Structure of modules induced from simple modules with minimal annihilator

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Abstract

We study the structure of generalized Verma modules over a semi-simple complex finite-dimensional Lie algebra, which are induced from simple modules over a parabolic subalgebra. We consider the case when the annihilator of the starting simple module is a minimal primitive ideal if we restrict this module to the Levi factor of the parabolic subalgebra. We show that these modules correspond to proper standard modules in some parabolic generalization of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ and prove that the blocks of this parabolic category are equivalent to certain blocks of the category of Harish-Chandra bimodules. From this we derive, in particular, an irreducibility criterion for generalized Verma modules. We also compute the composition multiplicities of those simple subquotients, which correspond to the induction from simple modules whose annihilators are minimal primitive ideals.

1 Introduction

Let $\mathfrak{g}$ be a semi-simple complex finite-dimensional Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$ and $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. With every $\mathfrak{p}$-module V one can associate the *parabolically induced* module $M_{\mathfrak{p}}(V) = U(\mathfrak{g})\otimes_{U(\mathfrak{p})}(V)$. If V is simple, the module $M_{\mathfrak{p}}(V)$ is called a *generalized Verma module* (GVM in the sequel). Every generalized Verma module $M_{\mathfrak{p}}(V)$ has the unique simple quotient, denoted by $L_{\mathfrak{p}}(V)$. The usual Verma modules are obtained in the case when $\mathfrak{p}$ is a Borel subalgebra of $\mathfrak{g}$. Generalized Verma modules can be applied for example to study the structure of usual Verma modules for different algebras arising in theoretical physics, [Se]. The structure of various classes of generalized Verma modules was investigated by many authors. In particular, Verma submodules of Verma modules and simple subquotients of Verma modules were described by Bernstein, I.Gelfand and S.Gelfand in [BGG], the multiplicity problem in this case (Kazhdan-Lusztig conjecture) was solved by Brylinski-Kashiwara, [BK], and Beilinson-Bernstein, [BB], for integral case, and this result was extended to the general case by Soergel in [S1]. The generalized Verma modules induced from finite-dimensional modules were studied by Jantzen in [J1] nd by Rocha-Caridi in [R]. The generalized Verma modules induced from infinite-dimensional weight modules, in particular from Gelfand-Zetlin modules, were studied by Futorny, Khomenko, Mazorchuk and Ovsienko in [FM, KM1, KM2, KM3, MO] (see also references therein). The generalized Verma modules induced from Whittaker modules were studied by McDowell in [Mc1, Mc2], by Miličić and Soergel in [MS1, MS2] and by Backelin in [Ba]. For these special cases of simple modules the structure of the corresponding GVMs is now relatively well-understood. Although these families of modules are rather big, "almost all" simple $\mathfrak{g}$-modules are not of this type. Unfortunately, for general V the only known result is the simplicity criterion for GVMs associated with $sl(2,\mathbb{C})$-induction (the case when the semi-simple part of the Levi factor of $\mathfrak{p}$ is isomorphic to $sl(2,\mathbb{C})$) obtained in [KM3].

If one compares all the classes of simple modules studied in the papers listed above (with the exception of finite-dimensional modules), then it is not difficult to see that all these modules have one thing in common: their annihilators are smallest possible, which means that they are minimal primitive ideals of the universal enveloping algebra for the semi-simple part $\mathfrak{a}$ in the Levi factor $\tilde{\mathfrak{a}}$ of $\mathfrak{p}$ (we also denote by $\mathfrak{h}_{\mathfrak{a}}$ the center of $\tilde{\mathfrak{a}}$). In [MS1] the authors proved that the GVMs they consider belong to some category of $\mathfrak{g}$-modules, which is equivalent to a certain category of Harish-Chandra bimodules in the sense of [BG]. An analogous result for GVMs induced from generic Gelfand-Zetlin modules was proved in [FKM2, KoM1, KoM2]. The similarity of these results was a good motivation to try to find a general approach to this problem, which will be free from the necessity to restrict consideration to special simple modules. We hope to do this in the present paper by extending the arguments of Miličić and Soergel from [MS1]. We will see that these arguments work smoothly only under assumption that the simple module we start with has a minimal possible annihilator. In particular, these arguments do not work for finite-dimensional modules, however, this case is already handled. Moreover, there is a price to pay for the ambitions to work with arbitrary simple module (even with a fixed minimal annihilator). It happens that the structure of the corresponding GVM can be studied only "roughly", that is up to subquotients induced from simple modules with bigger annihilators. After the example, constructed by Stafford in [St], it is theoretically possible that some GVMs have infinite length. However, this phenomena is not visible on our "rough" level.

Associated with $M_{\mathfrak{p}}(V)$ are two full subcategories in the category of $\mathfrak{g}$-modules: the first one consists of all subquotients of modules $F \otimes M_{\mathfrak{p}}(V)$, $\dim(F) < \infty$, and the second one consists of those modules, which has a presentation by certain projective modules from the first one. The second category carries information about the "rough" structure of modules. We construct a right exact functor, $\mathfrak{C}$, from the first category to the second one, which preserves the "rough" structure of a module. The restriction of this functor to $\mathcal{O}$ is the twist of the (global) Enright's completion functor, see [KoM1], by duality. Using this construction we prove:

Theorem 1. *Let V be a simple $\mathfrak{p}$-module with a minimal possible annihilator (over $\mathfrak{a}$). Then there exists a simple $\mathfrak{p}$-module, $\tilde{V}$, such that the module $\mathfrak{C}(M_{\mathfrak{p}}(V))$ is a proper standard* *module in the parabolic category of modules, presentable by modules of the form* $F \otimes M_{\mathfrak{p}}(\tilde{V})$, $\dim(F) < \infty$. *This category has a block decomposition with blocks equivalent to certain blocks of the category of Harish-Chandra bimodules.*

In particular, all GVMs have finite length in the blocks given by the above theorem, moreover, one can get complete information about simple subquotients of GVMs corresponding to modules induced from simples with minimal annihilators. Nevertheless, this information is enough to derive an irreducibility criterion for GVMs, which is free from all this "rough" business. In fact, having V, we can consider a simple Verma module, $M(\lambda)$, over $\tilde{\mathfrak{a}}$ with the same annihilator. In a natural way $M(\lambda)$ extends to a $\mathfrak{p}$-module with the trivial action of the nilpotent radical $\mathfrak{n}$ of $\mathfrak{p}$. In the sequel we will often use this construction to get GVMs from simple $\tilde{\mathfrak{a}}$-modules. Our main result is the following statement:

Theorem 2. *Let* V *be a simple module with a minimal possible annihilator. Then* $M_{\mathfrak{p}}(V)$ *is simple if and only if* $M_{\mathfrak{p}}(M(\lambda))$ *is simple.*

Moreover, we even can compute some composition multiplicities of $M_{\mathfrak{p}}(V)$. Although $M_{\mathfrak{p}}(V)$ might have an infinite lengths, it is easy to see that these composition multiplicities of "rough" simple subquotients are well-defined and finite.

Theorem 3. *Let* V_i, $i = 1, 2$, *be simple modules with minimal possible annihilators. Let* $M_{\mathfrak{p}}(M(\lambda_i))$ *be corresponding Verma modules described above. Assume that the inequality* $[M_{\mathfrak{p}}(V_1) : L_{\mathfrak{p}}(V_2)] > 0$ *holds. Then*

$$[M_{\mathfrak{p}}(V_1) : L_{\mathfrak{p}}(V_2)] = [M_{\mathfrak{p}}(M(\lambda_1)), L_{\mathfrak{p}}(M(\lambda_2))].$$

We also provide an analogue of the classical BGG-criterion for $[M_{\mathfrak{p}}(V_1) : L_{\mathfrak{p}}(V_2)]$ to be positive reducing it to the corresponding question in the category $\mathcal{O}$. This covers and generalizes the result of Miličić and Soergel on Whittaker modules ([MS1]), the result of König and Mazorchuk on GVMs, induced from generic Gelfand-Zetlin modules ([KoM1, KoM2]), and the result of the authors on modules induced from dense $sl(2)$ modules ([KM5]).

In the case of $sl(2, \mathbb{C})$-induction our results are most general. Indeed, in this case any simple module is either finite-dimensional or has a minimal possible annihilator. So, the combination of our results with those of Rocha-Caridi gives a "rough" classification of the categories of parabolically induced modules, generated by a simple module. We have to note that in this case all GVMs always have finite length as $\mathfrak{g}$-modules. This follows from the fact that all simple $sl(2, \mathbb{C})$-modules are holonomic as well as all their tensor products with finite-dimensional modules.

The "rough" and precise structures of GVMs we consider coincide provided that the simple module V we started with satisfies the following condition: the length of the module $F \otimes V$ is equal to $\dim(F)$ for every finite-dimensional module F. This is the case for instance for Whittaker or Gelfand-Zetlin modules. However, this is not true in general, and as an example one can take simple Verma modules with regular integral weights.

The paper is organized as follows: in Section 2 we adjust the arguments of Miličić and Soergel (which are modifications of the arguments due to Bernstein and Gelfand, see [BG]) to our situation. In Section 3 we apply them to study the categories of $\mathfrak{a}$-modules, which are obtained if one tensors a given simple module having a minimal annihilator with finite-dimensional modules. Section 4 is devoted to the study of the categories of modules, obtained via parabolic induction from the categories studied in Section 3. We prove that these categories have a block decomposition with blocks being equivalent to the module categories of finite-dimensional algebras, moreover, they are also equivalent to suitably chosen categories of Harish-Chandra bimodules. In Section 5 we derive some corollaries of our results which we apply to study GVMs. In Section 6 we give a "rough" classification of categories of modules, which can be obtained from $sl(2,\mathbb{C})$ via parabolic induction. We complete the paper discussing the case of other annihilators in Section 7, in particular, we present some examples which show that the structure of the categories appearing in this case differs from that in previous cases. For example, we show that these categories of induced modules are no longer described neither by quasi-hereditary nor by properly stratified algebras (here we refer the reader to [CPS, Dl] for details on these classes of associative algebras). In fact the following is true:

Proposition 1. *Let V be arbitrary simple $\mathfrak{a}$-module, which is projective in the full subcategory of $\mathfrak{g}$-modules consisting of all subquotients of modules $F \otimes V$, $\dim(F) < \infty$. Then the category of modules, presentable by $F \otimes V$, F finite dimensional, has a block decomposition with blocks equivalent to the module categories of finite-dimensional associative self-injective algebras.*

In general, the algebras given by the above theorem are neither semi-simple nor local and hence the blocks of the corresponding categories of induced modules can not be equivalent to the blocks of Harish-Chandra bimodules. However, using [GM] one still can get some information about these categories, for example derive an analogue of the BGG-reciprocity. The further study of these cases seems to be an interesting and challenging problem.

Finally, we would like to compare our results with those obtained in [KM3] in the case $\mathfrak{a} = sl(2,\mathbb{C})$. In the present paper we extend those results, for instance, by partial description the multiplicities of simple subquotients in a GVM. However, the settings in the present paper are much more restrictive: we work only with semi-simple finite-dimensional Lie algebras, whereas in [KM3] the case of arbitrary contragradient Lie algebra was considered.

2 Equivalence of *coker*-categories and Harish-Chandra bimodules

In this section we heavily rely on [MS1, BG] and mostly rewrite some results from these two papers, adjusting them to our situation. We try to keep the notation from [MS1]. If nothing is mentioned, all homomorphisms and tensor products are taken over $\mathbb{C}$.

For a Lie algebra, $\mathfrak{L}$, we denote by $\mathcal{F}_{\mathfrak{L}}$ the category of all finite-dimensional $\mathfrak{L}$-modules. For an $\mathfrak{L}$-module, V, we denote by $\langle \mathcal{F}_{\mathfrak{L}} \otimes V \rangle$ and $\mathrm{coker}(\mathcal{F}_{\mathfrak{L}} \otimes V)$ the full subcategory of the category of $\mathfrak{L}$-modules consisting of all subquotients of modules $F \otimes V$, $F \in \mathcal{F}_{\mathfrak{L}}$, and all $\mathfrak{g}$-modules N which admit a two-step resolution, $E \otimes V \to F \otimes V \to N \to 0$, $E, F \in \mathcal{F}_{\mathfrak{L}}$, respectively. If the algebra $\mathfrak{L}$ is semi-simple with a fixed Cartan subalgebra and the corresponding root system Δ, we decompose $\mathcal{F}_{\mathfrak{L}}$ into a direct sum of two full subcategories, $\mathcal{F}^0_{\mathfrak{L}}$ and $\mathcal{F}^1_{\mathfrak{L}}$, where the first one consists of all finite dimensional modules, whose weights belong to $\mathbb{Z}\Delta$, and the second one consists of all finite dimensional modules, whose supports do not intersect $\mathbb{Z}\Delta$. We define $\langle \mathcal{F}^i_{\mathfrak{L}} \otimes V \rangle$ and $\mathrm{coker}(\mathcal{F}^i_{\mathfrak{L}} \otimes V)$, $i = 0, 1$, in the obvious way as above.

For an arbitrary $U(\mathfrak{L})$-bimodule, B, we denote by B^{ad} the $\mathfrak{L}$-module, obtained by considering the adjoint action $ub - bu$, $b \in B$, $u \in U(\mathfrak{L})$, of $U(\mathfrak{L})$ on B. We denote by B_{adf} the subset of all elements $b \in B$ such that the adjoint action of $U(\mathfrak{L})$ on b is finite. For an $\mathfrak{L}$-module, V, we consider the algebra $\mathrm{End}(V)$ of all $\mathbb{C}$-endomorphisms of V, which is an $U(\mathfrak{L})$-bimodule under the action $u_1 f u_2(v) = u_1 \cdot f(u_2 \cdot v)$, $v \in V$, $u_1, u_2 \in U(\mathfrak{L})$.

Define the category $\mathcal{H} = \mathcal{H}_{\mathfrak{L}}$ as the category of all finitely generated $U(\mathfrak{L})$-bimodules B satisfying $B = B_{adf}$. If $\mathfrak{L}$ is semi-simple, then the category $\mathcal{H}$ in a natural way decomposes into a direct sum of two full subcategories $\mathcal{H}^0$ and $\mathcal{H}^1$, where $\mathcal{H}^i$, $i = 0, 1$, consists of all $B \in \mathcal{H}$ such that B^{ad} is a direct sum of modules from $\mathcal{F}^i_{\mathfrak{L}}$. For a two-sided ideal, $I \in U(\lambda)$, we let $\mathcal{H}(I)$, $\mathcal{H}^i(I)$, $i = 0, 1$, be the full subcategories of $\mathcal{H}$, $\mathcal{H}^i$, $i = 0, 1$, respectively, consisting of all B such that $BI = 0$.

Theorem 4. *Let $\mathfrak{L}$ be semi-simple, $I \subset U(\mathfrak{L})$ be a two-sided ideal and V be an $\mathfrak{L}$-module such that $IV = 0$. Assume that*

1. *For every $F \in \mathcal{F}^0_{\mathfrak{L}}$ the multiplication $U(\mathfrak{L}) \to \mathrm{End}(V)$ induces an isomorphism*

$$\mathrm{Hom}_{\mathfrak{L}}(F, (U(\mathfrak{L})/IU(\mathfrak{L}))^{ad}) \simeq \mathrm{Hom}_{\mathfrak{L}}(F, (\mathrm{End}(V))^{ad}).$$

2. *V is projective in $\langle \mathcal{F}^0_{\mathfrak{L}} \otimes V \rangle$.*

Then the functor $\_ \otimes_{U(\mathfrak{L})} V : U(\mathfrak{L})\text{-mod-}U(\mathfrak{L}) \to \mathfrak{L}\text{-mod}$ induces an equivalence of categories $\mathcal{H}^0(I)$ and $\mathrm{coker}(\mathcal{F}^0_{\mathfrak{L}} \otimes V)$.

Proof. Mutatis mutandis [MS1, Theorem 3.1] with the substitution of $\mathcal{F}$ in [MS1] with $\mathcal{F}^0$ and $\mathcal{H}$ with $\mathcal{H}^0$. □

3 Admissible category $\Lambda(V)$

Until Section 6 we fix a simple $\tilde{\mathfrak{a}}$-module, V, whose annihilator in $U(\mathfrak{a})$ is a minimal primitive ideal. According to [FKM1, FKM2] the first step in the study of categories of parabolically induced modules is constructing of certain admissible categories for the Levi factor of the parabolic subalgebra. In this section we show that, for the module V as above, there exists a simple object, $\tilde{V}$, in $\langle \mathcal{F}_{\mathfrak{a}} \otimes V \rangle$, such that the category $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes \tilde{V})$ qualifies for these purposes.

For an $\mathfrak{a}$-module, M, we define the *rough length* $\mathrm{RL}(M)$ of M as the number (possibly infinite) of simple subquotients of M whose annihilators are minimal primitive ideals. The same notion can be defined for $\mathfrak{g}$-modules and we will use $\mathrm{RL}_{\mathfrak{g}}(M)$ in this case. It happens that this invariant behaves well under tensoring with finite-dimensional modules.

Lemma 1. *Assume that M is an $\mathfrak{a}$-module of finite rough length. Then for very finite-dimensional $\mathfrak{a}$-module F we have* $\mathrm{RL}(F \otimes M) = \dim(F)\,\mathrm{RL}(M)$.

Proof. Using exactness of the tensor product with F we first reduce the statement to the case $\mathrm{RL}(M) = 1$, hence we can assume that M is a simple module with minimal annihilator. Further it is enough to study the behavior of the rough length under translation functors (we refer to [J2, GJ] for the definition and properties of these functors). Because of the minimality of the annihilator of M, this does not depend on M. Indeed, the standard properties of the translation functors show that the regular translations (those described in [BG, Theorem 4.1]) and translations to walls send simples to simples ([BeGi, Proposition 3.1], here we also use the minimality of the annihilator of M) and thus does not change the rough length. Having this we can translate from the wall and then back to the wall without crossing other walls, which will be a direct sum of some copies of the identity functor. Hence the rough length of the result does not depend on M. This means that we can check our statement for example on simple Verma modules, for which it is obvious. □

Theorem 5. *There exists an $\mathfrak{a}$-module V' of rough length* 1, *which surjects on V, such that*

1. *The category* $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes V')$ *decomposes into a direct sum of full subcategories each of which is equivalent to the module category of a finite-dimensional associative local self-injective algebra.*
2. *There exists a natural abelian structure on* $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes V')$ *such that the tensor product with finite-dimensional $\mathfrak{a}$-modules is an exact functor with respect to this structure.*

Proof. We construct V' and prove all the statements using an auxiliary module $\tilde{V}$, mentioned in the beginning of this section, which is defined in the following way. Let θ be the central character of V (which exists by Quillens lemma) and W_θ be the integral Weyl group of θ (see e.g. [J2, 2.5]). We define $\tilde{V}$ to be the translation of V to the most degenerate central character with respect to W_θ. Then we use Lemma 1 to get that $\tilde{V}$ is a simple $\mathfrak{a}$-module of rough length 1. Moreover, because of the choice of the central character for $\tilde{V}$, this module is projective in $\langle \mathcal{F}_{\mathfrak{a}} \otimes \tilde{V} \rangle$.

Let us first consider the category $\langle \mathcal{F}_{\mathfrak{a}}^0 \otimes \tilde{V} \rangle$. Denote by I the annihilator of $\tilde{V}$ in $U(\mathfrak{a})$.

Lemma 2. *For every $F \in \mathcal{F}_{\mathfrak{a}}^0$ the multiplication $U(\mathfrak{a}) \to \mathrm{End}(\tilde{V})$ induces an isomorphism*

$$\mathrm{Hom}_{\mathfrak{a}}(F, (U(\mathfrak{a})/I)^{ad}) \simeq \mathrm{Hom}_{\mathfrak{a}}(F, (\mathrm{End}(\tilde{V}))^{ad}).$$

Proof. The injectivity of this map follows from [J2, 6.8]. By Kostants Theorem and Lemma 1, we have for $F \in \mathcal{F}_{\mathfrak{a}}^0$ the equality $\dim\big(\mathrm{Hom}_{\mathfrak{a}}(F, (U(\mathfrak{a})/I)^{ad})\big) = \dim(F_0)$, where F_0 is the zero weight space of F. On the other hand, by [J2, 6.8] we also have:

$$\dim\left(\mathrm{Hom}_{\mathfrak{a}}(F, (\mathrm{End}(\tilde{V}))^{ad})\right) = \dim\left(\mathrm{Hom}_{\mathfrak{a}}(\tilde{V}, F^* \otimes \tilde{V})\right).$$

The latter is equal to $\dim(F_0)$ since the projection of $\langle \mathcal{F}_{\mathfrak{a}}^0 \otimes \tilde{V}\rangle$ on the block containing $\tilde{V}$ is a semi-simple category by the choice of the central character for $\tilde{V}$. □

Applying Theorem 4 we get that $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}}^0 \otimes \tilde{V})$ is equivalent to the corresponding category $\mathcal{H}^0(I)$ of Harish-Chandra bimodules. So, this category has a block decomposition, with each block being the module category for a finite-dimensional associative algebra. Moreover, these algebras are local since in our case each block clearly has only one simple object. That they are self-injective follows from the fact that the projection of $\langle \mathcal{F}_{\mathfrak{a}}^0 \otimes \tilde{V}\rangle$ on the block containing $\tilde{V}$ is semi-simple, hence self-dual, and every finite-dimensional module is self-dual with respect to the duality on $\mathcal{O}$.

To complete the proof we have to deal with $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}}^1 \otimes \tilde{V})$. Let $M \in \langle \mathcal{F}_{\mathfrak{a}}^1 \otimes \tilde{V}\rangle$ be a simple module of rough length 1. From $\mathrm{Hom}_{\mathfrak{g}}(N_1, F \otimes N_2) = \mathrm{Hom}_{\mathfrak{g}}(F^* \otimes N_1, N_2)$ for every $F \in \mathcal{F}_{\mathfrak{a}}$ and $N_1, N_2 \in \mathfrak{g}$-mod we get that either M is isomorphic to some module from $\langle \mathcal{F}_{\mathfrak{a}}^0 \otimes \tilde{V}\rangle$ or $\mathrm{Hom}_{\mathfrak{g}}(M, M') = 0$ for every $M' \in \langle \mathcal{F}_{\mathfrak{a}}^0 \otimes \tilde{V}\rangle$. The latter means, in particular, that $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}}^0 \otimes M)$ is a direct summand of $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes \tilde{V})$. Using a regular translation with respect to some finite-dimensional module from $\mathcal{F}_{\mathfrak{a}}^1$ we get that the indecomposable block of $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}}^1 \otimes \tilde{V})$ containing M is equivalent to some direct summand of $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}}^0 \otimes \tilde{V})$.

We remark that, using the equivalence established during the proof above, we get a natural abelian structure on $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes \tilde{V})$, induced from the one on $\mathcal{H}(I)$.

Consider the block in $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes \tilde{V})$ corresponding to θ. It is non-zero since we can translate $\tilde{V}$ to this block. Using our equivalence, this block is the module category over some finite-dimensional associative local algebra, hence we will get a simple object, V', in this block. Clearly, $\mathrm{RL}(V') = 1$ and V' surjects onto V. This completes the proof of the first statement.

The second statement now follows from the first one and the corresponding result for the category $\mathcal{H}(I)$. □

In the sequel we will use modules V' and $\tilde{V}$ constructed in Theorem 5. From [S1, Endomorphismensatz] we immediately get.

Corollary 1. *The finite-dimensional associative algebra describing a block of* $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}}^0 \otimes \tilde{V})$ *is either the coinvariant algebra or the algebra of invariants in the coinvariant algebra.*

4 The category $\mathcal{O}(\mathfrak{p}, \Lambda)$

After describing the category $\Lambda = \Lambda(V) = \mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes \tilde{V})$ in the previous section, we can apply procedure from [FKM1] to construct the corresponding parabolic generalization

$\mathcal{O}(\mathfrak{p},\Lambda)$. First, we extend this category to the category $\tilde{\Lambda}$ of $\tilde{\mathfrak{a}}$-modules consisting of modules from Λ with diagonal action of $\mathfrak{h}_{\mathfrak{a}}$. We define $\mathcal{O}(\mathfrak{p},\Lambda)$ as the full subcategory of the category of all finitely generated $\mathfrak{g}$-modules, consisting of all those M, which are $\mathfrak{n}$-finite and decompose into a direct sum of objects from $\tilde{\Lambda}$, when viewed as $\tilde{\mathfrak{a}}$-modules. Our main result in this section is the following theorem.

Theorem 6. *Let V be a simple $\mathfrak{a}$-module with a minimal annihilator and V' be as in Section 3. Then the category* $\mathrm{coker}(\mathcal{F}_{\mathfrak{g}}\otimes M_{\mathfrak{p}}(V'))$ *decomposes into a direct sum of full subcategories each of which is equivalent to an appropriate block of $\mathcal{H}(I)$, where I is a minimal primitive ideal in $U(\mathfrak{g})$.*

To prove this theorem we will need some auxiliary lemmas.

Lemma 3. *Let F be a finite-dimensional $\mathfrak{g}$-module. Then the module $F\otimes M_{\mathfrak{p}}(V)$ admits a filtration,*

$$0=M_0\subset M_1\subset\cdots\subset M_{\dim(F)}=F\otimes M_{\mathfrak{p}}(V),$$

such that $M_i/M_{i-1}\simeq M_{\mathfrak{p}}(V_i)$, where V_i is an $\tilde{\mathfrak{a}}$-module of rough length 1.

Proof. Follows by standard arguments from Lemma 1 and the fact that $M_{\mathfrak{p}}(V)$ is free over $\sigma(\mathfrak{n})$, where σ denotes the Chevalley involution. □

Lemma 4. *The annihilator of $M_{\mathfrak{p}}(V)$ is generated by the annihilating ideal of the central character of $M_{\mathfrak{p}}(V)$.*

Proof. From Lemma 3 we get the existence of a simple subquotient of $M_{\mathfrak{p}}(V)$, which is not annihilated by translations through all walls. Hence the annihilator of this simple subquotient is generated by the annihilating ideal of its central character. Now the lemma follows from the fact that $M_{\mathfrak{p}}(V)$ has a central character by [DFO, Theorem 1]. □

Now we have to find a projective "substitution" for $M_{\mathfrak{p}}(V)$. For this we define an $\tilde{\mathfrak{a}}$-module structure on $\tilde{V}$ as follows. We only have to construct the action of $\mathfrak{h}_{\mathfrak{a}}$. Consider the S-homomorphism of Harish-Chandra ([DFO]), corresponding to the subalgebra $\mathfrak{p}$. Application of this homomorphism provides a finite number of linear $\mathfrak{h}_{\mathfrak{a}}$-actions, which give rise to the central character of $M_{\mathfrak{p}}(V)$. Among these actions we choose the maximal one with respect to the standard partial order, see [DFO, Section 1]. In a trivial way we then extend $\tilde{V}$ to a $\mathfrak{p}$-module and consider the corresponding GVM $M_{\mathfrak{p}}(\tilde{V})$. The standard highest weight arguments immediately imply that $M_{\mathfrak{p}}(\tilde{V})$ is projective in the category $\mathrm{coker}(\mathcal{F}\otimes M_{\mathfrak{p}}(V))$.

Lemma 5. *Let $\tilde{V}$ be as above. Then for every $F\in\mathcal{F}^0_{\mathfrak{g}}$ the multiplication $U(\mathfrak{g})\to$* $\mathrm{End}(M_{\mathfrak{p}}(\tilde{V}))$ *induces an isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(F,(U(\mathfrak{g})/I)^{ad})\simeq\mathrm{Hom}_{\mathfrak{g}}(F,(\mathrm{End}(M_{\mathfrak{p}}(\tilde{V})))^{ad}).$$

Proof. The injectivity of this map still follows from Lemma 4 and [J2, 6.8]. For $F \in \mathcal{F}_{\mathfrak{a}}^0$ we again have $\dim\left(\mathrm{Hom}_{\mathfrak{a}}(F,(U(\mathfrak{a})/I)^{ad})\right) = \dim(F_0)$, where F_0 is the dimension of the zero weight space of F. On the other hand, from [J2, 6.8] we obtain:

$$\dim\left(\mathrm{Hom}_{\mathfrak{a}}(F,(\mathrm{End}(M_{\mathfrak{p}}(\tilde{V})))^{ad})\right) = \dim\left(\mathrm{Hom}_{\mathfrak{a}}(M_{\mathfrak{p}}(\tilde{V}), F^* \otimes M_{\mathfrak{p}}(\tilde{V}))\right).$$

Using the projectivity of $M_{\mathfrak{p}}(\tilde{V})$ we deduce

$$\dim\left(\mathrm{Hom}_{\mathfrak{a}}(M_{\mathfrak{p}}(\tilde{V}), F^* \otimes M_{\mathfrak{p}}(\tilde{V}))\right) = [F^* \otimes M_{\mathfrak{p}}(\tilde{V})) : L_{\mathfrak{p}}(\tilde{V})].$$

The latter is equal to $\dim(F_0)$ because of Lemma 3 and the definition of $\tilde{V}$. □

Proof of Theorem 6. Using Lemmas 4 and 5 the proof is similar to that of Theorem 5. □

Corollary 2. *The finite-dimensional associative algebra describing an indecomposable block of* $\mathrm{coker}(\mathcal{F}_{\mathfrak{g}}^0 \otimes M_{\mathfrak{p}}(\tilde{V}))$ *is either the quasi-hereditary algebra describing a block of* $\mathcal{O}$ *or the properly stratified algebra describing a block of a subcategory of Enright-complete modules in* $\mathcal{O}$*, see [KoM1].*

5 Application to the study of generalized Verma modules

In this section we are going to apply the above results to the study of properties of generalized Verma modules $M_{\mathfrak{p}}(V)$ in the case when the module V has a minimal annihilator as an $\mathfrak{a}$-module. The last is assumed throughout this section. We will also use modules V' and $\tilde{V}$ from the previous sections.

Let M be a $\mathfrak{g}$ module and $\lambda \in \mathfrak{h}_{\mathfrak{a}}^*$. Set $M_\lambda = \{m \in M : h(m) = \lambda(h)m \text{ for all } h \in \mathfrak{h}_{\mathfrak{a}}\}$. If $M = \oplus_{\lambda \in \mathfrak{h}_{\mathfrak{a}}^*} M_\lambda$ we define the *rough* $\mathfrak{a}$*-character* of M as the function $\mathrm{ch}_{\mathfrak{a}}^M : \mathfrak{h}_{\mathfrak{a}}^* \to \mathbb{Z} \cup \{\infty\}$ such that $\mathrm{ch}_{\mathfrak{a}}(\lambda) = \mathrm{RL}(M_\lambda)$. We note that $\mathrm{ch}_{\mathfrak{a}}^{M_{\mathfrak{p}}(V)}(\lambda) < \infty$ for all $\lambda \in \mathfrak{h}_{\mathfrak{a}}^*$.

We start with the following two results generalizing the corresponding classical properties of Verma modules ([D, Proposition 7.6.3, Theorem 7.6.6]).

Proposition 2. $M_{\mathfrak{p}}(V)$ *has a simple socle.*

Proof. Under the assumption that V has a minimal annihilator, one can adopt the classical growth arguments. By standard arguments it follows from Lemma 1 that the growth of $\mathrm{ch}_{\mathfrak{a}}^{M_{\mathfrak{p}}(V)}$ is polynomial and that all GVMs of the form $M_{\mathfrak{p}}(Y)$, where Y is a simple subquotient with rough length 1 of some $F \otimes V$, $F \in \mathcal{F}_{\tilde{\mathfrak{a}}}$, have the same rough $\mathfrak{a}$-character up to a shift.

Further we note that the socle of every $F \otimes V$ as above contains only simples of rough length 1. Indeed, this is trivial for translations through the walls and then this extends to every $F \otimes \_$. It follows that each submodule of $M_{\mathfrak{p}}(V)$ contains in its turn a submodule, isomorphic to some $M_{\mathfrak{p}}(Y)$ with Y as above. Since the leading coefficient of the growth polynomial for the rough $\mathfrak{a}$-character of $M_{\mathfrak{p}}(V)$ behaves additively with respect to the direct sums, we get that the socle of $M_{\mathfrak{p}}(V)$ can contain only one copy of $M_{\mathfrak{p}}(Y)$ and hence is simple. □

Proposition 3. *Let V_i, $i = 1, 2$, be two simple $\tilde{\mathfrak{a}}$-modules with minimal annihilators. Then the dimension of* $\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1), M_{\mathfrak{p}}(V_2))$ *is at most one and every non-zero element of this space is injective.*

Proof. Mutatis mutandis [D, Theorem 7.6.6], using Proposition 2. □

From Theorem 6 and [KM4] one also derives the following:

Proposition 4. *The module $M_{\mathfrak{p}}(V')$ is rigid as object of* $\mathrm{coker}(\mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V}))$.

Now we would like to associate a Verma module to the GVM $M_{\mathfrak{p}}(V)$. For this we denote by $\mathfrak{f}(V)$ the Verma module $M(\lambda)$ over $\tilde{a}$ such that it has the same central character as V and λ belongs to the closure of the antidominant Weyl chamber. Clearly, this one is uniquely defined. Then the module $M_{\mathfrak{p}}(M(\lambda)) = M_{\mathfrak{p}}(\mathfrak{f}(V))$ is a Verma module over $\mathfrak{g}$.

Theorem 7. *Let V_i, $i = 1, 2$, be two simple $\tilde{\mathfrak{a}}$-modules with minimal annihilators. Then the following conditions are equivalent:*

1. $\dim(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1), M_{\mathfrak{p}}(V_2))) = 1$.
2. $[M_{\mathfrak{p}}(V_2), L_{\mathfrak{p}}(V_1)] > 0$.
3. $V_1 \in \langle \mathcal{F}^0 \otimes V_2 \rangle$ *and* $\dim(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(\mathfrak{f}(V_1)), M_{\mathfrak{p}}(\mathfrak{f}(V_2)))) = 1$.

To prove this theorem we will need the following result.

Proposition 5. *Let V_i, $i = 1, 2$, be two simple $\tilde{\mathfrak{a}}$-modules with minimal annihilators. Then* $\dim(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1), M_{\mathfrak{p}}(V_2))) = \dim(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1'), M_{\mathfrak{p}}(V_2')))$, *where V_i' are as constructed in Theorem 5.*

We prove Proposition 5 in two steps.

Lemma 6. *There is a natural injection from the space* $\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1'), M_{\mathfrak{p}}(V_2'))$ *to the space* $\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1), M_{\mathfrak{p}}(V_2))$.

Proof. We start with two exact sequences:

$$0 \to N_1 \to M_{\mathfrak{p}}(V_1') \to M_{\mathfrak{p}}(V_1) \to 0, \tag{1}$$
$$0 \to N_2 \to M_{\mathfrak{p}}(V_2') \to M_{\mathfrak{p}}(V_2) \to 0, \tag{2}$$

where $\mathrm{ch}_{\mathfrak{a}}^{N_1} = \mathrm{ch}_{\mathfrak{a}}^{N_2} = 0$. Applying $\mathrm{Hom}_{\mathfrak{g}}(\_, M_{\mathfrak{p}}(V_2))$ to (1) we get

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1), M_{\mathfrak{p}}(V_2)) \to \mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1'), M_{\mathfrak{p}}(V_2)) \to \mathrm{Hom}_{\mathfrak{g}}(N_1, M_{\mathfrak{p}}(V_2)).$$

Here $\mathrm{Hom}_{\mathfrak{g}}(N_1, M_{\mathfrak{p}}(V_2)) = 0$ since $\mathrm{ch}_{\mathfrak{a}}^{N_1} = 0$ while the socle of $M_{\mathfrak{p}}(V_2)$ consists only of modules of rough length 1 (as $\tilde{a}$-modules). Hence we get the equality $\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1), M_{\mathfrak{p}}(V_2)) = \mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1'), M_{\mathfrak{p}}(V_2))$. Now we apply $\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1'), \_)$ to (2) and get

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1'), N_2) \to \mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1'), M_{\mathfrak{p}}(V_2')) \to \mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1'), M_{\mathfrak{p}}(V_2)),$$

Where $\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1'), N_2) = 0$ by the same arguments as above applied to the top of $M_{\mathfrak{p}}(V_1')$. Combining these two results we get an injection from $\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1'), M_{\mathfrak{p}}(V_2'))$ to $\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1), M_{\mathfrak{p}}(V_2))$. □

To get an estimate in the opposite direction we now construct a *completion functor*, $\mathfrak{C}$, from the category $\langle \mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V})\rangle$ to the category $\mathrm{coker}\left(\mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V})\right)$. For $M \in \langle \mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V})\rangle$ we denote by $\hat{M}$ the trace of all projective modules from $\mathrm{coker}\left(\mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V})\right)$ in M. These are direct summands of the modules $F \otimes M_{\mathfrak{p}}(\tilde{V})$, $F \in \mathcal{F}_{\mathfrak{g}}$. Now denote by $f : P_M \to \hat{M}$ a projective cover of $\hat{M}$. Define $\mathfrak{C}(M)$ as $P_M/\widehat{\ker(f)}$. It is easy to see that $\mathfrak{C}(M)$ does not depend on the choice of P_M. If $\varphi : M_1 \to M_2$ is a homomorphism, it can be restricted to $\hat{M}_1$ and obviously $\varphi(\hat{M}_1) \subset \hat{M}_2$. By standard arguments φ canonically extends to a unique homomorphism, $\mathfrak{C}(\varphi) : \mathfrak{C}(M_1) \to \mathfrak{C}(M_2)$. It is easy to see that $\mathfrak{C}$ indeed defines a covariant functor from $\langle \mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V})\rangle$ to $\mathrm{coker}\left(\mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V})\right)$. We refer the reader to [KoM1, Section 2] for abstract description of analogous functors.

Lemma 7. *1. $\mathfrak{C}$ is right exact.*

2. $\mathfrak{C}$ preserves the rough $\mathfrak{a}$-character.

3. Let $\varphi : M_1 \to M_2$ be such that $\varphi|_{\hat{M}_1} \neq 0$ then $\mathfrak{C}(\varphi) \neq 0$.

4. $\mathfrak{C}(M_{\mathfrak{p}}(V)) = M_{\mathfrak{p}}(V')$.

Proof. The first and the third statements follow immediately from the construction of $\mathfrak{C}$. The second statement follows from the fact that all simple $\mathfrak{a}$-submodules of rough length 1 in the modules from $\langle \mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V})\rangle$ can be covered by projective module from $\mathrm{coker}\left(\mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V})\right)$. Since $M_{\mathfrak{p}}(V')$ surjects onto $M_{\mathfrak{p}}(V)$ with kernel being a module of rough length zero, the last statement follows from the second one and the equality $\mathfrak{C}(M_{\mathfrak{p}}(V')) = M_{\mathfrak{p}}(V')$, which is clear from the construction of $M_{\mathfrak{p}}(V')$. □

Proof of Proposition 5. Lemma 6 gives an injection between the spaces of homomorphisms in one direction and the functor $\mathfrak{C}$ together with Lemma 7 gives an injection in another direction. □

Proof of Theorem 7. By Proposition 5 the condition $\dim\left(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1), M_{\mathfrak{p}}(V_2))\right) = 1$ is equivalent to the condition $\dim\left(\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(V_1'), M_{\mathfrak{p}}(V_2'))\right) = 1$. By Theorem 6 this is equivalent to the existence of homomorphisms between corresponding proper standard modules in the category $\mathcal{H}(I)$. The equivalence of categories established in Theorem 6 sends, by construction, $M_{\mathfrak{p}}(V_i)$ and $M_{\mathfrak{p}}(\mathfrak{f}(V_i))$ to the same modules. Hence the first and the third statements of our theorem are equivalent.

Clearly the first statement implies the second one. The inverse implication can be proved as follows. Since the projective cover P of $M_{\mathfrak{p}}(V_1')$ is projective both in $\langle \mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V})\rangle$ and coker $\left(\mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V})\right)$, we have that the condition $[M_{\mathfrak{g}}(V_2) : L_{\mathfrak{p}}(V_1)] > 0$ is equivalent to $\mathrm{Hom}_{\mathfrak{g}}(P, M_{\mathfrak{g}}(V_2)) \neq 0$. The functor $\mathfrak{C}$ transfers this to $\mathrm{Hom}_{\mathfrak{g}}(P, M_{\mathfrak{g}}(V_2')) \neq 0$, which is equivalent to the condition $[M_{\mathfrak{g}}(V_2'), S] > 0$, where S is the simple top of P in coker $\left(\mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V})\right)$. By Theorem 6 and the classical BGG-Theorem [D, Theorem 7.6.23] the latter is equivalent to $M_{\mathfrak{g}}(V_1') \subset M_{\mathfrak{g}}(V_2')$ and this implies $M_{\mathfrak{g}}(V_1) \subset M_{\mathfrak{g}}(V_2)$ by Proposition 5 and Proposition 3. □

Now we are ready to prove our main result, which is Theorem 2 from Introduction.

Theorem 8. *The module $M_{\mathfrak{p}}(V)$ is simple if and only if $M_{\mathfrak{p}}(\mathfrak{f}(V))$ is simple.*

Proof. From Proposition 2 and Theorem 7 it follows that $M_{\mathfrak{p}}(V)$ is simple if and only if $M_{\mathfrak{p}}(V')$ is a simple object in $\mathrm{coker}(\mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V}))$. The same arguments show that $M_{\mathfrak{p}}(\mathfrak{f}(V))$ is simple if and only if its completion $\mathfrak{C}(M_{\mathfrak{p}}(\mathfrak{f}(V)))$ is simple as an object of the corresponding coker-subcategory in $\mathcal{O}$ (there is an alternative description in terms of $\mathcal{S}$-subcategories in $\mathcal{O}$ in [FKM2], where completion $\mathfrak{C}$ is substituted with $\mathfrak{a}$-Enright's completion). Applying Theorem 6 twice we get an exact equivalence of categories sending $M_{\mathfrak{p}}(V')$ to $\mathfrak{C}(M_{\mathfrak{p}}(\mathfrak{f}(V)))$, which completes the proof. □

Theorem 9. *Let V_1 and V_2 be as in Theorem 7 and $V_1 \in \langle \mathcal{F}^0 \otimes V_2\rangle$. Then*

$$[M_{\mathfrak{p}}(V_1) : L_{\mathfrak{p}}(V_2)] = [M_{\mathfrak{p}}(\mathfrak{f}(V_1)) : L_{\mathfrak{p}}(\mathfrak{f}(V_2))].$$

Proof. Let P be a projective cover of $M_{\mathfrak{p}}(V_2)$ in $\langle \mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V})\rangle$. We start with $[M_{\mathfrak{p}}(V_1) : L_{\mathfrak{p}}(V_2)] = \dim(\mathrm{Hom}_{\mathfrak{g}}(P, M_{\mathfrak{p}}(V_1)))$. Applying $\mathfrak{C}$ and Lemma 7 we get the following inequality: $\dim(\mathrm{Hom}_{\mathfrak{g}}(P, M_{\mathfrak{p}}(V_1))) \leqslant \dim(\mathrm{Hom}_{\mathfrak{g}}(P, M_{\mathfrak{p}}(V_1')))$. The inverse inequality is obtained by applying $\mathrm{Hom}_{\mathfrak{g}}(P, \_)$ to the exact sequence $0 \to N \to M_{\mathfrak{p}}(V_1') \to M_{\mathfrak{p}}(V_1) \to 0$ and noticing that $\mathrm{Hom}_{\mathfrak{g}}(P, N) = 0$. Hence $\dim(\mathrm{Hom}_{\mathfrak{g}}(P, M_{\mathfrak{p}}(V_1))) = \dim(\mathrm{Hom}_{\mathfrak{g}}(P, M_{\mathfrak{p}}(V_1')))$. The latter number is equal to $[M_{\mathfrak{p}}(V_1') : S]$, where S is the simple top of P in $\mathrm{coker}(\mathcal{F}_{\mathfrak{g}} \otimes M_{\mathfrak{p}}(\tilde{V}))$. The equivalence from Theorem 6 implies that $[M_{\mathfrak{p}}(V_1') : S] = [M_{\mathfrak{p}}(\mathfrak{f}(V_1)) : L_{\mathfrak{p}}(\mathfrak{f}(V_2))]$, which completes the proof. □

We have to remark that Theorem 8 does not give the complete information about the subquotients of $M_{\mathfrak{p}}(V_1)$. It describes only the multiplicities of simple subquotients with non-zero rough $\mathfrak{a}$-character. However, if the length of the $\mathfrak{a}$-module $E \otimes \tilde{V}$ is equal to its length as an object of the category $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes \tilde{V})$ for all finite dimensional modules E, then the categories $\langle \mathcal{F}_{\mathfrak{a}} \otimes \tilde{V}\rangle$ and $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes \tilde{V})$ coincide. In particular, the Verma module $M_{\mathfrak{p}}(V)$ has finite length and Theorem 9 describes all composition multiplicities of this module.

6 Case of induction from $sl(2,\mathbb{C})$

In this section we consider the case when $\mathfrak{a} \simeq sl(2,\mathbb{C})$. It happens that our results now imply a "rough" classification of categories of $\mathfrak{g}$-modules parabolically induced from simple $\mathfrak{a}$-modules in this case.

We start with an arbitrary simple $\mathfrak{a}$-module V and consider the corresponding category $\langle \mathcal{F}_{\mathfrak{a}} \otimes V\rangle$. This extends to the category $\tilde{\Lambda}$ of $\tilde{\mathfrak{a}}$-modules with a diagonal action of $\mathfrak{h}_{\mathfrak{a}}$. Pick an arbitrary projective $\tilde{V}$ in the category $\tilde{\Lambda}$ and consider the category $\Lambda = \mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes \tilde{V})$.

Theorem 10. *The parabolic category $\mathcal{O}(\mathfrak{p},\Lambda)$ decomposes into a direct sum of full subcategories each of which is equivalent to the module category of one of the following finite-dimensional associative algebras:*

1. *The quasi-hereditary algebra associated with a block of the parabolic category $\mathcal{O}_S$ of Rocha-Caridi, [R] , where S consists of the simple root of $\mathfrak{a}$.*

2. *The quasi-hereditary algebra associated with a block of the category $\mathcal{O}$ for $\mathfrak{g}$.*

3. *The properly stratified algebra associated with as block of the parabolic $\mathcal{S}$-subcategory in $\mathcal{O}$, associated with $\mathfrak{p}$, as in [FKM2].*

Proof. If V is finite-dimensional we immediately arrive in the situation, considered by Rocha-Caridi. Otherwise we will have $\mathrm{RL}(V) = 1$ since $\mathfrak{a} \simeq sl(2,\mathbb{C})$ and the statement follows from Corollary 2. □

We would like to list one more peculiar feature of this case.

Proposition 6. *Let V be a simple $\tilde{\mathfrak{a}}$-module. Then the module $M_{\mathfrak{p}}(V)$ has finite length as a $\mathfrak{g}$-module.*

Proof. If V is finite-dimensional the statement follows from [R]. Otherwise we decompose $M_{\mathfrak{p}}(V) = \oplus_{\lambda\in\mathfrak{h}_{\mathfrak{a}}^*} M_{\mathfrak{p}}(V)_\lambda$. As an $\mathfrak{a}$-module, the module $M_{\mathfrak{p}}(V)_\lambda$ equals $F \otimes V$ for some finite-dimensional V and hence is holonomic. Thus it has finite length. Now the proof can be completed similarly to [D, Proposition 7.6.1] using the S-Harish-Chandra homomorphism, [DFO]. □

It would be very interesting to compute the multiplicities of all simple subquotients in $M_{\mathfrak{p}}(V)$. By Proposition 6 they are finite, but Theorem 9 gives an answer only for simple subquotients of the form $L_p(V_1)$, where V_1 is infinite-dimensional (in our case this is equivalent to the condition that the rough $\mathfrak{a}$-character is non-zero).

7 Case of simple modules with bigger annihilators

After discussions above it is a natural question to consider the category $\langle \mathcal{F}_{\mathfrak{a}} \otimes V\rangle$ in the case of arbitrary $\mathfrak{a}$ and arbitrary simple $\mathfrak{a}$-module V. There are two extreme cases. The first one is when V is finite-dimensional. This was studied in [R] and we can say that this case is more or less known. In the present paper we have studied the second extreme case when the module V has a minimal annihilator. What will happen if V is neither finite-dimensional nor with minimal annihilator? This question seems to be non-trivial. In this section we establish some basic results for this case, try to underline where, from our point of view, the difficulties arise and present an example, which shows that the categories of induced modules, arising in the general case, go beyond quasi-hereditary and properly stratified algebras. We start with the following routine observation.

Proposition 7. *Let V be an arbitrary simple $\mathfrak{a}$-module, which is projective in $\langle \mathcal{F}_{\mathfrak{a}} \otimes V \rangle$. Then* $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes V)$ *has a block decomposition with blocks equivalent to the module categories of finite-dimensional associative self-injective algebras.*

Proof. The functor $F \otimes \_$ is exact and has two-sided adjoint $F^* \otimes \_$. Hence it sends projectives to projectives. This gives us enough projectives in $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes V)$. The latter implies that the decomposition with respect to the action of the center provides a block decomposition with blocks being equivalent to the module categories of finite-dimensional associative algebras.

Since V is simple and projective, it generates a semi-simple block and hence V is injective as well. The functor $F \otimes \_$ sends injectives to injectives and thus all projective modules in $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes V)$ are injective, completing the proof. $\square$

Now we are going to present an example, from which it will follow, that the algebras, appearing in Proposition 7 are not necessarily local, in contrast to what we had in Theorem 5.

We consider the principal block of the category $\mathcal{O}$ for $\mathfrak{a} = sl(3, \mathbb{C})$. There are six simple modules in this block denoted by 1, 2, 3, 4, 5, 6. We choose the enumeration so that the corresponding Verma modules $M(i)$, $i \in \{1, 2, 3, 4, 5, 6\}$, have the following Loewy series (written as radical series):

$M(1)$	$M(2)$	$M(3)$	$M(4)$	$M(5)$	$M(6)$
1	2	3	4	5	6
2 3	4 5	4 5	6	6	
4 5	6	6			
6					

We can choose simple roots α and β so that the action of the coherent translation functors θ_α and θ_β through the corresponding walls on simple modules is as follows (all modules are given by their Loewy series):

$$\theta_\alpha(1)=0,\quad \theta_\alpha(2)=0,\quad \theta_\alpha(3)=\begin{array}{c}3\\5\ 1\\3\end{array},\quad \theta_\alpha(4)=\begin{array}{c}4\\2\\4\end{array},\quad \theta_\alpha(5)=0,\quad \theta_\alpha(6)=\begin{array}{c}6\\5\\6\end{array};$$

$$\theta_\beta(1)=0,\quad \theta_\beta(2)=\begin{array}{c}2\\4\ 1\\2\end{array},\quad \theta_\beta(3)=0,\quad \theta_\beta(4)=0,\quad \theta_\beta(5)=\begin{array}{c}5\\3\\5\end{array},\quad \theta_\beta(6)=\begin{array}{c}6\\4\\6\end{array}.$$

Let V denote the translation of 5 to the β-wall. This module is the only $X_{\pm\beta}$-finite simple module in the corresponding block of $\mathcal{O}$ and hence it is projective in $\langle \mathcal{F}_\mathfrak{a}\otimes V\rangle$. Now, what are the projective modules in the intersection of $\langle \mathcal{F}_\mathfrak{a}\otimes V\rangle$ with the principal block? The first one is the translation of V back to the principal block, which coincides with $\theta_\beta(5)$. Now we can apply θ_α to $\theta_\beta(5)$ to get $\theta_\alpha(3)$. The modules 3 and 5 are the only infinite-dimensional $X_{\pm\beta}$-finite simple modules in the principal block, hence the projectives we got are all indecomposable projectives. Therefore the projective generator of the corresponding block of $\mathrm{coker}(\mathcal{F}_\mathfrak{a}\otimes V)$ is $\theta_\beta(5)\oplus\theta_\alpha(3)$. This module has the following Loewy series:

$$P=\begin{array}{c}5\\3\\5\end{array}\oplus\begin{array}{c}3\\5\ 1\\3\end{array}.$$

Obviously $\mathrm{End}_\mathfrak{g}(P)$ is connected and not local. Actually it is the algebra of the following quiver with relations:

$$\bullet \underset{y}{\overset{x}{\rightleftarrows}} \bullet \qquad xyx=yxy=0.$$

We would like to finish the paper with underlining the main problems, which, from our point of view, prevent the generalization of our arguments to simple modules with arbitrary annihilators. The first step in our arguments was a construction of the projective module $\tilde{V}$ starting from a simple $\mathfrak{a}$-module V. For this we have translated V into the most degenerate central character (the intersection of all walls). For this central character the following is true: tensor product with an arbitrary finite dimensional module, followed by the projection on this central character, is a direct sum of several identity functors. This property implies projectivity of $\tilde{V}$. Now consider a simple module, V, with an intermediate annihilator. This module is annihilated under translation to the most degenerate central character. However, we can consider the set of all possible translations and find there "the most degenerate one", $\tilde{V}$, which is still non-zero. We remark that the module $\tilde{V}$ is not uniquely defined in general. Although it is tempting to claim that the module $\tilde{V}$ is projective in $\langle \mathcal{F}_\mathfrak{a}\otimes V\rangle$, but we do not know how to prove this. The above arguments with translation functors do not work any more, at least in the obvious way.

One more problem appears when one tries to generalize Lemma 2. Although the statement might be true, the arguments are not transferable to the case of smaller annihilators.

The third point is the study of corresponding GVM even in the category $\mathcal{O}$ that is the case, when V is a simple highest weight module. What are the composition multiplicities of the corresponding $M_{\mathfrak{p}}(V)$? When this module is simple? Are there any analogues of the BGG-Theorem?

Finally, we would like to mention that Proposition 7 shows the connection between the categories $\mathrm{coker}(\mathcal{F}_{\mathfrak{a}} \otimes M_{\mathfrak{p}}(\tilde{V}))$ and the finite-dimensional algebras, studied in [GM]. The main result of [GM] gives a BGG-type reciprocity for these categories, showing that some analogues of the classical results still can be obtained for the general case.

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