A note on simplicity of Generalized Verma modules

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Mathematical Subject Classification: Primary 17B10; Secondary 22E47.

Abstract

With each generalized Verma module over a complex simple finite-dimensional Lie algebra we associate a Verma module over the same algebra. We conjecture that simplicity of this Verma module implies the simplicity of the generalized Verma module. We prove this conjecture in a generic case.

1 Introduction

Generalized Verma modules (GVM) are the natural generalization of the famous Verma modules ([D, BGG]). They were studied intensively by several authors, see [RC, CF, FM, MO] and references therein. However, their structure is understood only in a few number of particular cases, see [FM, KM1, KM2, M, MO]. In this note we present one general conjecture on the structure of arbitrary GVM and prove it in a generic case. In fact, with each GVM we associate a Verma module and state that the simplicity of this Verma module implies the simplicity of the GVM.

2 Notations

For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ its universal enveloping algebra and by $Z(\mathfrak{A})$ the center of $U(\mathfrak{A})$.

Let $\mathfrak{G}$ be a simple complex finite-diminsional Lie algebra with a fixed Cartan subalgebra $\mathfrak{H}$ and the corresponding root system Δ with the chosen base π. This data lead to the standard triangular decomposition of $\mathfrak{G}$ and to the decomposition of Δ into positive and negative roots. For $\alpha \in \Delta$ let $\mathfrak{G}_\alpha$ denote the root space of $\mathfrak{G}$ corresponding to the root α. Let W be the Weyl group of Δ and let $\cdot$ denote its action on $\mathfrak{H}^*$. Let ρ denote the half-sum of all positive roots. For $\lambda \in \mathfrak{H}^*$ let $M(\lambda)$ denote the Verma module associated with λ and π ([D, Chapter 7]).

Fix a subset $S \subset \pi$. Let $\mathfrak{G}^S$ be a (semi-simple) subalgebra of $\mathfrak{G}$ generated by all $\mathfrak{G}_{\pm\alpha}$, $\alpha \in S$. Let Δ^S be the root subsystem of Δ with the base S and W^S be the corresponding Weyl group. Let $\mathfrak{H}_S$ denote the subspace of all $h \in \mathfrak{H}$ such that $\alpha(h) = 0$ for all $\alpha \in S$. Consider a subalgebra $\mathfrak{N}_\pm(S)$ generated by all $\mathfrak{G}_\alpha$, where α is a positive root in Δ which does not belong to Δ^S, and set $\mathfrak{B}(S) = \mathfrak{G}^S \oplus \mathfrak{H}_S \oplus \mathfrak{N}_+(S)$. Let ρ^S be the half-sum of all positive roots from Δ^S and $\rho_S = \rho - \rho^S$.

A $\mathfrak{G}$-module V is said to be an S-weight module if the action of $\mathfrak{H}_S$ is diagonalizable on V. For S-weight modules we retain all standard notations. A non-zero S-weight element v in an S-weight $\mathfrak{G}$-module will be called S-primitive if $\mathfrak{N}_+(S)v = 0$.

Consider a simple $\mathfrak{G}^S$ module V (not necessarily weight) and $\lambda \in \mathfrak{H}_S^*$. Setting $hv = (\lambda - \rho_S)(h)v$ and $av = 0$ for $h \in \mathfrak{H}_S$, $a \in \mathfrak{N}_+(S)$ and $v \in V$ we turn V into a $\mathfrak{B}(S)$-module. The $\mathfrak{G}$-module

$$M(\lambda, V) = U(\mathfrak{G}) \bigotimes_{U(\mathfrak{B}(S))} V$$

is called generalized Verma module (GVM) associated with V and λ. Basic properties of GVM can be found, for example, in [CF]. We note only that $M(\lambda, V)$ is an S-weight $\mathfrak{G}$-module and all S-primitive generators of $M(\lambda, V)$ are $v \in M(\lambda, V)_{\lambda - \rho_S} \simeq V$.

Unlike the classical case of Verma modules it is not clear if for any GVM $M(\lambda, V)$ and any $\mu \in \mathfrak{H}_S^*$ the corresponding $\mathfrak{G}^S$-module $M(\lambda, V)_\mu$ is of finite length or not. Using [K, Theorem 3.5] we can state that $M(\lambda, V)_\mu$ is of finite length in the case when V is a Harish-Chandra module (i.e. one coming from a group representation).

3 Basic construction

Let $M(\lambda, V)$ be a GVM. Then V has a central character, say χ_V, as a $\mathfrak{G}^S$-module by Quillen's lemma. By [D, Proposition 7.4.8] there exists a Verma module $M(\mu)$ over $\mathfrak{G}^S$ for some $\mu \in (\mathfrak{H}^S)^*$ having the same central character (χ_V). By [D, Proposition 7.4.7], a Verma module $M(\nu)$ over $\mathfrak{G}^S$, $\nu \in (\mathfrak{H}^S)^*$ has central character χ_V if and only if $\nu \in W^S(\mu)$. Among these Verma modules we choose $M(\nu)$ with ν lying in the anti-dominant Weyl chamber (with respect to S). Under this choice $M(\nu)$ is always simple. Consider a GVM $M(\lambda, M(\nu))$. One can easily show that $M(\lambda, M(\nu)) \simeq M(\lambda + \nu)$. Set $M(\lambda + \nu) = \mathfrak{f}(M(\lambda, V))$.

4 Main result

Conjecture 1. *If $\mathfrak{f}(M(\lambda, V))$ is simple then $M(\lambda, V)$ is simple.*

Remark 1. *We believe that in this conjecture one can replace $\mathfrak{G}$ by a symmetrizable Kac-Moody Lie algebra (but $\mathfrak{G}^S$ should remain finite dimensional one).*

Unfortunately we can not prove this conjecture in the complete generality. In what follows we will prove it in the so-called *generic* situation. In fact we will prove the following:

Theorem 1. *Suppose that the support of $M(\lambda + \nu) = \mathfrak{f}(M(\lambda, V))$ intersects the set $(W \cdot (\lambda + \nu)) - \rho$ only in the highest weight. Then $M(\lambda, V)$ is simple.*

Remark 2. *From the BGG criterion ([BGG, Theorem 2,3]) it follows that $M(\lambda+\nu)$ is simple under the conditions of Theorem 1.*

We call this situation generic since $(W\cdot\xi)-\rho$ intersect the support of $M(\xi)$ only in the highest weight for almost all Verma modules. Thus the above theorem can be applied in almost all cases and states that almost all GVMs are simple.

Proof. Consider $M(\lambda+\nu)$ as a GVM $M(\lambda, M(\nu))$. In this proof all weight spaces are assumed to be S-weight spaces and all weights are assumed to be S-weights. Suppose that $M(\lambda, V)$ is not simple and have a non-trivial submodule N. Clearly, N contains a non-zero S-primitive element $v\in M(\lambda, V)_\mu$, $\lambda\neq\mu$. Obviously, the $\mathfrak{G}^S$ module $M(\lambda, V)_\mu$ is obtained from V via tensor product with a with a finite dimensional module, say F. Thus by [K, Theorem 5.1] we can assume that $Z(\mathfrak{G}^S)$ acts on v via some character, say θ.

Consider a $\mathfrak{G}^S$ module $M(\lambda, M(\nu))_\mu$. Clearly, it is obtained from $M(\nu)$ via tensor product with F. Moreover, since $M(\nu)$ is a Verma module, then $M(\nu)\otimes F$ has a Verma flag ([D, Lemma 7.6.14]). Comparing the filtration in [D, Lemma 7.6.14] with [K, Theorem 5.1] we conclude that necessarily one of Verma sub-quotients of $M(\lambda, M(\nu))_\mu$, say $M(\xi)$, has the central character θ.

Consider the Verma module $M(\xi+\mu)$ as $M(\mu, M(\xi))$ and let w be its canonical generator. Since w is an S-primitive element we can calculate the action of $Z(\mathfrak{G})$ on w in terms of θ and μ using the S-Harish-Chandra homomorphism ([DFO, Proposition 2]). Moreover, the same can be done for v. This implies that the central characters of $M(\xi+\mu)$ and $M(\lambda, V)$ and thus the central characters of $M(\xi+\mu)$ and $M(\lambda+\nu)$ coincides. From Harish-Chandra theorem ([D, Theorem 7.4.7]) we obtain immediately that $\xi+\mu\in W\cdot(\lambda+\nu)$, which contradicts conditions of our theorem. This completes the proof. $\square$

Remark 3. *Using the S-Harish-Chandra homomorphism ([DFO, Proposition 2]) and [K, Theorem 3.5] one can also prove that $M(\lambda, V)$ has a (finite) composition series if V is a Harish-Chandra module.*

5 Acknowledgments

For the second author this research was partially supported by CRDF grant UM1-327. The paper was completed during the stay of the second author in Bielefeld University as an Alexander von Humboldt research fellow. The financial supports of CRDF and Humboldt Foundation are gratefully acknowledged.

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