(M)$ is the quotient of M modulo the maximal submodule, which does not contains simple subquotients of type Υ. From the definition of $\mathfrak{b}_\Upsilon$ it follows that the module $\mathfrak{b}_\Upsilon(M)$ is Υ-cogenerated for any M, that is $I_{\mathfrak{b}_\Upsilon(M)} \in add(I_\gamma)$.

On the second step we proceed with $\mathfrak{b}_\Upsilon(M)$. We define the module $\mathfrak{c}_\Upsilon(M)$ as the intersection of the kernels of all maps $I_{\mathfrak{b}_\Upsilon(M)} \to I_2$ for all choices of $I_2 \in add(I_\Upsilon)$, which send $\mathfrak{b}_\Upsilon(M)$ to zero. It is easy to see that, sending M to $\mathfrak{c}_\Upsilon(M)$, is idempotent and functorial. The resulting functor $\mathfrak{c}_\Upsilon(\_)$ is called the *approximation functor with respect to* Υ. The object $\mathfrak{c}_\Upsilon(M)$ together with the natural map $\mathfrak{c}_\Upsilon^{nat} : M \to \mathfrak{c}_\Upsilon(M)$, defined as the composition of the natural projection $M \to \mathfrak{b}_\Upsilon(M)$, followed by $\mathfrak{z} : \mathfrak{b}_\Upsilon(M) \to I_{\mathfrak{b}_\Upsilon(M)}$ (whose image obviously lies inside $\mathfrak{c}_\Upsilon(M)$), is the right approximation of M (see [AR]) in the category of Υ-copresented modules. If a module M is already Υ-cogenerated, say $M \subset I$, then $\mathfrak{c}_\Upsilon(M)$ is the largest submodule of I which contains M and such that all the composition factors of the quotient $\mathfrak{c}_\Upsilon(M)/M$ are not of type Υ. That is, $\mathfrak{c}_\Upsilon(M)$ is obtained from M by maximal coextension with non-Υ composition factors. By [Au, Section 3] the functor $\mathfrak{c}_\Upsilon$ is isomorphic to the composition of $\mathrm{Hom}_A(P(\Upsilon), \_)$ followed by $\mathrm{Hom}_{A_\Upsilon}(\mathrm{Hom}_A(P(\Upsilon), A), \_)$.

Since the functor $\mathfrak{c}_\Upsilon$ is idempotent, the modules M satisfying $M \cong \mathfrak{c}_\Upsilon(M)$ are called Υ-*complete*.

Dually, considering presentations with respect to $P(\Upsilon)$, one defines the *coapproximation functor* $\tilde{\mathfrak{c}}_\Upsilon$ and gets for every M the natural map $\tilde{\mathfrak{c}}_\Upsilon^{nat} : \tilde{\mathfrak{c}}_\Upsilon(M) \to M$, making $\tilde{\mathfrak{c}}_\Upsilon(M)$ the left approximation of M (with the properties, dual to those of right approximation). If A has a duality, D say, then it is easy to see that $\tilde{\mathfrak{c}}_\Upsilon \cong D \circ \mathfrak{c}_\Upsilon \circ D$.

By [Au, Section 3] the functors $\tilde{\mathfrak{c}}_\Upsilon$ is isomorphic to the composition of the functor $\mathrm{Hom}_A(P(\Upsilon), \_)$ followed by the functor $P(\Upsilon)\otimes_{A_\Upsilon} \_$, which is left adjoint to $\mathrm{Hom}_A(P(\Upsilon), \_)$.

Lemma 7. *The functor* $\tilde{\mathfrak{c}}_\Upsilon : A\text{-mod} \to A\text{-mod}$ *is left adjoint to the functor* $\mathfrak{c}_\Upsilon : A\text{-mod} \to A\text{-mod}$.

Proof. By adjointness, we have the following natural isomorphisms for all $M, N \in A\text{-mod}$:

$$\begin{aligned}\mathrm{Hom}_A\big(P(\Upsilon) \otimes_{A_\Upsilon} \mathrm{Hom}_A\big(P(\Upsilon), M\big), N\big) &= \\ \mathrm{Hom}_{A_\Upsilon}\big(\mathrm{Hom}_A\big(P(\Upsilon), M\big), \mathrm{Hom}_A\big(P(\Upsilon), N\big)\big) &= \\ \mathrm{Hom}_A\big(M, \mathrm{Hom}_{A_\Upsilon}\big(\mathrm{Hom}_A\big(P(\Upsilon), A\big), \mathrm{Hom}_A\big(P(\Upsilon), N\big)\big)\big).&\end{aligned}$$

□

Let us illustrate the action of $\mathfrak{c}_\Upsilon$ and $\tilde{\mathfrak{c}}_\Upsilon$ on the same $\mathfrak{sl}(2,\mathbb{C})$-example as before, choosing $\Upsilon = \{s_\alpha \cdot \lambda\}$.

Lemma 8. *In the notation of Subsection 2.1 we have:*

1. $\mathfrak{c}_{\{s_\alpha\cdot\lambda\}}(L(\lambda)) = 0$ *and* $\tilde{\mathfrak{c}}_{\{s_\alpha\cdot\lambda\}}(L(\lambda)) = 0$;
2. $\mathfrak{c}_{\{s_\alpha\cdot\lambda\}}(M(s_\alpha \cdot \lambda)) = M(\lambda)$ *and* $\tilde{\mathfrak{c}}_{\{s_\alpha\cdot\lambda\}}(M(s_\alpha \cdot \lambda)) = I(\lambda)$;

3. $\mathfrak{c}_{\{s_\alpha\cdot\lambda\}}(M(\lambda)) = M(\lambda)$ *and* $\tilde{\mathfrak{c}}_{\{s_\alpha\cdot\lambda\}}(M(\lambda)) = I(\lambda)$;

4. $\mathfrak{c}_{\{s_\alpha\cdot\lambda\}}(I(\lambda)) = M(\lambda)$ *and* $\tilde{\mathfrak{c}}_{\{s_\alpha\cdot\lambda\}}(I(\lambda)) = I(\lambda)$;

5. $\mathfrak{c}_{\{s_\alpha\cdot\lambda\}}(P(s_\alpha\cdot\lambda)) = P(s_\alpha\cdot\lambda)$ *and* $\tilde{\mathfrak{c}}_{\{s_\alpha\cdot\lambda\}}(P(s_\alpha\cdot\lambda)) = P(s_\alpha\cdot\lambda)$.

Proof. We will give a proof for $\mathfrak{c}_{\{s_\alpha\cdot\lambda\}}$, for $\tilde{\mathfrak{c}}_{\{s_\alpha\cdot\lambda\}}$ the statement follows using the duality $\star$ on $\mathcal{O}$. Since $\lambda \notin \Upsilon$ we have $\mathfrak{b}_{\{s_\alpha\cdot\lambda\}}(L(\lambda)) = 0$ and hence $\mathfrak{c}_{\{s_\alpha\cdot\lambda\}}(L(\lambda)) = 0$. Since $P(s_\alpha\cdot\lambda) = I(s_\alpha\cdot\lambda)$, the last statement is obvious.

The module $P(s_\alpha\cdot\lambda) = I(s_\alpha\cdot\lambda)$ is uniserial and has two subquotients of type Υ, namely in the socle and in the top (see Subsection 2.1). The maximal image of either $M(s_\alpha\cdot\lambda)$ or $M(\lambda)$ or $I(\lambda)$ in $P(s_\alpha\cdot\lambda)$ covers exactly the socle of $P(s_\alpha\cdot\lambda)$ (which is of type Υ). Since the composition subquotient in the middle is not of type Υ, we get $\mathfrak{c}_{\{s_\alpha\cdot\lambda\}}(M(s_\alpha\cdot\lambda)) = \mathfrak{c}_{\{s_\alpha\cdot\lambda\}}(M(\lambda)) = \mathfrak{c}_{\{s_\alpha\cdot\lambda\}}(I(\lambda)) = \mathrm{Rad}(P(s_\alpha\cdot\lambda)) = M(\lambda)$. □

Remark 1. Using Lemma 5 and Lemma 8 one gets by direct computation that the functors $\mathcal{C}_M$ and $\mathfrak{c}_{\{s_\alpha\cdot\lambda\}}$ on the regular block of the category $\mathcal{O}$ for $\mathfrak{sl}(2,\mathbb{C})$ are isomorphic.

2.5 Partial approximation with respect to an injective module

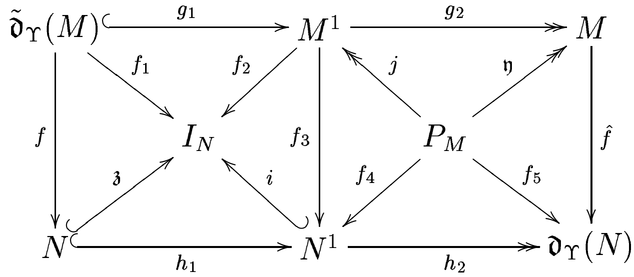
In what follows we will also need a modified version of the approximation functor defined in the previous subsection, which we keep all the notation from. We denote by $\mathfrak{d}_\Upsilon(\_)$ the functor, which is the composition of the maximal coextension with non-Υ composition factors followed by $\mathfrak{b}_\Upsilon$. To get the image of the first map on a module, M say, we realize M as a submodule of its injective envelope, I_M say, that is we start from $\mathfrak{z} : M \to I_M$. Now we can compute the maximal coextension M^1 of M with non-Υ composition subquotients as described in Subsection 2.4 (the intersection of the kernels of all possible maps from I_M to Υ-injectives, which annihilate M). This map is well-defined on modules, but not on morphisms. Indeed, if $f : M \to N$ is a homomorphism, then one can lift f to a map, $\hat{f} : M^1 \to N^1$ in a non-unique way but up to the choice of a map from M^1/M to N^1. In fact, since all composition subquotients of M^1/M are not of type Υ, the image of any map from M^1/M to N^1 is contained in the maximal submodule of N^1, which does not have any composition subquotients of type Υ. The latter one is then killed by $\mathfrak{b}_\Upsilon$. This observation implies that the map $\mathfrak{d}_\Upsilon(\_)$ is indeed functorial. Somehow, the functor $\mathfrak{d}_\Upsilon$ is obtained by "switching" the order of the two procedures, which contribute to the functor $\mathfrak{c}_\Upsilon$ as described in Subsection 2.4. We will call $\mathfrak{d}_\Upsilon(\_)$ the functor of a *partial approximation with respect to* Υ.

It is clear that $\mathfrak{d}_\Upsilon^2 \neq \mathfrak{d}_\Upsilon$ in general. However, it follows immediately from the definition that $\mathfrak{d}_\Upsilon^3 = \mathfrak{d}_\Upsilon^2 = \mathfrak{c}_\Upsilon$. It is also clear that the functor $\mathfrak{d}_\Upsilon$ comes together with a natural map $\mathfrak{d}_\Upsilon^{nat} : M \to \mathfrak{d}_\Upsilon(M)$. This map is a composition of $\mathfrak{z} : M \to I_M$, whose image lies inside M^1, followed by the natural projection of M^1 onto $\mathfrak{b}_\Upsilon(M^1)$. Moreover, it is easy to see that the kernel of the natural map $M \to \mathfrak{d}_\Upsilon(M)$ coincides with the maximal submodule of M, which does not have composition subquotients of type Υ. If the module M is injective, its coextension coincides with M and thus the map $M \to \mathfrak{d}_\Upsilon(M)$ is surjective.

Using the dual construction one defines the functor $\tilde{\mathfrak{d}}_\Upsilon$ of a *partial coapproximation with respect to* Υ. For every module M one gets a natural morphism $\tilde{\mathfrak{d}}_\Upsilon^{nat} : \tilde{\mathfrak{d}}_\Upsilon(M) \to M$ dualizing the above construction. The properties of this map are dual to those of $\mathfrak{d}_\Upsilon$. In particular, for all M the cokernel of this morphism is the maximal quotient of M, which does not have composition subquotients of type Υ. Further, $\tilde{\mathfrak{d}}_\Upsilon^{nat}$ is injective for projective M. If A has a duality, D say, then $\tilde{\mathfrak{d}}_\Upsilon \cong D \circ \mathfrak{d}_\Upsilon \circ D$.

Lemma 9. *The functor* $\tilde{\mathfrak{d}}_\Upsilon : A\text{-mod} \to A\text{-mod}$ *is left adjoint to the functor* $\mathfrak{d}_\Upsilon : A\text{-mod} \to A\text{-mod}$.

Proof. Let $M, N \in A$-mod and $f : \tilde{\mathfrak{d}}_\Upsilon(M) \to N$ be a homomorphism. We construct the diagram



in the following way: the module N^1 (resp. M^1) is the maximal coextension (resp. extension) of N (resp. M) with subquotients, which are not of type Υ. From the definition of N^1 and M^1 we have the maps $h_1 : N \to N^1$, $i : N^1 \to I_N$ such that $i \circ h_1 = \mathfrak{z}_N$, and the maps $g_2 : M^1 \to M$, $j : P_M \to M^1$ such that $g_2 \circ j = \mathfrak{y}_M$. From the definition of $\tilde{\mathfrak{d}}_\Upsilon$ and $\mathfrak{d}_\Upsilon$ we also have injection $g_1 : \tilde{\mathfrak{d}}_\Upsilon(M) \to M^1$ and surjection $h_2 : N^1 \to \mathfrak{d}_\Upsilon(N)$ such that $g_2 \circ g_1 = \tilde{\mathfrak{d}}_\Upsilon^{nat}$ and $h_2 \circ h_1 = \mathfrak{d}_\Upsilon^{nat}$ respectively.

Now we proceed to the construction of the maps f_i, $i = 1, 2, \dots, 5$, and $\hat{f}$. Set $f_1 = \mathfrak{z}_N \circ f$. Since g_1 is an injective map and I_N is an injective module, there exists $f_2 : M \to I_N$, making the corresponding triangle commutative. We remark that f_2 is not unique in general, but is defined only up to the maps from $\mathrm{Coker}(g_1)$ to I_N. Since all simple subquotients of $\mathrm{Coker}(g_1)$ are not of type Υ and N^1 is the maximal coextension of N with such subquotients, the image of f_2 belongs to N^1, giving us the map $f_3 : M^1 \to N^1$. The map f_3 depends on the choice of f_2 and thus is not uniquely defined by f. However, since the socle of $\mathfrak{d}_\Upsilon(N)$ consists only of simples, which are not of type Υ, the composition $h_2 \circ f_3$ is in fact independent of the choice of f_2 and hence is uniquely determined by f. We define $f_4 = f_3 \circ j$ and $f_5 = h_2 \circ f_4$. We have $\mathrm{Ker}(j) \subset \mathrm{Ker}(f_5)$ by construction. Further, $\mathrm{Ker}(g_2)$ contains only simple subquotients of type Υ, and the socle of $\mathfrak{d}_\Upsilon(N)$ does not contain such subquotients. This implies $\mathrm{Ker}(\mathfrak{y}_M) \subset \mathrm{Ker}(f_5)$ and thus there exists unique $\hat{f}$, which finally makes the whole diagram commutative. The commutativity of the diagram and the fact that $h_2 \circ f_3$ does not depend on the choice of f_2, implies that $\hat{f}$ is uniquely determined by f. Now one easily checks that the dual construction sends $\hat{f}$ back to f, thus providing the necessary isomorphism $\mathrm{Hom}_A(\tilde{\mathfrak{d}}_\Upsilon(M), N) = \mathrm{Hom}_A(M, \mathfrak{d}_\Upsilon(N))$. Naturality of this isomorphism follows easily from the construction. □

Corollary 2. *The functor* $\tilde{\mathfrak{d}}_\Upsilon : A\text{-mod} \to A\text{-mod}$ *is right exact and the functor* $\mathfrak{d}_\Upsilon : A\text{-mod} \to A\text{-mod}$ *is left exact.*

And again we would like to illustrate the action of $\mathfrak{d}_\Upsilon$ and $\tilde{\mathfrak{d}}_\Upsilon$ on the same $\mathfrak{sl}(2,\mathbb{C})$-example as before, choosing $\Upsilon = \{s_\alpha \cdot \lambda\}$.

Lemma 10. *In the notation of Subsection 2.1 we have:*

1. $\mathfrak{d}_{\{s_\alpha\cdot\lambda\}}(L(\lambda)) = 0$ *and* $\tilde{\mathfrak{d}}_{\{s_\alpha\cdot\lambda\}}(L(\lambda)) = 0$*;*
2. $\mathfrak{d}_{\{s_\alpha\cdot\lambda\}}(M(s_\alpha\cdot\lambda)) = M(\lambda)$ *and* $\tilde{\mathfrak{d}}_{\{s_\alpha\cdot\lambda\}}(M(s_\alpha\cdot\lambda)) = I(\lambda)$*;*
3. $\mathfrak{d}_{\{s_\alpha\cdot\lambda\}}(M(\lambda)) = M(\lambda)$ *and* $\tilde{\mathfrak{d}}_{\{s_\alpha\cdot\lambda\}}(M(\lambda)) = M(s_\alpha\cdot\lambda)$*;*
4. $\mathfrak{d}_{\{s_\alpha\cdot\lambda\}}(I(\lambda)) = M(s_\alpha\cdot\lambda)$ *and* $\tilde{\mathfrak{d}}_{\{s_\alpha\cdot\lambda\}}(I(\lambda)) = I(\lambda)$*;*
5. $\mathfrak{d}_{\{s_\alpha\cdot\lambda\}}(P(s_\alpha\cdot\lambda)) = P(s_\alpha\cdot\lambda)$ *and* $\tilde{\mathfrak{d}}_{\{s_\alpha\cdot\lambda\}}(P(s_\alpha\cdot\lambda)) = P(s_\alpha\cdot\lambda)$*.*

Proof. We again give the arguments for $\mathfrak{d}_{\{s_\alpha\cdot\lambda\}}$ and the arguments for $\tilde{\mathfrak{d}}_{\{s_\alpha\cdot\lambda\}}$ are dual.

It is clear that $L(\lambda)$ can not be coextended with non-Υ composition factors, so again we have $\mathfrak{b}_{\{s_\alpha\cdot\lambda\}}(L(\lambda)) = 0$ and hence $\mathfrak{d}_{\{s_\alpha\cdot\lambda\}}(L(\lambda)) = 0$.

Since $\mathrm{Soc}(P(s_\alpha\cdot\lambda)) = \mathrm{Soc}(M(s_\alpha\cdot\lambda)) = \mathrm{Soc}(M(\lambda)) = L(s_\alpha\cdot\lambda)$, we have $\mathfrak{d}_{\{s_\alpha\cdot\lambda\}}(M(s_\alpha\cdot\lambda)) = \mathfrak{c}_{\{s_\alpha\cdot\lambda\}}(M(s_\alpha\cdot\lambda))$, $\mathfrak{d}_{\{s_\alpha\cdot\lambda\}}(M(\lambda)) = \mathfrak{c}_{\{s_\alpha\cdot\lambda\}}(M(\lambda))$ and $\mathfrak{d}_{\{s_\alpha\cdot\lambda\}}(P(s_\alpha\cdot\lambda)) = \mathfrak{c}_{\{s_\alpha\cdot\lambda\}}(P(s_\alpha\cdot\lambda))$.

Finally, the module $I(\lambda)$ is injective and thus cannot be coextended. The maximal image of it in $I(s_\alpha\cdot\lambda)$ coincides with $M(s_\alpha\cdot\lambda)$, which completes the proof. □

Remark 2. Using Lemma 6 and Lemma 10 one shows by direct computation that on the regular block of the category $\mathcal{O}$ for $\mathfrak{sl}(2,\mathbb{C})$ we have the following:

1. The functors $\mathcal{C}_J$ and $\mathfrak{d}_{\{s_\alpha\cdot\lambda\}}$ are isomorphic.
2. The functors $\mathcal{A}$ and $\tilde{\mathfrak{d}}_{\{s_\alpha\cdot\lambda\}}$ are isomorphic.
3. The functor $\mathcal{C}_J$ is right adjoint to the functor $\mathcal{A}$.

3 A realization for $\mathcal{C}_M$

A realization of the functor $\mathcal{C}_M$ in terms of the approximation functor with respect to a suitably chosen injective module can be deduced from [KM]. However, it is not stated there, so we include the arguments. We fix a block, $\mathcal{O}^\lambda_{int}$.

Theorem 5. *Let* Υ *be the set of all* $\mu \in W\cdot\lambda$ *satisfying* $(\mu,\alpha) \leqslant 0$*. Then the functors* $\mathfrak{c}_\Upsilon$ *and* $\mathcal{C}^\alpha_M$ *are isomorphic.*

Proof. Denote by $\mathfrak{C}(\lambda)$ the full subcategory of $\mathcal{O}_{int}^{\lambda}$ whose objects are Υ-injectively copresented modules. By [Au, Section 5], the category $\mathfrak{C}(\lambda)$ is equivalent to A_{Υ}-mod and has an abelian structure, given by this equivalence. By [KM, Theorem 1] we have that $\mathcal{C}_M^{\alpha}(M) \in \mathfrak{C}(\lambda)$ for any $M \in \mathcal{O}_{int}^{\lambda}$. Moreover, from a straightforward $\mathfrak{sl}(2,\mathbb{C})$-computation (Remark 1) it follows that $\mathfrak{c}_{\Upsilon}(M) \in \mathfrak{C}(\lambda)$ for any $M \in \mathcal{O}_{int}^{\lambda}$. Let $\mathfrak{i} : \mathfrak{C}(\lambda) \to \mathcal{O}_{int}^{\lambda}$ be the inclusion functor. To complete the proof it is sufficient to show that both $\mathcal{C}_M^{\alpha}$ and $\mathfrak{c}_{\Upsilon}$ are left adjoint to $\mathfrak{i}$ (we note that for $\mathfrak{c}_{\Upsilon}$ this follows from [Au, Section 3]). Let $M \in \mathcal{O}_{int}^{\lambda}$ and $N \in \mathfrak{C}(\lambda)$. Then $X_{-\alpha}$ acts injectively on N and hence $\mathrm{Hom}_{\mathfrak{g}}(M, \mathfrak{i}(N)) = \mathrm{Hom}_{\mathfrak{g}}(M/K, \mathfrak{i}(N))$, where K is the maximal submodule of M on which $X_{-\alpha}$ acts locally nilpotent. In particular, $\mathcal{C}_M^{\alpha}(M) = \mathcal{C}_M^{\alpha}(M/K)$. But then all simple subquotients of K do not have type Υ and hence $\mathfrak{c}_{\Upsilon}(M) = \mathfrak{c}_{\Upsilon}(M/K)$. Further, one has the following canonical maps:

$$\mathrm{Hom}_{\mathfrak{g}}(\mathcal{C}_M^{\alpha}(M/K), \mathcal{C}_M^{\alpha}(\mathfrak{i}(N)))) \leftarrow \mathrm{Hom}_{\mathfrak{g}}(M/K, \mathfrak{i}(N)) \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{c}_{\Upsilon}(M/K), \mathfrak{c}_{\Upsilon}(\mathfrak{i}(N)))$$

Both these maps are injective because the module M/K is Υ-cogenerated, and, by construction, both $\mathcal{C}_M^{\alpha}$ and $\mathfrak{c}_{\Upsilon}$ do not annihilate morphisms from Υ-cogenerated modules. Moreover, the maps are also surjective because the canonical maps $M/K \to \mathcal{C}_M^{\alpha}(M/K)$ and $M/K \to \mathfrak{c}_{\Upsilon}(M/K)$ are injective on Υ-cogenerated modules. Since $N \in \mathfrak{C}(\lambda)$ we have $N = \mathcal{C}_M^{\alpha}(\mathfrak{i}(N)) = \mathfrak{c}_{\Upsilon}(\mathfrak{i}(N))$, so one gets natural isomorphisms

$$\mathrm{Hom}_{\mathfrak{g}}(\mathcal{C}_M^{\alpha}(M), N) \cong \mathrm{Hom}_{\mathfrak{g}}(M, \mathfrak{i}(N)) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{c}_{\Upsilon}(M), N)$$

which completes the proof. □

Remark 3. From the proof of Theorem 5 it follows that the functor $\mathfrak{c}_{\Upsilon} : \mathcal{O}_{int}^{\lambda} \to \mathfrak{C}(\lambda)$ is right exact. At the same time, considered as a functor $\mathfrak{c}_{\Upsilon} : \mathcal{O}_{int}^{\lambda} \to \mathcal{O}_{int}^{\lambda}$ it is left exact by Lemma 7.

It is very well-known that functors $\mathcal{C}_M^{\alpha}$, $\alpha \in \pi$, do not satisfy braid relations on the whole $\mathcal{O}_{int}^{\lambda}$ (see [Jo1, 3.15] or [KM, Section 6]). However, they satisfy braid relations on an appropriate subcategory of $\mathcal{O}$ (this was first shown in [Bo, De]). Using Theorem 5, we reduce the check of braid relations to the functors $\mathfrak{c}_{\Upsilon}$. Namely, we have:

Theorem 6. *Let $\varnothing \neq \pi_1 \subset \pi$ and $\mathcal{O}_{int}^{\lambda,\pi_1}$ be the full subcategory in $\mathcal{O}_{int}^{\lambda}$, which consists of all modules without torsion with respect to all X_{β}, $\beta \in \pi_1$. Let W_1 be the subgroup of W generated by S_{β}, $\beta \in \pi_1$, and let Υ_1 denote the set of all $\mu = w \cdot \lambda$, where w is the longest coset representative of W/W_1. Take any reduced decomposition $\mathfrak{l} : w_0^1 = s_{\alpha_1} \dots s_{\alpha_k}$ of the longest element w_0^1 of W_1, and for $i = 1, \dots, k$ set $\Upsilon_{(i)} = \{\mu \in W \cdot \lambda : (\mu, \alpha_i) \leqslant 0\}$. Then the functor $\mathfrak{c}_{\mathfrak{l}} = \mathfrak{c}_{\Upsilon_{(k)}} \circ \dots \circ \mathfrak{c}_{\Upsilon_{(1)}}$ is isomorphic to the functor $\mathfrak{c}_{\Upsilon_1}$. In particular, the functors $\mathfrak{c}_{\Upsilon_{(i)}}$ (and hence the functors $\mathcal{C}_M^{\alpha}$, $\alpha \in \pi_1$) satisfy braid relations on $\mathcal{O}_{int}^{\lambda,\pi_1}$.*

For completeness we give the corresponding argument.

Proof. Assuming $\mathfrak{c}_{\mathfrak{l}}(M) \cong \mathfrak{c}_{\Upsilon_1}(M)$ for all $M \in \mathcal{O}$, one just follows the arguments in the proof of Theorem 5 and shows that both functors are left adjoint to the inclusion functor of $\mathfrak{c}_{\mathfrak{l}}(\mathcal{O}^{\lambda}_{int})$ into $\mathcal{O}^{\lambda}_{int}$. Hence it is sufficient to show that $\mathfrak{c}_{\mathfrak{l}}(M)$ is isomorphic to $\mathfrak{c}_{\Upsilon_1}(M)$ for any $M \in \mathcal{O}$. We have natural maps $f : M \to \mathfrak{c}_{\mathfrak{l}}(M)$ and $g : M \to \mathfrak{c}_{\Upsilon_1}(M)$. Denote by K_1 and K_2 the corresponding kernels. We know that K_2 is the maximal submodule of M, which does not contain simple subquotients of type Υ_1, and that K_1 is the maximal submodule of M, which does not contain subquotients of type $\cap_i \Upsilon_{(i)}$. Clearly $\Upsilon_1 = \cap_i \Upsilon_{(i)}$ and hence $K_1 = K_2$. Denote by M_1 the module M/K_1 and we have $f(M) \cong g(M) \cong M_1$. In particular, the injective envelopes of $\mathfrak{c}_{\mathfrak{l}}(M)$ and $\mathfrak{c}_{\Upsilon_1}(M)$ coincide. Denote this envelope by I and assume that both $\mathfrak{c}_{\mathfrak{l}}(M)$ and $\mathfrak{c}_{\Upsilon_1}(M)$ are realized as submodules of I such that $f(M) = g(M)$. By the definition of approximation map we get that $\mathfrak{c}_{\mathfrak{l}}(M)/f(M)$ does not have simple subquotients of type Υ_1. By definition, $\mathfrak{c}_{\Upsilon_1}(M)$ is the maximal submodule of I, which contains $g(M)$ and such that all other simple subquotients of this module are not of type Υ_1. Hence $\mathfrak{c}_{\mathfrak{l}}(M) \subset \mathfrak{c}_{\Upsilon_1}(M)$ and to prove that these modules are isomorphic it is enough to show, say, that their characters coincide.

Let $\mathfrak{g}_1$ denote the semi-simple Lie subalgebra of $\mathfrak{g}$, generated by $X_{\pm\alpha}$, $\alpha \in \pi_1$. To prove that the characters of modules $\mathfrak{c}_{\mathfrak{l}}(M)$ and $\mathfrak{c}_{\Upsilon_1}(M)$ coincide it is sufficient to show that the restriction of both modules to the reductive algebra $\mathfrak{t} = \mathfrak{g}_1 + \mathfrak{h}$ are isomorphic. Denote by $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}}$ the restriction map.

By [KM, Lemma 4], the module $I(\Upsilon_1)$, restricted to $\mathfrak{g}_1$, is a direct sum of projective-injective $\mathfrak{g}_1$-modules from the corresponding category $\mathcal{O}$. Using the arguments, analogous to those in [KM, Lemma 4], one gets that the module $I(\Upsilon_{(i)})$, restricted to $\mathfrak{g}_1$, is a direct sum of injective $\mathfrak{g}_1$ modules, which correspond to the highest weights, defined by the longest coset representatives of of Weyl group for $\mathfrak{g}_1$ modulo the subgroup, generated by s_{α_i}. Let us denote this set by $\Upsilon'_{(i)}$.

Lemma 11. *The functors* $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}} \circ \mathfrak{c}_{\Upsilon_{(i)}}$ *and* $\mathfrak{c}_{\Upsilon'_{(i)}} \circ \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}}$ *are isomorphic as functors from the category* $\mathcal{O}$ *to* $\mathfrak{t}$-mod.

Proof. Both functors $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}} \circ \mathfrak{c}_{\Upsilon_{(i)}}$ and $\mathfrak{c}_{\Upsilon'_{(i)}} \circ \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}}$ are left exact. Moreover, from the definition of the approximation functor we also have for every module $M \in \mathcal{O}$ natural maps from $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}}(M)$ to both $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}} \circ \mathfrak{c}_{\Upsilon_{(i)}}(M)$ and $\mathfrak{c}_{\Upsilon'_{(i)}} \circ \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}}(M)$. Hence one verifies that all conditions of the Comparison Lemma (Lemma 1) with $F = \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}} \circ \mathfrak{c}_{\Upsilon_{(i)}}$, $G = \mathfrak{c}_{\Upsilon'_{(i)}} \circ \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}}$ and $H = \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}}$ are satisfied and thus the application of the Comparison Lemma completes the proof. □

Obviously, Lemma 11 (and induction in the rank of the algebra) reduces the check of the necessary statement to the rank two case for $\mathfrak{g} = \mathfrak{g}_1$, which we now assume until the end of the proof. In particular, we assume that $\pi_1 = \{\alpha, \beta\}$. To proceed we will need one more technical notion.

For a $\mathfrak{g}$-module, N, which does not have simple subquotients of type $\Upsilon_1 = \cap_i \Upsilon_{(i)} = \{w_0 \cdot \lambda\}$, we will call the $\mathfrak{l}$-*height* of N the length m of a shortest filtration $0 = N_0 \subset N_1 \subset \dots \subset N_m = N$ such that there is an increasing function $f : \{1, \dots, m\} \to \{1, \dots, k\}$ such that for every $i \in \{1, \dots, m\}$ the quotient N_i/N_{i-1} does not have simple subquotients of type $\Upsilon_{(f(i))}$. Such filtrations will be called $\mathfrak{l}$-*admissible.*

For example, it is easy to see that the $\mathfrak{l}$-*height* of the module $M(\lambda)/M(w_0 \cdot \lambda)$ equals $l(w_0)$. From the definition of $\Upsilon_{(i)}$ it follows that the category of all modules in $\mathcal{O}$, which do not have simple subquotients of type $\Upsilon_{(i)}$, is stable under translation functors. Since all projectives can be obtained via translations of $M(\lambda)$ and the top of any translation of the simple Verma module $M(w_0 \cdot \lambda)$ is a direct sum of simple Verma modules, we get that the $\mathfrak{l}$-height of all maximal quotients without simple subquotients of type $\{w_0 \cdot \lambda\}$ of all projective modules in $\mathcal{O}$ is not greater than that of the projective Verma module. This implies that $\mathfrak{l}$-height of any module without subquotients of type Υ_1 is at most $l(w_0)$.

As we are now in the rank two case, we have only two translations through the walls, namely θ_α and θ_β. It is easy to see that the first subquotient in an admissible filtration for $M(\lambda)$ is annihilated by one of the translations and the last subquotient is annihilated by another one. This implies that for all indecomposable projective modules, which are not isomorphic to $M(\lambda)$ the $\mathfrak{l}$-height is in fact at most $l(w_0) - 1$.

Now we take the simple root $\gamma \in \pi_1$, which is different from α_k. Remark that we obviously have $\mathfrak{c}_{\Upsilon_{(k)}} \circ \mathfrak{c}_\mathfrak{l}(M) \cong \mathfrak{c}_\mathfrak{l}(M)$. If $\mathfrak{c}_\gamma \circ \mathfrak{c}_\mathfrak{l}(M) \cong \mathfrak{c}_\mathfrak{l}(M)$, then the module $\mathfrak{c}_\mathfrak{l}(M)$ is Υ_1-complete and we are done. Otherwise we get a filtration

$$0 \subset M \subset \mathfrak{c}_{\Upsilon_{(1)}}(M) \subset \cdots \subset \mathfrak{c}_\mathfrak{l}(M) \subset \mathfrak{c}_\gamma \circ \mathfrak{c}_\mathfrak{l}(M),$$

where all the inclusion except the first one ($M \subset \mathfrak{c}_{\Upsilon_{(1)}}(M)$) must be proper (in the case of a non-proper inclusion we already get a Υ_1-complete module). Hence the $\mathfrak{l}$-height of the module $N = \mathfrak{c}_\gamma \circ \mathfrak{c}_\mathfrak{l}(M)/M$ must be either $l(w_0) + 1$ (in the case of all proper inclusions) or $l(w_0)$ (if the first inclusion is not proper). The first case is impossible as we know that the $\mathfrak{l}$-height is at most $l(w_0)$. From the above arguments it follows that the only quotient of a maximal $\mathfrak{l}$-height of an indecomposable projective in $\mathcal{O}$ is $M(\lambda)/M(w_0 \cdot \lambda)$. Hence, N must have a submodule, which is isomorphic to $M(\lambda)/M(w_0 \cdot \lambda)$. And this must extend some submodule of M. Since $M(\lambda)$ is projective, this submodule can only be a direct summand, isomorphic to $M(w_0 \cdot \lambda)$. However, since we know that $\mathfrak{c}_\gamma \circ \mathfrak{c}_\mathfrak{l}(M(w_0 \cdot \lambda)) \cong \mathfrak{c}_\mathfrak{l}(M(w_0 \cdot \lambda))$, for this submodule the process should have terminated already after $\mathfrak{c}_\mathfrak{l}$. A contradiction. This completes the proof. □

Let Υ be as above, $\hat{\mathfrak{C}}(\lambda)$ be the full subcategory of $\mathcal{O}_{int}^\lambda$ whose objects are Υ-projectively presented modules, and $\hat{\mathfrak{i}} : \hat{\mathfrak{C}}(\lambda) \to \mathcal{O}_{int}^\lambda$ be the inclusion functor. Dualizing arguments in the proof of Theorem 5 one gets that $(\mathfrak{c}_\Upsilon(\_)^\star))^\star$ is right adjoint to $\hat{\mathfrak{i}}$. On the other hand the category $\hat{\mathfrak{C}}(\lambda)$ is equivalent to subcategory ${}^\infty_{I_\lambda}\mathcal{H}^1_{I_\nu}$ of the category of Harish-Chandra bimodules, where $I_\mu = \mathrm{Ann}_{U(\mathfrak{g})} M(\mu)$ for $\mu \in \mathfrak{h}^*$ and $\nu \in \mathfrak{h}^*$ is an integral weight satisfying $s_\alpha \cdot \nu = \nu$ and $(\nu, \beta) > 0$ for all $\beta \in \Delta^+ \setminus \{\alpha\}$ (see e.g. [Ja]). The equivalence is given by the functor $\_ \otimes_{U(\mathfrak{g})} M(\nu)$, whose right adjoint functor is $\mathcal{L}(M(\nu), \_)$. Now, using Theorem 5, one gets another description of $\mathcal{C}_M$.

Theorem 7. *Let $\nu \in \mathfrak{h}^*$ be an integral weight satisfying $s_\alpha \cdot \nu = \nu$ and $(\nu, \beta) > 0$ for all $\beta \in \Delta^+ \setminus \{\alpha\}$. Then the functors $\mathcal{C}_M$ and $\big(\mathcal{L}(M(\nu), (\_)^\star) \otimes_{U(\mathfrak{g})} M(\nu)\big)^\star$ are isomorphic as endofunctors of $\mathcal{O}_{int}^\lambda$.*

4 A realization for $\mathcal{C}_J$

As we have already mentioned, it is not difficult to deduce the results of the previous section from [KM]. In the present section we are going to discuss some results, which seem to be new. They are mainly inspired by Remark 2. As in the previous section, we fix a block, $\mathcal{O}_{int}^{\lambda}$, but now assume that λ is regular.

Theorem 8. *Let Υ be the set of all $\mu \in W \cdot \lambda$ satisfying $(\mu, \alpha) \leqslant 0$. Then the functors $\mathfrak{d}_\Upsilon$ and $\mathcal{C}_J^\alpha$ are isomorphic.*

Proof. To prove the theorem we will use the Comparison Lemma, and hence we would now like to check all the necessary assumptions. We start with the remark that both functors are left exact. Now let us check that for any injective module I we have $\mathcal{C}_J(I) \cong \mathfrak{d}_\Upsilon(I)$.

We start with $\mathcal{C}_J(M(\lambda)^\star) \cong \mathfrak{d}_\Upsilon(M(\lambda)^\star)$. To see this we first remark that $\mathcal{C}_J(M(\lambda)^\star) \cong M(s_\alpha \cdot \lambda)^\star$ ([Jo2, Lemma 2.5]) and that the module $N = M(\lambda)/M(s_\alpha \cdot \lambda)$ is the maximal $X_{-\alpha}$-locally finite quotient of $M(\lambda)$. Hence the dual $N^\star$ of N is a maximal $X_{-\alpha}$-locally finite submodule in the injective module $M(\lambda)^\star$. Since $M(\lambda)^\star$ is injective, we get that $\mathfrak{d}_\Upsilon(M(\lambda)^\star) \cong M(\lambda)^\star/N^\star$ and thus $\mathfrak{d}_\Upsilon(M(\lambda)^\star) \cong M(s_\alpha \cdot \lambda)^\star \cong \mathcal{C}_J(M(\lambda)^\star)$.

We know that $\mathcal{C}_J$ commutes (on the whole category $\mathcal{O}_{int}$) with the functor $E\otimes\_$ for any finite-dimensional $\mathfrak{g}$-module E. Let us now show that $\mathfrak{d}_\Upsilon$ also commutes with $E\otimes\_$ (on $\mathcal{O}_{int}$). We can formulate this as the following lemma, where we abuse notation and denote by $\mathfrak{d}_\Upsilon$ the endofunctor on $\mathcal{O}_{int}$, which is a direct sum of corresponding endofunctors on all blocks:

Lemma 12. *Let E be a finite dimensional $\mathfrak{g}$-module. The functors $\mathfrak{d}_\Upsilon(E\otimes\_)$ and $E\otimes\mathfrak{d}_\Upsilon(\_)$ from $\mathcal{O}_{int}$ to $\mathcal{O}_{int}$ are isomorphic.*

Proof. Both functors $\mathfrak{d}_\Upsilon(E\otimes\_)$ and $E\otimes\mathfrak{d}_\Upsilon(\_)$ are left exact as compositions of left exact $\mathfrak{d}_\Upsilon$ with exact $E\otimes\_$. We also have natural transformations $\big(E\otimes\mathrm{Id}\big)\circ\mathfrak{d}_\Upsilon^{nat} : E\otimes\_ \to E\otimes\mathfrak{d}_\Upsilon(\_)$ and $\mathfrak{d}_\Upsilon^{nat}\circ\big(E\otimes\mathrm{Id}\big) : E\otimes\_ \to \mathfrak{d}_\Upsilon(E\otimes\_)$.

Let M be an injective module and N be the maximal submodule of M, which contains only subquotients not of type Υ, or, equivalently, the maximal $X_{-\alpha}$-locally finite submodule of M. As M is injective, there is a short exact sequence

$$0 \to N \to M \overset{\mathfrak{d}_\Upsilon^{nat}}{\to} \mathfrak{d}_\Upsilon(M) \to 0.$$

Applying to this sequence the exact functor $E\otimes\_$ gives the short exact sequence

$$0 \to E\otimes N \to E\otimes M \overset{\mathrm{Id}\otimes\mathfrak{d}_\Upsilon^{nat}}{\to} E\otimes\mathfrak{d}_\Upsilon(M) \to 0,$$

and the module $E\otimes N$ is the maximal $X_{-\alpha}$-locally finite submodule of $E\otimes M$.

On the other hand $E\otimes M$ is injective and we have the following short exact sequence:

$$0 \to K \to E\otimes M \overset{\mathfrak{d}_\Upsilon^{nat}}{\to} \mathfrak{d}_\Upsilon(E\otimes M) \to 0,$$

where K is the maximal $X_{-\alpha}$-locally finite submodule of $E \otimes M$. Hence $K = E \otimes N$ and we get that $E \otimes \mathfrak{d}_{\Upsilon}(M) \cong \mathfrak{d}_{\Upsilon}(E \otimes M)$ and $\mathrm{Ker}(\mathrm{Id} \otimes \mathfrak{d}_{\Upsilon}^{nat}(M)) = \mathrm{Ker}(\mathfrak{d}_{\Upsilon}^{nat}(\mathrm{Id} \otimes M)) = E \otimes N$.

Now the statement follows from the Comparison Lemma for $F = \mathfrak{d}_{\Upsilon}(E \otimes \_)$, $G = E \otimes \mathfrak{d}_{\Upsilon}(\_)$, $H = E \otimes \_$. □

Now we know that $\mathcal{C}_J(M(\lambda)^{\star}) \cong \mathfrak{d}_{\Upsilon}(M(\lambda)^{\star})$ and that both $\mathcal{C}_J$ and $\mathfrak{d}_{\Upsilon}$ commute with $E \otimes \_$ for any finite-dimensional $\mathfrak{g}$-module E. Translating $M(\lambda)^{\star}$ we can get (as a direct summand) any indecomposable projective in $\mathcal{O}_{int}^{\lambda}$ and hence, using standard induction with respect to the natural order on $\mathfrak{h}^*$, we get that $\mathcal{C}_J(I(\mu)) \cong \mathfrak{d}_{\Upsilon}(I(\mu))$ for any indecomposable injective module $I(\mu) \in \mathcal{O}_{int}^{\lambda}$. From this we derive that $\mathcal{C}_J(I) \cong \mathfrak{d}_{\Upsilon}(I)$.

From the definition of $\mathfrak{d}_{\Upsilon}$ it follows that the natural morphism $\mathfrak{d}_{\Upsilon}^{can} : I \to \mathfrak{d}_{\Upsilon}(I)$ is surjective on injective modules (see Subsection 2.5). In Subsection 2.3 we also saw that the same statement holds for $\mathcal{C}_J$ as well. Moreover, in Subsection 2.5 and Subsection 2.3 we also mentioned that the kernels of the these natural morphisms for any module M always coincide with the maximal $X_{-\alpha}$-locally finite submodule of M. Hence all conditions of the Comparison Lemma are satisfied for $F = \mathcal{C}_J$, $G = \mathfrak{d}_{\Upsilon}$, $H = \mathrm{Id}$, and the application of this lemma completes the proof. □

Following Theorem 6, this result possess the following straightforward modification:

Theorem 9. *Let $\varnothing \neq \pi_1 \subset \pi$, W_1 be the subgroup of W generated by S_β, $\beta \in \pi_1$, and let Υ_1 denote the set of all $\mu = w \cdot \lambda$, where w is the longest coset representative of W/W_1. Take any reduced decomposition $w_0^1 = s_{\alpha_1} \dots s_{\alpha_k}$ of the longest element w_0^1 of W_1, and for $i = 1, \dots, k$ set $\Upsilon_{(i)} = \{\mu \in W \cdot \lambda : (\mu, \alpha_i) \leqslant 0\}$. Then the functor $\mathfrak{d}_{\mathfrak{l}} = \mathfrak{d}_{\Upsilon_{(k)}} \circ \dots \circ \mathfrak{d}_{\Upsilon_{(1)}}$ is isomorphic to the functor $\mathfrak{d}_{\Upsilon_1}$ (as endofunctors on $\mathcal{O}_{int}^{\lambda}$). In particular, the functors $\mathfrak{c}_{\Upsilon_{(i)}}$ (and hence the functors $\mathcal{C}_M^{\alpha}$, $\alpha \in \pi_1$) satisfy braid relations.*

Proof. Both functors are left exact and one computes that they have isomorphic values on $M(\lambda)^{\star}$. From Lemma 12 we get that both functors $\mathfrak{d}_{\mathfrak{l}}$ and $\mathfrak{d}_{\Upsilon_1}$ commute with $E \otimes \_$ and hence one derives that $\mathfrak{d}_{\mathfrak{l}}(I) \cong \mathfrak{d}_{\Upsilon_1}(I)$ for all injective I. Moreover, it is easy to see that the corresponding natural morphisms $\mathfrak{d}_{\mathfrak{l}}(I) \to I$ and $\mathfrak{d}_{\Upsilon_1}(I) \to I$ are surjective. One checks that the kernels of these morphisms coincide and the statement follows from the Comparison Lemma applied to $F = \mathfrak{d}_{\mathfrak{l}}$, $G = \mathfrak{d}_{\Upsilon_1}$ and $H = \mathrm{Id}$. □

As another corollary we get the following functorial generalization of [Jo1, 2.12]:

Corollary 3. *The functors $\mathcal{C}_M^{\alpha}$, $\alpha \in \pi$, and $\mathcal{C}_J^{\alpha}$, considered as endofunctors on the full subcategory $\mathcal{O}_{int}^{\lambda,\alpha}$ of $\mathcal{O}_{int}^{\lambda}$, consisting of $X_{-\alpha}$ torsion-free modules, are isomorphic.*

Proof. Follows directly from Theorems 5, 8 and observation that the functors $\mathfrak{c}_{\Upsilon_1}$ and $\mathfrak{d}_{\Upsilon}$ are obviously isomorphic on $\mathcal{O}_{int}^{\lambda,\alpha}$. □

This gives us an alternative proof of Theorem 6 and it is quite interesting to compare its complexity with our original proof by direct arguments:

Corollary 4. *The functors $\mathcal{C}_M^{\alpha}$, $\alpha \in \pi$, satisfy braid relations on $\cap_{\alpha \in \pi} \mathcal{O}_{int}^{\lambda,\alpha}$.*

Proof. Follows from Theorem 9 and Corollary 3. □

One more interesting result, which we get immediately from Theorem 8 and Lemma 9 is the following.

Corollary 5. *The endofunctor $\mathcal{C}_J^\alpha$, $\alpha \in \pi$, on $\mathcal{O}_{int}^\lambda$ is right adjoint to the functor $\star \circ \mathcal{C}_J^\alpha \circ \star = (\mathcal{C}_J((\_)^\star))^\star$.*

Remark 4. Originally Enright constructed the completion functors for $\mathfrak{sl}(2,\mathbb{C})$ and defined the completion functor for $\mathfrak{g}$-modules by restricting to $\mathfrak{sl}(2,\mathbb{C})$-subalgebra (generated by $X_{\pm\alpha}$), applying completion on the $\mathfrak{sl}(2,\mathbb{C})$ level and reconstructing the $\mathfrak{g}$-module structure in unique way (see [En, Proposition 3.6]). Obviously, the functor $\mathfrak{d}_\Upsilon$ can also be given such a description. Hence, using Theorem 8 one gets that Joseph's version of the completion functor can also be defined restricting the module to the $\mathfrak{sl}(2,\mathbb{C})$ subalgebra, completing it there and then canonically reconstructing the resulting $\mathfrak{g}$-module.

Remark 5. If λ is integral but singular and $s_\alpha \cdot \lambda = \lambda$, the functor $\mathcal{C}_J$ on $\mathcal{O}_{int}^\lambda$ becomes the identity. However, Theorem 8 allows one to extend the definition of $\mathcal{C}_J$ in an alternative way (using $\mathfrak{d}_\Upsilon$ instead) to all singular blocks of $\mathcal{O}_{int}$, keeping the properties, analogous to those of $\mathcal{C}_J$ for the regular block.

5 A realization for $\mathcal{A}$

In this section we present a realization for Arkhipov's functor $\mathcal{A}^\alpha$. It happens that here the answer reduces to the partial (co)approximations as in the previous section. As in Section 3, we fix a block, $\mathcal{O}_{int}^\lambda$.

Theorem 10. *Let Υ denote the set of all $\mu \in W \cdot \lambda$ satisfying $(\mu, \alpha) \leqslant 0$. Then the functors $\mathcal{A}^\alpha$ and $\tilde{\mathfrak{d}}_\Upsilon$ on $\mathcal{O}_{int}^\lambda$ are isomorphic.*

Proof. The functors $\mathcal{A}^\alpha$ and $\tilde{\mathfrak{d}}_\Upsilon$ are right exact and we are going to use the statement, dual to the Comparison Lemma. From Subsections 2.1 we know that if P is projective in $\mathcal{O}_{int}$, then the natural map $\mathfrak{r} : \mathcal{A}^\alpha(P) \to P$ is injective. In Subsections 2.5 we also saw that the map $\tilde{\mathfrak{d}}_\Upsilon^{nat} : \tilde{\mathfrak{d}}_\Upsilon(P) \to P$ is injective as well. Moreover, the cokernels of both maps coincide with the maximal $X_{-\alpha}$-locally finite quotient of P. In particular, $\tilde{\mathfrak{d}}_\Upsilon(P) \cong \mathcal{A}^\alpha(P)$. The statement now follows from the dual of the Comparison Lemma. □

Corollary 6. *Let λ be regular.*

1. *The functor $\mathcal{A}^\alpha$ is left adjoint to the functor $\mathcal{C}_J^\alpha$.*
2. *The functor $\mathcal{A}^\alpha$ is left adjoint to the functor $\star \circ \mathcal{A}^\alpha \circ \star = (\mathcal{A}((\_)^\star))^\star$.*
3. *The functor $\mathcal{A}^\alpha$ is isomorphic to the functor $\star \circ \mathcal{C}_J^\alpha \circ \star = (\mathcal{C}_J((\_)^\star))^\star$.*
4. *The functor $\mathcal{C}_J^\alpha$ is isomorphic to the functor $\star \circ \mathcal{A}^\alpha \circ \star = (\mathcal{A}((\_)^\star))^\star$.*

Proof. Follows from Theorem 10, Theorem 8 and Lemma 9. □

We remark that the second statement of Corollary 6 appears in [AS, Theorem 4.1] in a much more general setup. From the last corollary and Theorem 9 we immediately obtain the following (we will give an independent proof of this fact in the next section):

Corollary 7. *Let λ be regular. The functors $\mathcal{A}^{\alpha}$, $\alpha \in \pi$, satisfy braid relations on $\mathcal{O}_{int}^{\lambda}$.*

As one more corollary we can in some sense describe the value of $\mathcal{A}^{\alpha}$ on simple and injective modules:

Corollary 8. *Let Υ be as in Theorem 10. Then the following holds:*

1. *If $\lambda \notin \Upsilon$ then $\mathcal{A}^{\alpha}(L(\lambda)) = 0$.*
2. *If $\lambda \in \Upsilon$ then $\mathcal{A}^{\alpha}(L(\lambda)) = (\mathcal{C}_M(L(\lambda)))^{\star}$.*

Proof. The first statement is clear as $L(\lambda)$ for $\lambda \notin \Upsilon$ is locally $X_{-\alpha}$-finite. For the second statement we apply the third statement of Corollary 6 to get $\mathcal{A}^{\alpha}(L(\lambda)) = (\mathcal{C}_J((L(\lambda))^{\star}))^{\star}$. The module $L(\lambda)$ is self-dual. Moreover, for $\lambda \in \Upsilon$ the module $L(\lambda)$ is $X_{-\alpha}$ torsion-free and hence $\mathcal{C}_J((L(\lambda))^{\star}) \cong \mathcal{C}_J(L(\lambda)) \cong \mathcal{C}_M(L(\lambda))$ by Corollary 3. This completes the proof. □

Corollary 9. *If I is an injective module, then the module $\mathcal{A}^{\alpha}(I)$ is also injective. Moreover, the restriction of $\mathcal{A}^{\alpha}$ to the subcategory of all injective modules is isomorphic to the identity functor.*

Proof. The third statement of Corollary 6 and Theorem 8 reduce the statement to analogous statement for projective modules and the functor $\mathfrak{d}_{\Upsilon}$. Let us show that all projective modules in $\mathcal{O}_{int}$ are complete. If λ is dominant, then $M(\lambda) \subset I(w_0 \cdot \lambda)$ and the cokernel of this inclusion is filtered by Verma modules. This implies that $M(\lambda)$ is complete and the general statement for all projectives is obtained by applying $E\otimes\_$ to the inclusion above. Thus all projective modules in $\mathcal{O}_{int}$ are complete and the statement follows. □

Corollary 10. *Let λ be regular. The functor $\mathcal{A}^{\alpha}$ on $\mathcal{O}_{int}^{\lambda}$ is isomorphic to the functor $\mathcal{L}(M(\lambda), \_) \otimes_{U(\mathfrak{g})} M(s_{\alpha} \cdot \lambda)$.*

Proof. It is enough to show that $\mathcal{L}(M(\lambda), \_) \otimes_{U(\mathfrak{g})} M(s_{\alpha} \cdot \lambda)$ is left adjoint to $\mathcal{C}_J^{\alpha}$, the last being defined as $\mathcal{L}(M(s_{\alpha} \cdot \lambda), \_) \otimes_{U(\mathfrak{g})} M(\lambda)$. From [Ja, Kapitel 6] it follows that the functor $\_ \otimes_{U(\mathfrak{g})} M(\lambda)$ is an equivalence of categories with inverse $\mathcal{L}(M(\lambda), \_)$, and $\_ \otimes_{U(\mathfrak{g})} M(s_{\alpha} \cdot \lambda)$ is left adjoint to $\mathcal{L}(M(s_{\alpha} \cdot \lambda), \_)$. This completes the proof. □

6 Braid relations

Let α and β be two simple roots and $\mathfrak{n}(\alpha, \beta)$ be the Lie subalgebra in $\mathfrak{g}$, generated by $\mathfrak{g}_{\alpha}$ and $\mathfrak{g}_{\beta}$. Let α_j, $j = 1, \dots, k$, be a list of all roots of $\mathfrak{g}$ such that $\mathfrak{g}_{\alpha_j} \subset \mathfrak{n}(\alpha, \beta)$. In particular, both α and β occur in this list and $\mathfrak{n}(\alpha, \beta) = \oplus_{j=1}^{k} \mathfrak{g}_{\alpha_j}$. Since every X_{α_j}, $j = 1, \dots, n$, is locally ad-nilpotent on $U(\mathfrak{g})$, we get that the multiplicative subset T, generated by these elements, is an Ore subset in $U(\mathfrak{g})$, see [Ma, Lemma 4.2]. Denote by U_T the localization of $U(\mathfrak{g})$ with respect to T. For $j = 1, \dots, k$ we denote by $\mathcal{S}_{\alpha_j}$ the $U(\mathfrak{g}) - U(\mathfrak{g})$-bimodule $U_{\alpha_j}/U(\mathfrak{g})$, where U_{α_j} is the localization of $U(\mathfrak{g})$ with respect to the powers of X_{α_j}, see Subsection 2.1. Finally, let B denote the $U(\mathfrak{g}) - U(\mathfrak{g})$-subbimodule of U_T, generated by all monomials in U_T, which do not contain at least one of the elements $X_{\alpha_j}^{-1}$, $j = 1, \dots, k$ (alternatively, if for $i = 1, \dots, k$ we denote by T_i the multiplicative subset of $U(\mathfrak{g})$, generated by X_{α_j}, $j \neq i$, then $B = \sum_{i=1}^{k} U_{T_i} \subset U_T$). Denote by $\mathcal{S}(\alpha, \beta)$ the $U(\mathfrak{g}) - U(\mathfrak{g})$-bimodule U_T/B. We start with a PBW-type theorem for $\mathcal{S}(\alpha, \beta)$.

Lemma 13. *Let $Z_1, \dots, Z_{\dim(\mathfrak{g}) - \dim(\mathfrak{n}(\alpha,\beta))}$ be the elements from the Weyl-Chevalley basis in $\mathfrak{g}$, which are not contained in $\mathfrak{n}(\alpha, \beta)$. Then monomials*

$$Z_1^{i_1} Z_2^{i_2} \dots Z_l^{i_l} X_{\alpha_1}^{-a_1} \dots X_{\alpha_k}^{-a_k}, \tag{3}$$

where $i_1, i_2, \dots, i_l \in \mathbb{Z}_+$, $a_1, \dots, a_k \in \mathbb{N}$, form a basis of $\mathcal{S}(\alpha, \beta)$ over $\mathbb{C}$.

Proof. First we show that these monomials are linearly independent. For $j = 0, \dots, k$ let $\hat{T}_j$ denote the multiplicative subset of $U(\mathfrak{g})$, generated by X_{α_i}, $i \leqslant j$. Let B_j denote the $U(\mathfrak{g}) - U(\mathfrak{g})$-subbimodule of $U_{\hat{T}_j}$, generated by all monomials, which do not contain at least one of the elements $X_{\alpha_i}^{-1}$, $i = 1, \dots, j$. We show by induction in j that the monomials

$$Z_1^{i_1} Z_2^{i_2} \dots Z_l^{i_l} X_{\alpha_1}^{-a_1} \dots X_{\alpha_j}^{-a_j},$$

where $i_1, i_2, \dots, i_l \in \mathbb{Z}_+$, $a_1, \dots, a_j \in \mathbb{N}$, are linearly independent in $U_{\hat{T}_j}/B_j$.

Indeed, for $j = 0$ the statement follows from the classical PBW Theorem. Now let us prove the induction step. Assume that the statement is not true and we have a non-trivial linear combination of our monomials:

$$\sum c_{i_1,\dots,i_l,a_1,\dots,a_j} Z_1^{i_1} Z_2^{i_2} \dots Z_l^{i_l} X_{\alpha_1}^{-a_1} \dots X_{\alpha_j}^{-a_j} = 0 \tag{4}$$

Let l be the maximal positive integer such that $c_{i_1,\dots,i_l,a_1,\dots,a_{j-1},l} \neq 0$. Multiplying (4) with $X_{\alpha_j}^{l-1}$ from the right we get that, as element in $U_{\hat{T}_j}$,

$$f = \sum c_{i_1,\dots,i_l,a_1,\dots,a_{j-1},l} Z_1^{i_1} Z_2^{i_2} \dots Z_l^{i_l} X_{\alpha_1}^{-a_1} \dots X_{\alpha_{j-1}}^{-a_{j-1}} X_{\alpha_j}^{-1} \in B_j. \tag{5}$$

Further, since $B_{j-1} X_{\alpha_j}^{-1} \subset B_j$, we have the following isomorphisms of left $U(\mathfrak{g})$-modules:

$$\big(U_{\hat{T}_{j-1}} X_{\alpha_j}^{-1} + B_j\big)/B_j \overset{\psi}{\cong} U_{\hat{T}_{j-1}} X_{\alpha_j}^{-1} / B_{j-1} X_{\alpha_j}^{-1} \overset{\varphi}{\cong} U_{\hat{T}_{j-1}}/B_{j-1},$$

where the isomorphism φ is given by the right multiplication with X_{α_j} and it's inverse is the right multiplication with $X_{\alpha_j}^{-1}$. Now $f \in U_{\hat{T}_{j-1}} X_{\alpha_j}^{-1}$ and (5) imply

$$\varphi(\psi(f + B_j)) = \sum c_{i_1,\dots,i_l,a_1,\dots,a_{j-1},l} Z_1^{i_1} Z_2^{i_2} \dots Z_l^{i_l} X_{\alpha_1}^{-a_1} \dots X_{\alpha_{j-1}}^{-a_{j-1}} = 0 \text{ in } U_{\hat{T}_{j-1}}/B_{j-1},$$

which contradicts the inductive assumption. This shows that the monomials (3) are linearly independent.

Now let us prove that the monomials (3) span $\mathcal{S}(\alpha, \beta)$. Since T is an Ore subset, every element from U_T and hence from $\mathcal{S}(\alpha, \beta)$ can be written as us^{-1}, where $u \in U(\mathfrak{g})$ and $s \in T$. Further, in every monomial from $U(\mathfrak{g})$ we can collect the elements from T to the right. Let us first show that these elements from T can always be canceled.

We choose a filtration, $0 = F_0 \subset F_1 \subset \dots$ of $\mathfrak{n}(\alpha, \beta)$ such that each F_i is an ideal of codimension 1 in F_{i+1} and $F_i = F_{i-1} \oplus \mathfrak{g}_{\alpha_{\sigma(i)}}$, where σ is a permutation of $1, 2, \dots, k$.

Obviously, the elements $Z_1^{i_1} Z_2^{i_2} \dots Z_l^{i_l} X_{\alpha_1}^{a_1} \dots X_{\alpha_k}^{a_k} t$, where $i_1, i_2, \dots, i_l, a_1, \dots, a_k \in \mathbb{Z}_+$ and $t \in T$, span $\mathcal{S}(\alpha, \beta)$. Let us show by induction in i that the element $X_{\alpha_{\sigma(i)}}$ can be canceled. For $i = 1$ the element $X_{\alpha_{\sigma(1)}}$ commutes with $\mathfrak{n}(\alpha, \beta)$ and hence with all monomials in $X_{\alpha_j}^{-1}$ as well. Recall that non-zero monomials in $\mathcal{S}(\alpha, \beta)$ must contain all $X_{\alpha_j}^{-1}$ at least once by definition. Hence we can commute $X_{\alpha_{\sigma(1)}}$ to $X_{\alpha_{\sigma(1)}}^{-1}$ and cancel them. Now let us prove the induction step. By induction and classical PBW theorem we can assume that we have already canceled all $X_{\alpha_{\sigma(i)}}$, $i < j$, and that $X_{\alpha_{\sigma(j)}}$ now stays at the rightmost place. As for all s we have that $[X_{\alpha_{\sigma(j)}}, X_{\alpha_{\sigma(s)}}] = cX_{\alpha_{\sigma(t)}}$ for some $t < j$ and $t < s$, we get

$$[X_{\alpha_{\sigma(j)}}, X_{\alpha_{\sigma(s)}}^{-1}] = cX_{\alpha_{\sigma(s)}}^{-1} X_{\alpha_{\sigma(t)}} X_{\alpha_{\sigma(s)}}^{-1}. \qquad (6)$$

Using this we commute $X_{\alpha_{\sigma(j)}}$ to the corresponding $X_{\alpha_{\sigma(j)}}^{-1}$ and cancel them. The additional terms, which appear during this process, are dealt with by inductive assumption as $t < j$. Therefore we have shown that the elements $Z_1^{i_1} Z_2^{i_2} \dots Z_l^{i_l} t$, $t \in T$, span $\mathcal{S}(\alpha, \beta)$.

Finally, let us now show that we can always rearrange the elements $X_{\alpha_j}^{-1}$ in the necessary order. We use induction, analogous to the one above. As $X_{\alpha_{\sigma(1)}}$ commutes with $\mathfrak{n}(\alpha, \beta)$, the element $X_{\alpha_{\sigma(1)}}^{-1}$ commutes with all $X_{\alpha_{\sigma(j)}}^{-1}$ and hence can be placed at any position. Now, the equality $[X_{\alpha_{\sigma(j)}}, X_{\alpha_{\sigma(s)}}] = cX_{\alpha_{\sigma(t)}}$ implies

$$[X_{\alpha_{\sigma(j)}}^{-1}, X_{\alpha_{\sigma(s)}}^{-1}] = cX_{\alpha_{\sigma(j)}}^{-1} X_{\alpha_{\sigma(s)}}^{-1} X_{\alpha_{\sigma(t)}} X_{\alpha_{\sigma(s)}}^{-1} X_{\alpha_{\sigma(j)}}^{-1}.$$

Hence, the elements either commute (if $c = 0$) or after commutation we necessarily get an additional term of higher degree, but which contains $X_{\alpha_{\sigma(t)}}$ with $t < j$ and $t < s$. By induction and (6), this element can be commuted and canceled with $X_{\alpha_{\sigma(t)}}^{-1}$, decreasing the total degree with respect to $X_{\alpha_{\sigma(t)}}^{-1}$ with $t < j$. But as soon as one of $X_{\alpha_{\sigma(t)}}^{-1}$ disappears from the monomial, the result will be zero in $\mathcal{S}(\alpha, \beta)$. Hence, by induction in this degree, the process of commutation will successfully terminate in a finite number of steps in all additional terms. This completes the proof. □

Theorem 11. *The $U(\mathfrak{g}) - U(\mathfrak{g})$-bimodules $\mathcal{S}(\alpha, \beta) = U_T/B$ and*

$$\mathcal{S}_{\alpha_1} \otimes_{U(\mathfrak{g})} \mathcal{S}_{\alpha_2} \otimes_{U(\mathfrak{g})} \dots \otimes_{U(\mathfrak{g})} \mathcal{S}_{\alpha_k}$$

are canonically isomorphic. In particular, the second bimodule does not depend on the order of $\{\alpha_j\}$.

Proof. Multiplication induces a natural homomorphism of $U(\mathfrak{g})-U(\mathfrak{g})$-bimodules from $\mathcal{S}_{\alpha_1}\otimes_{U(\mathfrak{g})}\mathcal{S}_{\alpha_2}\otimes_{U(\mathfrak{g})}\cdots\otimes_{U(\mathfrak{g})}\mathcal{S}_{\alpha_k}$ to $\mathcal{S}(\alpha,\beta)=U_T/B$, which is bijective by Lemma 13. $\square$

Corollary 11. *The functors $\mathcal{A}^\alpha$, $\alpha\in\pi$, satisfy braid relations.*

Proof. Let $\dots s_\beta s_\alpha s_\beta=\dots s_\alpha s_\beta s_\alpha=w$ be a braid relation in the Weyl group. By [Ar2, Section 2.3], the functor $\mathcal{A}^\alpha$ is exactly the functor $\Theta_\alpha\circ\mathcal{S}_\alpha$. If we denote by Θ_w the twist with respect to the automorphism of $\mathfrak{g}$, which corresponds to w (this one certainly does not depend on the expression for w), we get that the left hand side of the braid relation for Arkhipov's functor reads

$$\Theta_w\circ\dots\circ(S_{s_\beta s_\alpha(\beta)}\otimes_{U(\mathfrak{g})}-)\circ(S_{s_\beta(\alpha)}\otimes_{U(\mathfrak{g})}-)\circ(S_\beta\otimes_{U(\mathfrak{g})}-).$$

So, up to the twist by Θ_w, we get a composition of tensor products with elementary Arkhipov's bimodules, which can be written as the tensor product with the $U(\mathfrak{g})-U(\mathfrak{g})$-bimodule

$$\mathcal{S}_{left}=\dots S_{s_\beta s_\alpha(\beta)}\otimes_{U(\mathfrak{g})}S_{s_\beta(\alpha)}\otimes_{U(\mathfrak{g})}S_\beta.$$

The latter one, in fact, corresponds to some ordering of the basis of $\mathfrak{n}(\alpha,\beta)$, consisting of Weyl-Chevalley generators. By Theorem 11, the $U(\mathfrak{g})-U(\mathfrak{g})$-bimodule $\mathcal{S}_{left}$ is isomorphic to $\mathcal{S}(\alpha,\beta)$. Certainly the same arguments apply to the right hand side of the braid relation as well and we get that, up to the Θ_w twist, both sides correspond to the functor of tensor product with the same bimodule. This completes the proof. $\square$

Corollary 11 gives an alternative proof for Theorem 9 and Theorem 6.

Corollary 12. *Let λ be dominant and integral. The functors $\mathcal{C}_J^\alpha$, $\alpha\in\pi$, satisfy braid relations on $\mathcal{O}_{int}^\lambda$.*

Proof. Follows from Lemma 6 and Corollary 11. $\square$

Corollary 13. *Let λ be dominant and integral. The functors $\mathcal{C}_M^\alpha$, $\alpha\in\pi$, satisfy braid relations on $\mathcal{O}_{int}^{\lambda,\alpha}$.*

Proof. Follows from Corollary 12 and Corollary 3. $\square$

Remark 6. One also gets that the functors $\mathfrak{d}_{\Upsilon_{(i)}}$ (notation as in Theorem 9) satisfy braid relations on $\mathcal{O}_{int}$.

7 Appendix: Θ_α is the identity if one "glues" $\mathcal{O}$ and $\Theta_\alpha(\mathcal{O})$

The arguments, used in the proof of Theorem 4, generalize in a natural way to the following situation: Denote by $\mathfrak{p}$ the parabolic subalgebra $\mathfrak{g}(\alpha)+\mathfrak{h}+\mathfrak{n}_+$ of $\mathfrak{g}$. Take $c\in\mathbb{C}$ such that $c\neq(n+1/2)^2$ and $c\neq n^2$ for all $n\in\mathbb{Z}$. Denote by $V^{(i)}(c)$, $i=1,2$, the simple weight dense $\mathfrak{g}(\alpha)$-module, which is uniquely defined by the following two conditions:

1. All weights of $V^{(1)}(c)$ are integral and all weights of $V^{(2)}(c)$ are integral plus 1/2 (these are usually called half-integral).

2. c is the eigenvalue on $V^{(i)}(c)$ of the "small" Casimir element $(H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha$.

The restriction on $V^{(i)}(c)$ to be dense is equivalent to the fact that $V^{(i)}(c)$ does not have neither highest nor lowest weight, which is, in turn, equivalent to the fact that $X_{-\alpha}$ (and X_α) acts bijectively on $V^{(i)}(c)$, see for example [FKM, Section 2] for more details. Let $\Lambda = \Lambda(V^{(i)}(c))$ be the full subcategory in the category of all $\mathfrak{g}(\alpha)$-modules, which consists of all subquotients of all modules having the form $E \otimes V^{(i)}(c)$, E finite-dimensional. Finally we denote by $\mathcal{O}(\mathfrak{p}, \Lambda)$ the full subcategory in the category of all $\mathfrak{g}$-modules, which consists of all modules M, which are finitely generated, $\mathfrak{h}$-diagonalizable, locally finite over the nilpotent radical of $\mathfrak{p}$, and which are direct sums (usually infinite) of modules from Λ, when viewed as $\mathfrak{g}(\alpha)$-modules (see for example again [FKM, Section 2]). This $\mathcal{O}(\mathfrak{p}, \Lambda)$ can be viewed as a result of "gluing together" some subcategories of $\mathcal{O}$ and $\Theta_\alpha(\mathcal{O})$.

It is clear that $\Theta_\alpha((H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha) = (H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha$. Since the H_α-eigenvalues on all modules in $\mathcal{O}(\mathfrak{p}, \Lambda)$ are either integers or half-integers, one gets that Θ_α preserves $\mathcal{O}(\mathfrak{p}, \Lambda)$, thus defining a natural covariant involutive equivalence on $\mathcal{O}(\mathfrak{p}, \Lambda)$. But one can even get a stronger result using the following observation.

It is quite obvious that the functor Θ_α on $\mathcal{O}(\mathfrak{p}, \Lambda)$ satisfies $\Theta_\alpha(P) \cong P$ for all projective modules P. Moreover, it is easy to see that the arguments, analogous to those, used in the proof of Theorem 4, also work for the category $\mathcal{O}(\mathfrak{p}, \Lambda)$, and one gets the following statement:

Theorem 12. *There is a natural morphism,* $\mathfrak{m} : \Theta_\alpha \to \mathrm{Id}$*, considered as functors on* $\mathcal{O}(\mathfrak{p}, \Lambda)$*. Moreover, for all modules M from* $\mathcal{O}(\mathfrak{p}, \Lambda)$ *one has that the natural map* $\mathfrak{m} : \Theta_\alpha(M) \to M$ *is an isomorphism. In particular, the functors* Θ_α *and* Id *are isomorphic.*

Acknowledgments

An essential part of the research which led to the results written in this paper was done during the visit of the authors to The Fields Institute for Research in Mathematical Sciences, whose financial support is gratefully acknowledged. The first author thanks the German Research Council (DFG) for financial support within the scope of the project "Projektive Funktoren und Hecke algebra". The second author was partially supported by The Royal Swedish Academy of Science and The Swedish Research Council. Some parts of the paper were written during the visit of the first author to Uppsala and of the second author to Leicester. We thank the host Universities for their hospitality and The Royal Swedish Academy of Sciences and EPSRC for financial support of these visits. We are especially indebted to Catharina Stroppel for many helpful and stimulating discussions and for various remarks, which led to the improvements in this paper. We also thank Wolfgang Soergel for helpful discussions and remarks.

References

[AL] H.H.Andersen, N.Lauritzen, Twisted Verma modules, in: Studies in Memory of Issai Schur, 1-26, v. 210, Progress in Math., Birkhäuser, Basel, 2002.

[AS] H.H.Andersen, C.Stroppel, Twisting functors on $\mathcal{O}_0$, Preprint 2003, available at: "http://home.imf.au.dk/mathha/preprints.html".

[Ar1] S.Arkhipov, Semi-infinite cohomology of associative algebras and bar duality. Internat. Math. Res. Notices 1997, no. 17, 833–863.

[Ar2] S.Arkhipov, Algebraic construction of contragradient quasi-Verma modules in positive characteristics, Preprint MPI 2001-34, Max-Planck Institute für Mathematik, 2001.

[Au] M.Auslander, Representation theory of Artin algebras I. Commun. in Alg. 1 (1974), 177-268.

[AR] M.Auslander and I.Reiten, Applications of contravariantly finite subcategories. Adv. Math. 86, 111-152 (1991).

[BGG] I.N.Bernstein, I.M.Gelfand, S.I.Gelfand, A certain category of $\mathfrak{g}$-modules. (Russian) Funkcional. Anal. i Prilozen. 10 (1976), no. 2, 1–8.

[BG] J.N. Bernstein and S.I. Gelfand, Tensor products of finite- and infinite-dimensional representations of semi-simple Lie algebras. Compositio Math. 41 (1980), 245–285.

[Bo] A.Bouaziz, Sur les représentations des algèbres de Lie semi-simples construites par T.Enright. In: Non-commutative harmonic analysis and Lie groups, Springer LN Mathematics 880 (1981), 57-68.

[De] V.V.Deodhar, On a construction of representations and a problem of Enright, Invent. Math. 57 (1980), 101-118.

[En] T.J.Enright, On the fundamental series of a real semisimple Lie algebra: their irreducibility, resolutions and multiplicity formulae, Annals Math., 110 (1979), 1-82.

[FKM] V.Futorny, S.König and V.Mazorchuk, A combinatorial description of blocks in $\mathcal{O}(P,\Lambda)$ associated with $\mathfrak{sl}(2,\mathbb{C})$-induction, J. Algebra 231 (2000), no. 1, 86–103.

[Ja] J.C.Jantzen, Einhüllende Algebren halbeinfacher Lie-Algebren. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 3. Springer-Verlag, Berlin-New York, 1983.

[Jo1] A. Joseph, The Enright functor on the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. Invent. Math. 67 (1982), no. 3, 423–445.

[Jo2] A. Joseph, Completion functors in the $\mathcal{O}$ category. Noncommutative harmonic analysis and Lie groups (Marseille, 1982), 80–106, Lecture Notes in Math., 1020, Springer, Berlin, 1983.

[Jo3] A. Joseph, Quantum groups and their primitive ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 29. Springer-Verlag, Berlin, 1995.

[KM] S.König, N.Mazorchuk, Enright's completions and injectively copresented modules. Trans. Amer. Math. Soc. 354 (2002), no. 7, 2725–2743.

[Ma] O.Mathieu, Classification of irreducible weight modules, Ann. Inst. Fourier (Grenoble), 50 (2000), 537–592.

[RW] A.Rocha-Caridi, N.R.Wallach, Projective modules over graded Lie algebras. I. Math. Z. 180 (1982), no. 2, 151–177.

[So1] W.Soergel, Charakterformeln für Kipp-Moduln über Kac-Moody-Algebren. Represent. Theory 1 (1997), 115–132 (electronic).

[So2] W.Soergel, Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. J. Amer. Math. Soc. 3 (1990), no. 2, 421–445.

O.K.: Mathematisches Institut der Universität Freiburg, Eckestrasse 1, D-79104, Freiburg im Breisgau, FRG, e-mail: oleksandr.khomenko@math.uni-freiburg.de

V.M.: Department of Mathematics, Uppsala University, Box 480, SE-751 06, Uppsala, SWEDEN, e-mail: mazor@math.uu.se