On the determinant of Shapovalov form for Generalized Verma modules

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Running Heads: Shapovalov form for GVM

Abstract

We define a generalization of the Shapovalov form for contragradient Lie algebras and compute its determinant for Generalized Verma modules induced from a well-embedded $sl(2,\mathbb{C})$ subalgebra. As a corollary we obtain a generalization of the BGG-theorem for Generalized Verma modules.

1 Introduction

The structure theory of Verma modules is classical part in representation theory of Lie algebras. First deep result in this direction was obtained in the original paper by I.Bernstein, I.Gelfand and S.Gelfand ([BGG]). This theorem (which we will call the BGG-theorem) provides some criterion for the existence of a non-trivial homomorphism between two Verma modules over a complex semisimple finite-dimensional Lie algebra. The original proof by BGG uses some deep results on the structure of the Weyl group of the Lie algebra and refers to Harish-Chandra theorem on central characters of Verma module. In eight years V.Kac and D.Kazhdan ([KK]) managed to generalize this result on Verma modules over arbitrary contragradient complex Lie algebra with symmetrizable Cartan matrix. The most amazing thing is that their proof was quite elementary. The main tool in that proof was special bilinear form defined on a Verma module by N.Shapovalov ([S]).

There are a lot of different generalizations of Verma modules. One of them, called α-stratified Generalized Verma modules (GVM), was studied intensively during last years (see for example [CF, FM, KM] and references therein). For example, an analogue of the BGG-theorem for α-stratified GVM over a simple complex finite-dimensional Lie algebra was obtained in [FM, KM]. The technique used to prove this generalization is analogous to that of BGG. Certainly, it seems to be impossible to generalize this result using BGG-method for infinite-dimensional algebras. Nevertheless, some information about the structure of GVMs over affine Lie algebra of type $A_1^{(1)}$ was obtained in [F] by introducing a generalization of Shapovalov form.

In the present paper we define certain analogue of the Shapovalov form on the enveloping algebra of a contragradient Lie algebra and use this form to study the structure of GVMs induced from a well-embedded $sl(2,\mathbb{C})$ subalgebra. The family of GVMs considered in this paper is a bit bigger than one of α-stratified GVMs considered for example in [CF]. Nevertheless, the irreducibility criterion remains valid in this general case, but it seems to be a very easy generalization of the classical α-stratified case. Results, obtained in this paper, cover and generalize all known facts about structure of GVMs ([FM, KM, F]).

The structure of the paper is the following: In Section 2 we collect all necessary notations and preliminary results on GVMs. In Section 3 we define a generalization of the Shapovalov form, investigate its basic properties and present a generalization of the determinant formula. In Section 4 we prove the determinant formula presented in Section 3. Finally, in Section 5 we obtain a criterion of irreducibility for a GVM and a generalization of BGG theorem.

2 Preliminaries

Let $\mathbb{C}$ denote the complex field, $\mathbb{Z}$ denote the set of integers and $\mathbb{N}$ denote the set of all positive integers. All the notations that will be used in this paper without preliminary definition can be found in [MP]. For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ its universal enveloping algebra.

Let $\mathfrak{G}$ be a complex contragradient Lie algebra (or Chevalley algebra) associated with a complex $(n\times n)$-matrix $A=(a_{ij})$ (see [KK]). We fix the standard triangular decomposition $(\mathfrak{G}_+,\mathfrak{H},\mathcal{Q}_+,\sigma)$ of $\mathfrak{G}$, where $\mathfrak{H}$ is Cartan subalgebra, $\mathcal{Q}_+$ is the set of roots of $\mathfrak{G}_+$ and σ is an antiinvolution on $\mathfrak{G}$ (see [MP, KK] for details).

Let $\mathcal{Q}$ be the set of roots of the algebra $\mathfrak{G}$ i.e. $\mathcal{Q}=\mathcal{Q}_+\cup-\mathcal{Q}_+$ ([MP]). For a root β let $\mathfrak{G}^\beta$ denote the corresponding root space. For the rest of the paper we fix a base π of $\mathcal{Q}_+$ and an element $\alpha\in\pi$ satisfying the following conditions: the subalgebra $\mathfrak{G}_\alpha$ of $\mathfrak{G}$ generated by $\mathfrak{G}^{\pm\alpha}$ should be isomorphic to $sl(2,\mathbb{C})$ and $\mathfrak{G}$ should be an integrable (i.e. direct sum of finite-dimensional modules) $\mathfrak{G}_\alpha$-module under the adjoint action. Let $\mathfrak{N}_\pm^\alpha=\sum_{\beta\in\mathcal{Q}_+\setminus\{\alpha\}}\mathfrak{G}^{\pm\beta}$, $\mathfrak{H}^\alpha=\{h\in\mathfrak{H}\,|\,\alpha(h)=0\}$, $\pi_\alpha=\pi\setminus\{\alpha\}$. Then we have the following decomposition: $\mathfrak{G}=\mathfrak{G}_\alpha\oplus\mathfrak{N}_-^\alpha\oplus\mathfrak{H}^\alpha\oplus\mathfrak{N}_+^\alpha$. For $\mathfrak{H}_\alpha=\mathfrak{G}_\alpha\cap\mathfrak{H}$ one obtains $\mathfrak{G}_\alpha=\mathfrak{G}^\alpha\oplus\mathfrak{H}_\alpha\oplus\mathfrak{G}^{-\alpha}$. Fix some Weyl-Chevalley basis H_α, $X_{\pm\alpha}$ in $\mathfrak{G}_\alpha$. We also fix the dual elements $H_\beta\in\mathfrak{H}$, $\beta\in\mathcal{Q}_+$.

Under the above choice of α a simple reflection s_α on $\mathfrak{H}^*$ is correctly defined and satisfies all the standard properties of a simple reflection. Let P denotes the standard Kostant partition function with respect to π and $\hat{P}$ denotes the standard Kostant partition function with respect to $s_\alpha(\pi)$. By a quasiroot we will mean any element $q\alpha\in\mathfrak{H}^*$, where $\alpha\in\mathcal{Q}_+$ and q is a positive rational number.

A $\mathfrak{G}$-module V is said to be weight module provided the action of $\mathfrak{H}$ is diagonalizable on V. Any weight $\mathfrak{G}$-module V admits a weight-space decomposition $V=\oplus V_\lambda$, where λ

runs through $\mathfrak{H}^*$ and V_λ is the weight subspace corresponding to λ (see [D]). For a weight module V by $\operatorname{ch} V$ we will denote its character ([D, Section 7.5]). A weight $\mathfrak{G}$-module V is called α-stratified ([CF]) if the actions of $X_{\pm\alpha}$ are injective on V. An element $v \neq 0$ of a weight $\mathfrak{G}$-module V will be called α-highest weight vector provided $v \in V_\lambda$ for some $\lambda \in \mathbb{C}$ and $\mathfrak{N}_+^\alpha v = 0$.

Consider the standard quadratic Casimir operator $c = (H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha$ in $U(\mathfrak{G}_\alpha)$. For any pair $a, b \in \mathbb{C}$ one can consider a $\mathfrak{G}^\alpha$-module $N(a, b)$ uniquely defined by the following conditions:

- b is the eigenvalue of c on $N(a, b)$;
- all weight spaces $N(a, b)_{a-2k}$, $k \in \mathbb{Z}$ are one dimensional;
- all non-zero weight spaces of $N(a, b)$ are exhaust by those listed above;
- $N(a, b)$ is generated by $N(a, b)_a$.

Since $\mathfrak{H} = \mathfrak{H}_\alpha \oplus \mathfrak{H}^\alpha$ we can rewrite arbitrary $\lambda \in \mathfrak{H}^*$ as $\lambda = \lambda_\alpha + \lambda^\alpha$, where $\lambda_\alpha \in \mathfrak{H}_\alpha$ and $\lambda^\alpha \in \mathfrak{H}^\alpha$. Let $a, b \in \mathbb{C}$ and $\lambda \in \mathfrak{H}^*$ such that $\lambda(H_\alpha) = \lambda_\alpha(H_\alpha) = a$. We can define a structure of an $\mathfrak{H}$-module on $N(a, b)$ by setting $hv = \lambda^\alpha(h)v$ for all $h \in \mathfrak{H}^\alpha$ and all $v \in N(a, b)$. Further, we can consider $N(a, b)$ as a $D = \mathfrak{H} + \mathfrak{G}_\alpha \oplus \mathfrak{N}_+^\alpha$-module by setting $\mathfrak{N}_+^\alpha N(a, b) = 0$. The $\mathfrak{G}$-module

$$M_\alpha(\lambda, b) = U(\mathfrak{G}) \bigotimes_{U(D)} N(a, b)$$

is called Generalized Verma module (GVM). One can easily prove that $M_\alpha(\lambda, b)$ is α-stratified if and only if $b \neq (a + 1 + 2l)^2$ for all $l \in \mathbb{Z}$ (see also [CF, Theorem 2.1]). An equivalent condition is that $N(a, b)$ is irreducible. For $M_\alpha(\lambda, b)$ we will denote by $L_\alpha(\lambda, b)$ its unique irreducible quotient. Since α is fixed we will omit it as an index in the subsequent notations of $M_\alpha(\lambda, b)$ and $L_\alpha(\lambda, b)$.

For a contragradient Lie algebra with a symmetrisable Cartan matrix let $(\cdot, \cdot)$ denote the bilinear form on $\mathfrak{G}$ ([K, MP]). The corresponding bilinear form on $\mathfrak{H}^*$ will be also denoted by $(\cdot, \cdot)$. For a restricted weight $\mathfrak{G}$-module V we introduce the action of the Kac-Casimir operator Ω ([KK]) on V as follows: for $v \in V_\mu$, $\mu \in \mathfrak{H}^*$ let

$$\Omega v = (\mu + 2\rho, \mu)v + 2 \sum_{\beta \in Q_+} \sum_i e_{-\beta}^{(i)} e_\beta^{(i)} v,$$

where ρ is an element in $\mathfrak{H}^*$ such that $(\rho, \gamma) = 1$ for all $\gamma \in \pi$, $e_\beta^{(i)}$ form a basis of $\mathfrak{G}^\beta$ and $e_{-\beta}^{(i)}$ form the dual basis of $\mathfrak{G}^{-\beta}$. One can easily check that the form $(\cdot, \cdot)$ on $\mathfrak{H}^*$ is invariant under s_α.

3 α-Shapovalov form and the determinant formula

Set $\mathfrak{H}(\alpha) = U(\mathfrak{H}) \otimes \mathbb{C}[c]$. Consider the following decomposition of $U(\mathfrak{G})$ ([F, page 88]):

$$U(\mathfrak{G}) = \left(\mathfrak{N}_{-}^{\alpha} U(\mathfrak{G}) + U(\mathfrak{G})\mathfrak{N}_{+}^{\alpha}\right) \oplus \mathfrak{H}(\alpha)\mathbb{C}[X_\alpha]X_\alpha \oplus \mathfrak{H}(\alpha)\mathbb{C}[X_{-\alpha}]X_{-\alpha} \oplus \mathfrak{H}(\alpha).$$

Let p be the projection of $U(\mathfrak{G})$ on $\mathfrak{H}(\alpha)$ with respect to the above decomposition. We define α-Shapovalov form (or generalized Shapovalov form) F_α on $U(\mathfrak{G})$ as a symmetric bilinear form with values in $\mathfrak{H}(\alpha)$ as follows (see also [F, KK, MP, S]):

$$F_\alpha(x, y) = p(\sigma(x)y), \quad x, y \in U(\mathfrak{G}).$$

It is straightforward that the graded components $U(\mathfrak{G})_\xi$, $\xi \in \mathbb{Z}\mathcal{Q}$ are orthogonal with respect to F_α. Moreover, F_α is contravariant, i.e. $F_\alpha(zx, y) = F_\alpha(x, \sigma(z)y)$ for all $x, y, z \in U(\mathfrak{G})$.

Consider a vectorsubspace

$$\mathcal{M} = U(\mathfrak{N}_{-}^{\alpha} \oplus \mathfrak{G}^{\alpha}) + U(\mathfrak{N}_{-}^{\alpha} \oplus \mathfrak{G}^{-\alpha})$$

in $U(\mathfrak{G})$. For $\xi \in \mathbb{Z}\mathcal{Q}$ we set $\mathcal{M}_\xi = \mathcal{M} \cap U(\mathfrak{G})_\xi$. Clearly, each $\mathcal{M}_\xi$ is finite-dimensional. To calculate the dimension of $\mathcal{M}_\xi$ we have to introduce the notion of Kostant α-function P_α (see [MO]).

For $\gamma = \sum_{\beta\in\pi} a_\beta \beta \in \mathcal{Q}$ set $\psi_\alpha(\gamma) = \sum_{\beta\in(\pi\setminus\{\alpha\})} a_\beta \beta$. Define the Kostant α-function P_α : $\mathfrak{H}^* \to \mathbb{N} \cup \{0\}$ as follows: for $\lambda \in \mathfrak{H}^*$ set $P_\alpha(\lambda)$ to be the maximum number of the decompositions

$$\lambda + n\alpha = \sum_{\beta\in\mathcal{Q}_+\setminus\{\alpha\}} n_\beta \psi_\alpha(\beta)$$

with non-negative integer coefficients, where n runs through all integers. It follows easily from the definition of P_α that $\dim \mathcal{M}_{-\xi} = P_\alpha(\xi)$.

For $\eta \in \mathbb{Z}\mathcal{Q}$ we denote by F_α^η the restriction of F_α on $\mathcal{M}_{-\eta}$.

Let $\lambda \in \mathfrak{H}^*$ and $b \in \mathbb{C}$. Clearly, from the construction of $N(a, b)$ it follows that GVM $M(\lambda, b)$ is generated by $M(\lambda, b)_\lambda$ as a $\mathfrak{G}$-module. Let $0 \neq v_{(\lambda,b)} \in M(\lambda, b)_\lambda$ be a canonical generator of $M(\lambda, b)$. It is well-known (see for example [CF]) that $\mathcal{M}v_{(\lambda,b)} = M(\lambda, b)$ since $M(\lambda, b)$ is generated by $v_{(\lambda,b)}$. We can naturally identify $\mathfrak{H}(\alpha)$ with the ring of polynomials on the $\mathbb{C}$-space $\{(\lambda, b) \mid \lambda \in \mathfrak{H}^*, b \in \mathbb{C}\}$ by setting $c^* = (0, 1)$. Thus we can define the value $F_\alpha^\eta((\lambda, b))$ of F_α^η in the point (λ, b).

Now we can define a bilinear $\mathbb{C}$-valued form $\hat{F}_\alpha$ on $M(\lambda, b)$ by setting

$$\hat{F}_\alpha(u_1 v_{(\lambda,b)}, u_2 v_{(\lambda,b)}) = F_\alpha(u_1, u_2)((\lambda, b)), \quad u_1, u_2 \in \mathcal{M}.$$

One can easily obtain the following standard properties of $\hat{F}_\alpha$:

Lemma 1. *1. The kernel of $\hat{F}_\alpha$ coincides with the unique maximal submodule in the module $M(\lambda, b)$.*

2. $\hat{F}_\alpha$ is non-degenerate on $M(\lambda, b)$ if and only if $M(\lambda, b)$ is irreducible.

3. All weight subspaces of $M(\lambda, b)$ are orthogonal with respect to $\hat{F}_\alpha$.

Proof. Proof is analogues to that for classical Shapovalov form (see for example [MP]). □

The main result of this paper is the following theorem which computes the determinant of F^η_α:

Theorem 1. *Let $\mathfrak{G}$ be a contragradient Lie algebra with a symmetrisable Cartan matrix. Then for any $\eta \in \mathfrak{H}^*$*

$$\det F^\eta_\alpha = \prod_{k=1}^{\infty} (X_{-a}X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))^{P(\eta-k\alpha)} \times$$
$$\times \prod_{k=1}^{\infty} (X_{-a}X_\alpha + (1-k)(H_\alpha + \rho(H_\alpha) - (1-k)))^{\hat{P}(\eta-k\alpha)} \times$$
$$\times \prod_{\substack{\beta \in \mathcal{Q}_+ \setminus \{\alpha\},\\ s_\alpha(\beta) = \beta}} \prod_{k=1}^{\infty} \left(H_\beta + \rho(H_\beta) - k\frac{(\beta,\beta)}{2}\right)^{P_\alpha(\eta-k\beta)} \times$$
$$\times \prod_{\substack{\{\beta, s_\alpha(\beta) | \\ \beta \in \mathcal{Q}_+ \setminus \{\alpha\},\\ s_\alpha(\beta) \neq \beta\}}} \prod_{k=1}^{\infty} \left(\left(H_\beta + \rho(H_\beta) - k\frac{(\beta,\beta)}{2}\right) \cdot\right.$$
$$\left. \cdot \left(H_{s_\alpha(\beta)} + \rho(H_{s_\alpha(\beta)}) - k\frac{(\beta,\beta)}{2}\right) + \alpha(H_\beta)\alpha(H_{s_\alpha(\beta)})X_{-\alpha}X_\alpha\right)^{P_\alpha(\eta-k\beta)}.$$

up to a non-zero constant factor, where all the roots β are taken with their multiplicities.

We note that the product in the last factor of the above formula runs through all non-orderd pairs $\{\beta, s_\alpha(\beta)\}$ such that $\beta \neq s_\alpha(\beta)$.

4 Proof of the determinant formula

Proof of Theorem 1 follows general line of the original proof in [KK]. Although, there are several differences and technical difficulties. To proceed we need the following lemmas.

Lemma 2. *Up to a non-zero constant factor,* $\det F^\eta_\alpha$ *is a product of factors having one of the following forms:*

1. $(X_{-a}X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))$;

2. $(X_{-a}X_\alpha + (1-k)(H_\alpha + \rho(H_\alpha) - (1-k)))$;

3. $\left(H_\beta + \rho(H_\beta) - k\frac{(\beta,\beta)}{2}\right)$, *where* β *is a quasiroot such that* $s_\alpha(\beta) = \beta$.

4. $\left(\left(H_\beta + \rho(H_\beta) - k\frac{(\beta,\beta)}{2}\right) \cdot \left(H_{s_\alpha(\beta)} + \rho(H_{s_\alpha(\beta)}) - k\frac{(\beta,\beta)}{2}\right) + \alpha(H_\beta)\alpha(H_{s_\alpha(\beta)})X_{-\alpha}X_\alpha\right)$, *where* β *is a quasiroot such that* $s_\alpha(\beta) \neq \beta$.

Proof. Consider a GVM $M(\lambda, b)$ generated by a non-zero element $v_{(\lambda,b)} \in M(\lambda,b)_\lambda$. First we note that the module $M(\lambda,b)$ is restricted ([KK]) and thus the action of Ω on it is well-defined. Applying Ω to $v_{(\lambda,b)}$ one obtains $\Omega v_{(\lambda,b)} = ((\lambda+2\rho,\lambda) + (b - ((\lambda,\alpha)+1)^2)/2)v_{(\lambda,b)}$ and thus Ω acts as $((\lambda+2\rho,\lambda) + (b-(\lambda+\rho,\alpha)^2)/2)id$ on $M(\lambda,b)$.

Consider the $\mathfrak{G}_\alpha$-module $N(a,b)$ from the definition of $M(\lambda,b)$. Note that $M(\lambda,b)$ can be reducible in two cases: if $N(a,b)$ is reducible or if there exists an α-highest weight vector in some $M(\lambda,b)_\mu$ with $\mu - \lambda \notin \mathbb{Z}\alpha$.

Suppose that $N(a,b)$ is reducible. This is possible if and only if for some $m \in \mathbb{N}$ $X_\alpha^m X_{-\alpha}^m v_{(\lambda,b)} = 0$ or $X_{-\alpha}^m X_\alpha^m v_{(\lambda,b)} = 0$ holds. By the direct calculations with $U(\mathfrak{G}_\alpha)$ we obtain

$$\prod_{k=1}^{m} \left(X_{-a}X_\alpha + k(H_\alpha + \rho(H_\alpha) - k)\right) v_{(\lambda,b)} = 0$$

or

$$\prod_{k=1}^{m} \left(X_{-a}X_\alpha + (1-k)(H_\alpha + \rho(H_\alpha) - (1-k))\right) v_{(\lambda,b)} = 0.$$

Further, suppose that there exists an α-highest weight vector w in $M(\lambda,b)_\mu$ for some $\mu \in \mathfrak{H}^*$ such that $\mu - \lambda \notin \mathbb{Z}\alpha$. Then the eigenvalues of Ω on $v_{(\lambda,b)}$ and w coincide and we obtain

$$(\lambda+2\rho,\lambda) + (b - ((\lambda,\alpha)+1)^2)/2 = (\mu+2\rho,\mu) + (b' - ((\mu,\alpha)+1)^2)/2 \qquad (1)$$

for some $b' \in \mathbb{C}$. Clearly, the difference $b'-b$ polynomialy depends on $\sqrt{b}$ after fixing λ^α and $\mu-\lambda$ (see [FM]). Thus the formula above can be applied to the case $(a+1+2n)^2 = b$, $n \in \mathbb{Z}$. For such $N(a,b)$ we get $M(\lambda,b)$ to be an extension of two Verma modules (with respect to different bases in $\mathcal{Q}$). Now, using the fact that the action of Ω on a Verma module can be calculated at the highest weight vector, we obtain that $b' = b + 2\sqrt{b}(\mu-\lambda,\alpha) + (\mu-\lambda,\alpha)^2$ (here $\sqrt{b}$ is complex square root function which has two different values as soon as $b \neq 0$).

If $(\mu-\lambda,\alpha) = 0$ the equality (1) reduces to $(\lambda+2\rho,\lambda) = (\mu+2\rho,\mu)$ and we can use the same arguments as in proof of [KK, Lemma 3.2] obtaining the factors $(H_\beta - \rho(H_\beta) - (\beta,\beta)/2)$ (here β is not necessary quasiroot).

If $(\mu - \lambda, \alpha) \neq 0$ we can take two equalities of the form (1) corresponding to different values b_1 and b_2 of $\sqrt{b}$, transfer everything in the left-hand side and multiply them. We obtain the following (here $\beta = \lambda - \mu$):

$$(2(\lambda+\rho,\beta) - (\beta,\beta) - (\lambda+\rho,\alpha)(\beta,\alpha))^2 - b(\beta,\alpha)^2 = 0.$$

The last equality can be rewritten in the form

$$(2(\lambda+\rho,\beta) - (\beta,\beta) - (\lambda+\rho,\alpha)(\beta,\alpha))^2 - (\beta,\alpha)^2(\lambda+\rho,\alpha)^2 - (\beta,\alpha)^2(b - (\lambda+\rho,\alpha)^2) = 0.$$

We note that

$$(\lambda+\rho,\beta) - (\lambda+\rho,\alpha)(\beta,\alpha) = (\lambda+\rho,\beta - (\beta,\alpha)\alpha) = (\lambda+\rho,\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha) = (\lambda+\rho, s_\alpha(\beta)).$$

From this it follows that

$$\begin{aligned}(2(\lambda+\rho,\beta) - (\beta,\beta))\,(2(\lambda+\rho,s_\alpha(\beta)) - (s_\alpha(\beta),s_\alpha(\beta))) +& \\ + 4(\frac{1}{4}(b - (\lambda+\rho,\alpha)^2))(\alpha,\beta)(\alpha,s_\alpha(\beta)) &= 0.\end{aligned}$$

Taking into account that $\frac{1}{4}(b - (\lambda+\rho,\alpha)^2)$ is an eigenvalue of the operator $X_{-\alpha}X_\alpha$, we obtain the factor of the form

$$\big((H_\beta + \rho(H_\beta) - (\beta,\beta)/2)\,\big(H_{s_\alpha(\beta)} + \rho(H_{s_\alpha(\beta)}) - (\beta,\beta)/2\big) + \alpha(H_\beta)\alpha(H_{s_\alpha(\beta)})X_{-\alpha}X_\alpha\big)$$

with the same arguments as in [KK, Lemma 3.2].

Now we only need to show that all β appeared above are quasiroots. Suppose not. Thus we will have some factor of the determinant of F_α corresponding to a non-quasiroot β. Calculating F_α on a Verma submodule for some reducible $N(a,b)$ we obtain a contradiction with [KK, Theorem 1]. Lemma is proved. $\square$

By PBW theorem we can define a new α-gradation on $U(\mathfrak{G})$ by setting the grade of $X_{\pm\alpha}$ and the grade of H_α to be 0 and all the grades of other base elements in $\mathfrak{G}$ to be 1.

Lemma 3. *Up to a factor of grade zero the leading term of* $\det F_\alpha^\eta$ *with respect to the* α*-gradation is equal to*

$$\prod_{\beta\in\mathcal{Q}_+\setminus\{\alpha\}} \prod_{k=1}^{\infty} H_\beta^{P_\alpha(\eta-k\beta)}.$$

Proof. From the classical Shapovalov determinant formula ([KK]) it follows that the above formula is correct for $\det F_\alpha^{\eta-l\alpha}$, where $l \in \mathbb{N}$ is big enough. To complete the proof it is sufficient to show that the leading term of $\det F_\alpha^\eta$ in the α-gradation does not depend on the shift on α.

Choose some PBW monomial base $v_1, \dots, v_t$ in $\mathcal{M}_{-\eta}$ and suppose that as soon as some v_i contains $X_{-\alpha}$ this monomial should start with this $X_{-\alpha}$. Consider the elements $X_\alpha v_1, \dots, X_\alpha v_t$ and let W be a linear span of these elements. For $1 \leqslant i \leqslant t$ set $\hat{v}_i = X_\alpha v_i$ if v_i does not contain $X_{-\alpha}$ and $\hat{v}_i = w_i$ if $v_i = X_{-\alpha} w_i$. Clearly, elements $\hat{v}_1, \dots, \hat{v}_t$ form a basis of $\mathcal{M}_{-\eta+\alpha}$. Moreover, it follows from the definition of $\hat{v}_i$ that up to a factor of zero degree the leading term of $\det F_\alpha^{\eta-\alpha}$ coincides with the leading term of the determinant of the form F_α restricted to W (we will denote it by $F_\alpha(W)$). Since the base change from $v_1, \dots, v_t$ to $X_\alpha v_1, \dots, X_\alpha v_t$ is defined by the elements of zero grade it follows that $\det F_\alpha^\eta$ differs from $\det F_\alpha(W)$ by a factor of grade zero. This implies that the leading term of $\det F_\alpha^\eta$ in the α-gradation does not depend on the shift on α. $\square$

To proceed we have to define a Jantzen filtration on $M(\lambda, b)$. Choose $z \in \mathfrak{H}^*$ such that $(z, \beta) \neq 0$ for all $\beta \in \mathbb{Z}\mathcal{Q}_+ \setminus 0$. Let t be an indeterminate. By standard technique we can extend $M(\lambda, b)$ to the module $\widetilde{M(\widetilde{\lambda, b})}$ over the algebra $\widetilde{U(\mathfrak{G})} = U(\mathfrak{G}) \otimes \mathbb{C}[t]$, where $\widetilde{(\lambda, b)} = (\lambda, b) + t(z, 1) \in \widetilde{\mathfrak{H}(\alpha)^*} = \mathfrak{H}(\alpha)^* \otimes \mathbb{C}[t]$. Further we can trivially extend σ on $\widetilde{U(\mathfrak{G})}$ and construct a bilinear form $\widetilde{F_\alpha}$. Using $\widetilde{F_\alpha}$ one can define a bilinear $\mathbb{C}[t]$-valued form $\widetilde{F_\alpha^t}$ on $\widetilde{M(\widetilde{\lambda, b})}$. Setting $\widetilde{M^i}$ to be equal to the set of all elements v in $\widetilde{M(\widetilde{\lambda, b})}$ such that $\widetilde{F_\alpha^t}(v, w)$ is divisible by t^i for all $w \in \widetilde{M(\widetilde{\lambda, b})}$ we define a Jantzen filtration

$$\widetilde{M(\widetilde{\lambda, b})} = \widetilde{M^0} \supset \widetilde{M^1} \supset \dots$$

on $\widetilde{M(\widetilde{\lambda, b})}$. The canonical epimorphism $\varphi : \widetilde{M(\widetilde{\lambda, b})} \to M(\lambda, b)$ $(t \to 0)$ induces a filtration

$$M(\lambda, b) = M^0 \supset M^1 \supset \dots$$

of $M(\lambda, b)$ which will be also called Jantzen filtration.

Proof of theorem 1. We have only to calculate the degrees in $\det F_\alpha^\eta$ of the factors described in Lemma 2. For a quasiroot β, which is not proportional to α, the proof of this fact is exactly the same as in [KK, Proof of Theorem 1] because of Lemma 3 and the remark that the functions $P_\alpha(x - y)$ $y \in \alpha^\perp$ are linearly independent (here $\alpha^\perp$ is taken with respect to $(\cdot, \cdot)$).

Thus we have only to calculate the degrees of the factors of the form

- $(X_{-a} X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))$;
- $(X_{-a} X_\alpha + (1 - k)(H_\alpha + \rho(H_\alpha) - (1 - k)))$.

We will do it for the first kind of factors. One can apply analogous arguments for the second case. Consider a factor $(X_{-a} X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))$ for some fixed $k \in \mathbb{N}$. Let $N(a, b)$ be such that it has the unique submodule starting at the highest weight $a - k\alpha$. We note that in this case $a \notin \mathbb{Z}$. One can easily choose $\lambda \in \mathfrak{H}^*$ $(\lambda(H_\alpha) = a)$ such that GVM $M(\lambda, b)$ has the unique non-trivial submodule N. Clearly, in the described case N is isomorphic to the Verma module $M(\lambda - k\alpha)$. From the definition of Jantzen filtration we have $M^0 = M(\lambda, b)$ and $M^1 = N$. Our goal is to prove that $M^2 = 0$. Since N is irreducible it follows that either $M^2 = N$ or $M^2 = 0$. Consider $\widetilde{U(\mathfrak{G})}$-modules $\widetilde{M(\lambda, b)}$ and $\widetilde{N}$ and let w be a canonical generator of $\widetilde{N}$. Use the definition of $\widetilde{F_\alpha}$ to calculate $\widetilde{F_\alpha^t}(w, w)$. By the direct application of $sl(2)$-theory we obtain that

$$\widetilde{F_\alpha^t}(w, w) = \prod_{i=1}^{k} f_k(t),$$

where $f_k(t) \in \mathbb{C}[t]$ such that $f_k'(0) \neq 0$ satisfy the following condition: the differences between constant terms in f_{k+1} and f_k is equal to $a - 2k$. Since a is not integer it follows that the product in the formula above is divisible at most by t. But it is divisible by t since $\widetilde{N}$ is a submodule. Thus the canonical generator of N belongs to M^1 and does not belong to M^2. Hence $M^2 = 0$. Now we can claim that from the construction of Jantzen filtration it follows immediately, that $\det F_\alpha^\eta$ is divisible exactly by $P(\eta - k\alpha)$-th power of $(X_{-a}X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))$ (see [KK, Proof of Theorem 1] and [MP, Section 6.6]). This completes our proof. $\square$

5 Structure of GVMs

As in the classical case, the determinant formula for F_α enables one to prove a generalization of the BGG-criterion for the embeddings of Verma modules (see [KK, Theorem 2] and [MP, Section 6.7]). In this section we will formulate and prove an analogous result for GVMs induced from $\mathfrak{G}_\alpha$.

For $\lambda, \mu \in \mathfrak{H}^*$ and $b_1, b_2 \in \mathbb{C}$ we set $(\lambda, b_1) \rightharpoonup (\mu, b_2)$ in one of the following cases:

1. $b_1 = b_2$ and $\lambda = \mu - k\alpha$ for some $k \in \mathbb{Z}$;

2. $b_1 = b_2 \pm 2\sqrt{b_2}(k\beta, \alpha) + (k\beta, \alpha)^2$ for $k \in \mathbb{N}$ and $\beta \in \mathcal{Q}_+ \setminus \{\alpha\}$ such that $\lambda = \mu - k\beta$ and

$$2(\lambda + \rho)(H_\beta) - k(\beta, \beta) - (\lambda + \rho)(H_\alpha)(\beta, \alpha) = \pm\sqrt{b_2}(\beta, \alpha).$$

   (here an analytic branch of $\sqrt{z}$ function is fixed).

Denote by $\prec$ the transitive closure of the relation $\rightharpoonup$ on $\mathfrak{H}^* \times \mathbb{C}$.

For each pair $\beta \neq s_\alpha(\beta)$ of roots in $\mathcal{Q}_+$ we fix some bijective map

$$\mathrm{sign} : \{\beta, s_\alpha(\beta)\} \to \{\pm 1\}.$$

We also set $\mathrm{sign}(\beta) = 0$ if $(\alpha, \beta) = 0$ and fix some analytic branch of $\sqrt{z}$ function. For $\beta \in \mathcal{Q}_+ \setminus \{\alpha\}$, $k \in \mathbb{N}$ and $b \in \mathbb{C}$ set $f_{\beta,k}(b) = b + 2\,\mathrm{sign}(\beta)\sqrt{b_2}(k\beta, \alpha) + (k\beta, \alpha)^2$.

First of all it worth nothing to formulate the following criterion of irreducibility of the module $M(\lambda, b)$ which follows immediately from Theorem 1 and Lemma 1.

Theorem 2. *$M(\lambda, b)$ is irreducible if and only if two following conditions are satisfied:*

1. *$((\lambda+\rho,\alpha)+2k)^2 \neq b$ for all $k \in \mathbb{Z}$.*

2. *$((2(\lambda+\rho,\beta)-k(\beta,\beta))\,(2(\lambda+\rho,s_\alpha(\beta))-k(s_\alpha(\beta),s_\alpha(\beta)))+(\alpha,\beta)(\alpha,s_\alpha(\beta))\,\cdot$
$\cdot\,(b-(\lambda+\rho,\alpha)^2)) \neq 0$ for all $\beta \in \mathcal{Q}_+ \setminus \{\alpha\}$ and for all $k \in \mathbb{N}$.*

Remark 1. *The first condition of the above theorem is equivalent to the condition that the module $N(a,b)$ (see definition of $M(\lambda,b)$) and thus the module $M(\lambda,b)$ is α-stratified. Hence for α-stratified modules one needs to check only the second condition.*

The following theorem is a generalization of BGG structure theorem for Verma modules (see [BGG, Theorem 2] and [KK, Theorem 2]).

Theorem 3. *The following statements are equivalent:*

1. *$L(\lambda,b_1)$ is a subquotient of $M(\mu,b_2)$.*

2. *$M(\lambda,b_1) \subset M(\mu,b_2)$;*

3. *$(\lambda,b_1) \prec (\mu,b_2)$.*

Proof. One can easily see that it is enough to prove that the first condition implies the third one. Other implications are easy. Using Theorem 1 all necessary steps can be done at the same way as in [KK, Theorem 2]. We will only outline the basic statements.

Consider the Jantzen filtration

$$M(\mu,b_2) = M^0 \supset M^1 \supset \dots$$

defined in the previous section. Clearly

$$\operatorname{ord} \widetilde{F_\alpha^t}^{\eta}(\mu,b_2) = \sum_{i\geqslant 1} \dim M^i_{\mu-\eta},$$

where ord denotes the maximal power of t dividing $\widetilde{F_\alpha^t}^{\eta}(\mu,b_2)$. Further, it follows by direct calculation that

$$\sum_{i\geqslant 1} \operatorname{ch} M^i = \sum_k \operatorname{ch} M(\mu-k\alpha) + \sum_k \operatorname{ch} M^\alpha(\mu+k\alpha) + \sum_{(\beta,k)} \operatorname{ch} M(\mu-k\beta, f_{\beta,k}(b_2)),$$

where the first sum is taken over positive integers k such that $((\lambda+\rho,\alpha)-2k)^2 = b$ and $M(\xi)$ denotes the Verma module with respect to π with the highest weight $\xi \in \mathfrak{H}^*$, the second sum is taken over positive integers k such that $((\lambda+\rho,\alpha)+2k)^2 = b$ and $M^\alpha(\xi)$ denotes the Verma module with respect to $s_\alpha(\pi)$ with the highest weight $\xi \in \mathfrak{H}^*$ and the last sum is taken over all pairs $(\beta,k) \in \mathcal{Q}_+ \times \mathbb{N}$, $\beta \neq \alpha$ such that $(\mu-k\beta, f_{\beta,k}(b_2)) \prec (\mu,b_2)$. Now proof of the theorem follows by standard arguments using induction in η (see [KK, Proof of Theorem 2]). $\square$

Remark 2. *One can easily obtain that the equivalence 1) $\Leftrightarrow$ 3) in theorem 3 remains valid even for GVMs $M(\lambda,b)$ that is not generated by $M(\lambda,b)_\lambda$ (this means that $N(a,b)$ is not generated by $N(a,b)_a$ and we can forget about this condition on $N(a,b)$). This case can be reduced easily to that where $N(a,b)$ is generated by $N(a,b)_a$.*

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