SIMPLE WEIGHT MODULES OVER TWISTED GENERALIZED WEYL ALGEBRAS

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Abstract

We introduce a twisted higher rank generalization of generalized Weyl algebras and study simple weight modules over such algebras.

1 Introduction

Generalized Weyl algebras (GWA) appeared in the original work by V.Bavula (see bibliography in [2]) and were studied intensively during last years (see [1, 2, 3, 5]). In fact, complete classification of simple modules over such algebras was obtained in [2] and complete classification of indecomposable weight modules was obtained in [5]. The class of generalized Weyl algebras contains a number of known algebras such as $U(sl(2,\mathbb{C}))$, $U_q(sl(2,\mathbb{C}))$, Weyl algebra and many other studied in literature ([7, 10, 11]). In [1] the higher rank GWA were introduced as a tensor product of rank 1 GWA, moreover, it was shown in [3] that the problem to describe indecomposable weight (generalized weight) modules over higher rank GWA is wild in general case (in the sense of [4]).

In the present paper we propose a construction of twisted higher rank generalized Weyl algebras and study their simple weight modules in a torsion-free case. It happened, that in our case the support of a finite-dimensional module might have more interesting geometrical structure than in the case of classical GWA. Moreover, we provide some analogue between the structure of such supports for twisted GWA and for finite-dimensional modules over classical simple Lie algebras. The same analogue can not be obtained for the classical higher rank GWA.

The paper is organized as follows: In section 2 we introduce twisted generalized Weyl algebras and establish their basic properties. In sections 3 we describe canonical simple weight modules for the case of torsion-free orbit and, finally, in section 4 we illustrate an analogue between the supports of simple weight modules over twisted GWA and simple Lie algebras by examples.

2 Definition of twisted GWA

Denote by $\mathbb{C}$ the complex field, by $\mathbb{Z}$ the ring of integers, by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}_+$ the set of all non-negative integers.

Let R be a commutative ring with a unit element. As in [5] we define a category $\mathcal{C}$ to be an R-category if its morphism sets $\mathcal{C}(i,j)$ are equipped with an R-bimodule structure for any objects i, j, the multiplication of its morphisms is R-linear with respect to both left and right R-module structure, and $(ar)b = a(rb)$ for any possible morphisms a, b and $r \in R$. If $\mathcal{C}$ contains only one object we will call it an R-ring.

Denote by $\operatorname{Max} R$ the set of maximal ideals $\mathbf{m} \subset R$. For $\mathbf{m} \in \operatorname{Max} R$ and an R-module V we set $V_{\mathbf{m}} = \{v \in V \mid \mathbf{m}v = 0\}$. For an R-category $\mathcal{C}$, any $\mathcal{C}$-module M and any object i, M_i becomes an R-module in a natural way. We call M the weight module if

$$M = \sum_{\mathbf{m} \in \operatorname{Max} R} M_{\mathbf{m}}.$$

For a weight module M we set $\operatorname{Supp} M = \{\mathbf{m} \in \operatorname{Max} R \mid M_{\mathbf{m}} \neq 0\}$.

Let Γ be a finite non-oriented tree with a set of vertices Γ_0 and a set of edges Γ_1. Suppose that σ_i, $i \in \Gamma_0$ is a set of pairwise commuting automorphisms of R. Fix some elements $0 \neq t_i \in R$, $i \in \Gamma_0$ satisfying the following relations:

- $t_i t_j = \sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i)$, $(i,j) \in \Gamma_1$;
- $\sigma_i(t_j) = t_j$, $(i,j) \notin \Gamma_1$.

Let $\mathfrak{A}'$ be an R-ring generated over R by the indeterminates X_i, Y_i, $i \in \Gamma_0$ subject to the following relations:

- $X_i r = \sigma_i(r) X_i$ for any $r \in R$, $i \in \Gamma_0$;

- $Y_i r = \sigma_i^{-1}(r) Y_i$ for any $r \in R$, $i \in \Gamma_0$;
- $X_i Y_j = Y_j X_i$ for any $i, j \in \Gamma_0$, $i \neq j$;
- $Y_i X_i = t_i$, $i \in \Gamma_0$;
- $X_i Y_i = \sigma_i(t_i)$, $i \in \Gamma_0$.

Lemma 1. *Algebra $\mathfrak{A}'$ is non-trivial.*

Proof. Since, by [2, 5], $\langle X_i, Y_i, R \rangle$ is a free R-module with the base 1, X_i^k, Y_i^k, $k \in \mathbb{N}$ we need only to check that the conditions $X_i Y_j = Y_j X_i$, $i \neq j$, and $Y_i X_i = t_i$ do not contradict each other. It is obvious for $(i, j) \notin \Gamma_1$. For $(i, j) \in \Gamma_1$ it follows easily from the identity

$$t_i t_j = \sigma_i^{-1}(t_j) \sigma_j^{-1}(t_i).$$

□

Denote by $\mathfrak{i} : \mathfrak{A}' \to \mathfrak{A}'$ the canonical anti-involution on $\mathfrak{A}'$ defined as follows: $\mathfrak{i}(r) = r$ for any $r \in R$, $\mathfrak{i}(X_i) = Y_i$ and $\mathfrak{i}(Y_i) = X_i$ for $i \in \Gamma_0$.

Algebra $\mathfrak{A}'$ possesses a natural structure of $\mathbb{Z}^n$ graded algebra ($n = |\Gamma_0|$) by setting $\deg R = 0$, $\deg X_i = e_i$, $\deg Y_i = -e_i$, $i \in \Gamma_0$, where e_i, $i = 1, \dots, n$ denote the standard generators of $\mathbb{Z}^n$. Clearly, R coincides with the zero component with respect to this grading.

Lemma 2. *Among graded two-sided ideals of $\mathfrak{A}'$ intersecting R trivially there exists a unique maximal ideal I.*

Proof. Since a sum of graded ideals is again a graded ideal and R is a graded component of $\mathfrak{A}'$, it follows that I is the sum of all graded two-sided ideals that do not intersect R. □

We define a twisted GWA $\mathfrak{A} = \mathfrak{A}(R, \sigma_1, \dots, \sigma_n, t_1, \dots, t_n)$ of rank $n = |\Gamma_0|$ as a quotient ring $\mathfrak{A}'/I$, where I is the ideal given by Lemma 2. As it was done in [5] we retain the term "algebra" for all cases, even if there is no ground field at all. In all natural examples R is a k-algebra over some field k and thus $\mathfrak{A}$ is really a k-algebra.

Lemma 3. *The following identities hold in $\mathfrak{A}$:*

1. $X_i X_j = X_j X_i$ *for all* $(i, j) \notin \Gamma_1$.
2. $X_i X_j t_i = X_j X_i \sigma_j^{-1}(t_i)$ *for all* $i \neq j$, $(i, j) \in \Gamma_1$;
3. $X_i X_j v_i = X_j X_i v_j$ *if* $i \neq j$, $t_i = u_i v_i$, $t_j = u_j v_j$, *such that* $\sigma_i(u_j) = u_j$, $\sigma_j(u_i) = u_i$, $\sigma_i(v_i) = v_j$, $\sigma_j(v_j) = v_i$;

4. $Y_iY_j = Y_jY_i$ *for all* $(i,j) \notin \Gamma_1$,

5. $\sigma_j^{-1}(t_i)Y_iY_j = t_iY_jY_i$ *for all* $i \neq j$, $(i,j) \in \Gamma_1$.

6. $v_jY_iY_j = v_iY_jY_i$ *if* $i \neq j$, $t_i = u_iv_i$, $t_j = u_jv_j$, *such that* $\sigma_i(u_j) = u_j$, $\sigma_j(u_i) = u_i$, $\sigma_i(v_i) = v_j$, $\sigma_j(v_j) = v_i$;

Proof. Follows from the defining relations by direct calculations. □

$W = \langle \sigma_i \mid i \in \Gamma_0 \rangle$ acts on $\operatorname{Max} R$ in a natural way. Let Ω denotes the set of orbits under this action. For a weight $\mathfrak{A}$-module V and $\gamma \in \Omega$ set $M_\gamma = \oplus_{\mathbf{m} \in \gamma} M_{\mathbf{m}}$.

Lemma 4. *Let* M *be a weight* $\mathfrak{A}$*-module and* $\mathbf{m} \in \operatorname{Max} R$

1. $X_iM_{\mathbf{m}} \subset M_{\sigma_i(\mathbf{m})}$;

2. $Y_iM_{\mathbf{m}} \subset M_{\sigma_i^{-1}(\mathbf{m})}$;

3. $M = \oplus_{\gamma \in \Omega} M_\gamma$, *moreover,* $\operatorname{Supp} M_\gamma \subset \gamma$.

Proof. These are the standard properties of weight modules that follow immediately from the definition of $\mathfrak{A}$ ([5]). □

For $\gamma \in \Omega$ we will denote by $\mathcal{M}(\gamma)$ the full subcategory in the category of $\mathfrak{A}$-modules consisting of all weight modules M with $\operatorname{Supp} M \subset \gamma$. For a commutative ring K and its maximal ideal $\mathbf{m}$ we will denote by $K_{\mathbf{m}}$ the corresponding residue field.

Examples 1. *1. Let* Γ *be a tree without edges,* R *an arbitrary field,* σ_i *the identity map and* $t_i = 1$ *for* $i \in \Gamma_0$*. Then* $\mathfrak{A}$ *is isomorphic to the Laurent polynomial ring* $R[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, $n = |\Gamma_0|$.

2. *Let* Γ *be a tree without edges,* σ_i, t_i, $i \in \Gamma_0$ *arbitrary elements satisfying the defining relations. If* R *has no zero divisors then* $\mathfrak{A}$ *is a higher rank GWA. In fact,* $\mathfrak{A}$ *is isomorphic to a quotient ring of* $\mathfrak{A}'$ *by the two-sided ideal generated by* $X_iX_j - X_jX_i$, $Y_iY_j - Y_jY_i$, $i, j \in \Gamma_0$, $i \neq j$.

3. *Let* Γ *be the Coxeter graph of type* A_2*. Consider the polynomial algebra* $R = k[H]$ *over an arbitrary field* k *and automorphisms* σ_1, σ_2 *defined by* $\sigma_1(H) = H + 1$, $\sigma_2(H) = H - 1$*. Set* $t_1 = H$, $t_2 = H + 1$*. Then* $\mathfrak{A} = \mathfrak{A}(k[H], \sigma_1, \sigma_2, H, H + 1)$ *is a non-trivial twisted GWA since* $H(H + 1) = ((H - 1) + 1)(H + 1)$*. But in this presentation* $\mathfrak{A}$ *is not a higher rank GWA in the sense of [1].*

3 Simple weight modules for torsion-free case

In this section we will use one general result from [6] to obtain a classification of canonical simple weight modules having the supports on an orbit γ satisfying the conditions below. Fix a bijection of Γ_0 with the set $\{1,\ldots,n\}$, $n=|\Gamma_0|$. Let $\varphi:\mathbb{Z}^n\to W$ be a canonical epimorphism sending the i-th generator e_i of $\mathbb{Z}_n$ to σ_i. Let $K=\ker\varphi$ and $F(W)$ be a subset of Γ_0 consisting of all those $j\in\Gamma_0$ such that there exists $a\in K$, $a=\sum_{i=1}^n a_ie_i$ with $a_j\neq 0$. We will say that the pair W,γ, where $\gamma\in\Omega$, is torsion-free if the following conditions hold:

- The order of each σ_i is infinite in W;
- $w\mathbf{m}\neq\mathbf{m}$ for any $\mathbf{m}\in\gamma$ and all $w\in W$, $w\neq 1$;
- There are no $(i,j)\in\Gamma_1$ such that $i,j\in F(W)$.

An orbit $\gamma\in\Omega$ is called torsion-free if the pair W,γ is torsion-free. Throughout this section we fix a torsion-free orbit γ. For Corollary 1 and Theorem 2 we also assume that R is an algebra over an algebraically closed field. Our results are based on the following key observation.

Lemma 5. *Let A be a subalgebra of $\mathfrak{A}$ generated by all homogeneous $a\in\mathfrak{A}$ with $\deg a\in K$. Then A is commutative.*

Proof. For $l=(l_1,l_2,\ldots,l_n)\in\mathbb{Z}_n$ set $|l|=|l_1|+|l_2|+\ldots|l_n|$. A monomial $x\in\mathfrak{A}'$, $\deg x=l$ is said to be reduced if $x=Z_1Z_2\ldots Z_{|l|}$, where Z_i is one of the generators $X_1,\ldots,X_n$, $Y_1,\ldots,Y_n$, $1\leq i\leq|l|$. It follows from the definition of A that any reduced monomial generator x of A is of the form: $x=Z_1Z_2\ldots Z_{|\deg x|}$, where $Z_i\in\{X_j,Y_j\,|\,j\in F(W)\}$. Moreover, since $F(W)$ is a subgraph of Γ without edges, we see, by Lemma 3, that $x=Z_{\tau(1)}Z_{\tau(2)}\ldots Z_{\tau(\deg x)}$ for any permutation $\tau\in S_{|\deg x|}$. It is clear that the proof is completed by showing that $xy-yx\in I$ for any two reduced monomial generators of A viewing as elements of $\mathfrak{A}'$.

Let $x,y\in A$ be reduced monomials with $\deg x=l$ and $\deg y=l'$. We consider x and y as elements in $\mathfrak{A}'$. Suppose that $l=\sum_{i\in F(W)}l_ie_i$ and $l'=\sum_{i\in F(W)}l_i'e_i$ Consider a reduced monomial $z\in\mathfrak{A}'$ such that z is a product of some X_i, Y_j and $\deg z=-l-l'$. Let us calculate zxy. ¿From the disscussion above it follows that any X_i (Y_i), $i\in\Gamma_0$ occurring in zxy commutes with any factor $Z\in\{X_k,Y_k,k\in\Gamma_0\}$ of zxy such that $Z\neq Y_i$ ($Z\neq X_i$). Thus, we can write

$$z=\prod_{i\in F(W)}\left(Z_{i1}\ldots Z_{ik_i}\right),x=\prod_{i\in F(W)}\left(Z_{ik_i+1}\ldots Z_{im_i}\right),y=\prod_{i\in F(W)}\left(Z_{im_i+1}\ldots Z_{is_i}\right),$$

$$zxy = \prod_{i \in F(W)} (Z_{i1} Z_{i2} \dots Z_{is_i}),$$

where $Z_{ij} = X_i$ or $Z_{ij} = Y_i$ for all possible i, j. Fix $i \in F(W)$. Since $\deg zxy = 0$ we conclude that the number of X_i among Z_{ij}, $1 \le j \le s_i$ coincides with the number of Y_i among Z_{ij}, $1 \le j \le s_i$. Moreover,

$$Z_{i1} Z_{i2} \dots Z_{is_i} = Z(1)^{|l_i + l'_i|} Z(2)^{|l_i|} Z(3)^{|l'_i|},$$

where $Z(1)$ ($Z(2)$, $Z(3)$) equals X_i if $l_i + l'_i > 0$ (respectively $l_i > 0$, $l'_i > 0$) or equals Y_i if $l_i + l'_i < 0$ (respectively $l_i < 0$, $l'_i < 0$). Now one has to consider case by case all possibilities for the signs of l_i, l'_i and $l_i + l'_i$. We will consider only that with $l_i > 0$, $l'_i < 0$ and $l_i + l'_i < 0$. The same works for other cases.

According to the above argument, we have

$$Z_{i1} Z_{i2} \dots Z_{is_i} = X_i^{-l_i - l'_i} X_i^{l_i} Y_i^{-l'_i} = X_i^s Y_i^s,$$

where $s = -l'_i$. Note that $\sigma_i(t_j) = t_j$ for any $j \in F(W)$, $i \ne j$. Since $l, l' \in K$ we obtain $\sigma_i^{l_i}(t_i) = \sigma_i^{l'_i}(t_i) = t_i$ and hence $\sigma_i^s(t_i) = t_i$. Applying the defining relations to the monomial above we have

$$A = Z_{i1} Z_{i2} \dots Z_{is_i} = \prod_{j=1}^{s} \sigma_i^j(t_i).$$

On the other hand, we can do the same for the product zyx. By the same procedure we will obtain the following submonomial in zyx corresponding to the same i:

$$B = X_i^{-l_i - l'_i} Y_i^{-l'_i} X_i^{l_i} = X_i^{-l_i - l'_i} Y_i^{-l_i - l'_i} Y_i^{l_i} X_i^{l_i} = \prod_{j=1}^{-l_i - l'_i} \sigma_i^j(t_i) \cdot \prod_{j=-l_i+1}^{0} \sigma_i^j(t_i).$$

Taking into account that $\sigma_i^t(t_i) = t_i$ we obtain

$$\prod_{j=-l_i+1}^{0} \sigma_i^j(t_i) = \sigma_i^s \left(\prod_{j=-l_i+1}^{0} (\sigma_i^j(t_i)) \right) = \prod_{j=s-l_i+1}^{s} (\sigma_i^j(t_i)) = \prod_{j=-l'_i - l_i + 1}^{s} (\sigma_i^j(t_i)).$$

Hence $A = B$.

It follows now that $zxy = zyx$ for any reduced x, y, z as above. Thus $zxy - zyx = 0$ and $xy - yx \in I$. Hence $xy - yx = 0$ in $\mathfrak{A}$, since otherwise $xy - yx$ generates a non-trivial two-sided graded ideal intersecting R trivially. □

Corollary 1. *Let $M \in \mathcal{M}(\gamma)$ be a simple module, $\mathbf{m} \in \gamma$. Then* $\dim_{R_{\mathbf{m}}} M_{\mathbf{m}} \le 1$.

Proof. Follows from Lemma 5 and [6], Theorem 18. □

An element $\mathbf{m} \in \gamma$ will be called forward (backward) i-break ($i \in \Gamma_0$) if $t_i \in \mathbf{m}$ ($\sigma_i(t_i) \in \mathbf{m}$).

For $\mathbf{m} \in \gamma$ denote by $P_{\mathbf{m}}$ the set of all $\mathbf{n} \in \gamma$ such that there exists an element $w \in W$ such that

$$w = \sigma_{i_1}^{\varepsilon_1} \sigma_{i_2}^{\varepsilon_2} \dots \sigma_{i_k}^{\varepsilon_k}, \quad \varepsilon_l = \pm 1,\, 1 \leq l \leq k,$$

with $\mathbf{m} = w\mathbf{n}$ and each

$$\sigma_{i_{l+1}}^{\varepsilon_{l+1}} \dots \sigma_{i_k}^{\varepsilon_k} \mathbf{n}, \quad 1 \leq l \leq k,$$

is not a forward i_l-break if $\varepsilon_l = 1$ and is not a backward i_l-break if $\varepsilon_l = -1$. It follows directly from the definition that $P_{\mathbf{m}} = P_{\mathbf{n}}$ for any $\mathbf{n} \in P_{\mathbf{m}}$.

Lemma 6. *Let $M \in \mathcal{M}(\gamma)$ be a module with $M_{\mathbf{m}} \neq 0$. Then $P_{\mathbf{m}} \subset \operatorname{Supp} M$.*

Proof. Let $\mathbf{n} \in P_{\mathbf{m}}$. Assume that $\mathbf{n} \notin \operatorname{Supp} M$. It is sufficient to consider the case $\mathbf{m} = \sigma_i \mathbf{n}$ where $\mathbf{n}$ is not a forward i-break. Then $t_i \notin \mathbf{n}$ and $\sigma_i(t_i) \notin \mathbf{m}$. It follows immediately that $X_i Y_i M_{\mathbf{m}} = \sigma_i(t_i) M_{\mathbf{m}} \neq 0$. Thus $Y_i M_{\mathbf{m}} \neq 0$, and $Y_i M_{\mathbf{m}} \subset M_{\mathbf{n}}$ by Lemma 4. □

Lemma 7. *Let $\mathbf{m} \in \gamma$ be a forward i-break, $i \in \Gamma_0$ and $j \in \Gamma_0$, $i \neq j$. Then either $\sigma_i(\mathbf{m})$ is a forward j-break or $\sigma_j(\mathbf{m})$ is a forward i-break and either $\sigma_j^{-1}(\mathbf{m})$ is a forward j-break or $\sigma_j^{-1}(\mathbf{m})$ is a forward i-break*

Proof. Since $t_i \in \mathbf{m}$ we have $t_i t_j \in \mathbf{m}$ and thus $\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i) \in \mathbf{m}$ (we note that the identity $t_i t_j = \sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i)$ holds for any $i, j \in \Gamma_0$ $i \neq j$). Applying to the both sides of the last formula the automorphisms σ_i, σ_j we easily obtain that either $\sigma_i(\mathbf{m})$ is a forward j-break or $\sigma_j(\mathbf{m})$ is a forward i-break.

Suppose that $\sigma_j^{-1}(\mathbf{m})$ is not a forward i-break and forward j-break. Then t_i and t_j do not belong to $\sigma_j^{-1}(\mathbf{m})$ and thus $t_i t_j = \sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i) \notin \sigma_j^{-1}(\mathbf{m})$ which gives that $\sigma_i(\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i)) = t_i \sigma_j \sigma_i^{-1}(t_j) \notin \mathbf{m}$. This contradicts the assumption that $\mathbf{m}$ is a forward i-break and thus contains t_i. □

A simple module $M \in \mathcal{M}(\gamma)$ is said to be canonical if $\operatorname{Supp} M = P_{\mathbf{m}}$ for some $\mathbf{m} \in \gamma$ (and thus for any $\mathbf{m} \in \operatorname{Supp} M$). Let $\mathbf{n} \in P_{\mathbf{m}}$ be some i-break. We will call it inner break if $\sigma_i(\mathbf{n}) \in P_{\mathbf{m}}$ (for forward break $\mathbf{n}$) or $\sigma_i^{-1}(\mathbf{n}) \in P_{\mathbf{m}}$ (for backward break $\mathbf{n}$).

We call a set $S \subset \gamma$ regular if it does not contain inner breaks and the following condition holds:

- For any $i, j \in \Gamma_0$, $i \neq j$ the condition $\mathbf{n}, \sigma_i(\sigma_j^{-1}(\mathbf{n})) \in P_{\mathbf{m}}$ implies $\sigma_i(\mathbf{n}), \sigma_j^{-1}(\mathbf{n}) \in P_{\mathbf{m}}$.

Next proposition gives the complete description of $P_{\mathbf{m}}$:

Proposition 1. *$P_{\mathbf{m}}$ is regular.*

Proof. Consider the free abelian group $G = \mathbb{Z}^n$ and fix the standard basis $\{e_1, \dots, e_n\}$ in it. We can identify G with the set of points with integer coordinates in the Euclidean space. Let $\hat{G}$ be a non-oriented graph with G as the set of vertices and the set of edges defined as follows: vertices g and h are connected by an edge if and only if $g = h \pm e_i$ for some $i \in \{1, \dots, n\}$. Fix the canonical epimorphism φ of G onto W with the kernel K (see the beginning of the section). To each $g \in G$ there corresponds an element $\varphi(g)(\mathbf{m})$ of γ. Let $\hat{G}$ be the graph obtained from $\tilde{G}$ by erasing edges connecting g and $g + e_i$ for all $g \in G$, $i \in \Gamma_0$ such that $\varphi(g)(\mathbf{m})$ is a forward i-break. Clearly, up to K, $P_{\mathbf{m}}$ can be identified with a connected component of $\tilde{G}$. Thus it is sufficient to show that for any connected component C of $\tilde{G}$ the following holds:

1. C is a full subgraph of $\hat{G}$, i.e. if elements g, $g + e_i \in G$ are vertices from C then C contains the corresponding edge;

2. if C contains vertices g and $g + e_i - e_j$ for some $g \in G$ and $i \neq j \in \{1, \dots, n\}$ then it contains $g + e_i$ and $g - e_j$.

To show this we begin with some observations. From Lemma 7 we see that the following is true:

1. Let g, $g + e_i$, $g + e_j$, $g + e_i + e_j$ be the vertices of the graph $\tilde{G}$. Then the number of edges of $\tilde{G}$ connecting these four vertices can not be equal to 3. Moreover, if there are exactly two of them then they are either the edges $(g, g + e_i)$ and $(g + e_i, g + e_i + e_j)$ or the edges $(g, g + e_j)$ and $(g + e_j, g + e_i + e_j)$.

2. Let g, $g + e_i$, $g + e_i - e_j$, $g - e_j$ be a quadruple of vertices from $\tilde{G}$ such that $\tilde{G}$ does not contain any edge connecting $g - e_j$ with any of other three. Then $\tilde{G}$ contains at most one edge connecting two of these four vertices.

We will call these properties quadruple and triple properties respectively.

We now suppose that C contains g and $g + e_i$ for some $g \in G$ and $j \in \{1, \dots, n\}$ but it does not contain the corresponding edge. Since C is a connected component, there is a path a_0, a_1, $\dots$, a_k in G such that $a_0 = g$, $a_k = g + e_i$ and a_l is connected by an edge with a_{l+1} for all possible l. Suppose that this path has minimal possible length (k). By the quadruple property we have $k > 3$.

Consider the element $Z = Z_k Z_{k-1} \dots Z_1 \in \mathfrak{A}'$ defined in the following way: $Z_l = X_j$ if $a_l = a_{l-1} + e_j$ and $Z_l = Y_j$ if $a_l = a_{l-1} - e_j$. Using the relations in $\mathfrak{A}'$ we can rewrite Z as a product $Z = Z^1 Z^2 \dots Z^s$, where each factor Z^t is either a product of some X_i, $i \in \Gamma_0$ or Y_i, $i \in \Gamma_0$ i.e., $Z^t = X_{i_1} X_{i_2} \dots X_{i_{k(t)}}$, or $Z^t = Y_{i_1} Y_{i_2} \dots Y_{i_{k(t)}}$. Moreover, we will assume that if Z^t is a product of some elements X_i (Y_i respectively) then the factors Z^{t+1}, Z^{t-1} are products of Y_i (X_i respectively). Clearly, $s > 1$ since Z necessarily contains a factor from $\{X_1, \dots, X_n\}$ and a factor from $\{Y_1, \dots Y_n\}$. The last follows from the fact that $e_1, \dots, e_n$ are linearly independent. We can also suppose that s is minimal possible. Consider Z^1 and Z^2. One of them, say Z^1, is a product of some elements from $\{X_1, \dots, X_n\}$ and another is a product of elements from $\{Y_1, \dots, Y_n\}$. Moreover, $Z^1 Z^2 \neq Z^2 Z^1$ since s is minimal. Thus there is $j \in \{1, \dots, n\}$ such that X_j is a factor in Z^1 and Y_j is a factor in Z^2. Let $Z^2 = Y_{i_1} Y_{i_2} \dots Y_{i_{k(2)}}$ and p is the minimal such that X_{i_p} is a factor in Z^1. Then

$$Z^1 Z^2 = Y_{i_1} \dots Y_{i_{p-1}} Z^1 Y_{i_p} \dots Y_{i_{k(2)}}.$$

Commuting Z^1 with Y_{i_p} we obtain that Z contains a fragment $X_{i_p} Y_{i_p}$ which can be obviously erased. The last contradicts with the minimality of k. Thus C contains the edge connecting g with $g + e_i$ and hence C is a full subgraph of $\hat{G}$.

Assume now that C contains g and $g + e_i - e_j$ for some $g \in G$ and $i \neq j \in \{1, \dots, n\}$. We claim that if C contains also one of the elements $g + e_i$, $g - e_j$ then it does contain another one. Indeed, if this is not the case then according to the triple property we conclude, that C is not a full subgraph of $\hat{G}$. Assuming that C does not contain both of elements $g + e_i$ and $g - e_j$ and considering a path connecting g with $g + e_i - e_j$ one can obtain a contradiction in the same way as above. This completes the proof. □

Lemma 8. *Let M be a simple weight $\mathfrak{A}'$-module with a regular support. Then $IM = 0$.*

Proof. Suppose that $uv \neq 0$ for some homogeneous $u \in I$ and $v \in M_{\mathbf{m}}$. Then $\hat{u}v \neq 0$ for some monomial $\hat{u}$ of u. Let $\hat{u} = rA_1 A_2 \dots A_t$, where $r \in R$ and each $A_j = X_i$ or Y_i for some $i \in \Gamma_0$ (depending on j). Thus $A_j A_{j+1} \dots A_t v \neq 0$ for any $1 \leq j \leq t$ and we obtain $\mathfrak{i}(\hat{u})\hat{u}v \neq 0$ since $\operatorname{Supp} M$ is regular (and thus has no inner breaks). Moreover, the absence of inner breaks in $\operatorname{Supp} M$ and the inequality $\mathfrak{i}(\hat{u})\hat{u}v \neq 0$ forces $\mathfrak{i}(\hat{u})w \neq 0$ for any non-zero w having the same weight with $\hat{u}v$. Since u is homogeneous, we immediately obtain that $\mathfrak{i}(\hat{u})uv \neq 0$. But $\deg \mathfrak{i}(\hat{u})u = 0$ and $\mathfrak{i}(\hat{u})u \in I$ which implies $\mathfrak{i}(\hat{u})u = 0$, since $I \cap R = 0$. This contradiction completes our proof. □

Theorem 1. *Suppose that $\{\sigma_i \,|\, i \in \Gamma_0\}$ is a set of free generators of W as an abelian group. Then each simple weight module in $\mathcal{M}(\gamma)$ is canonical. Moreover, there is a $1-1$ correspondence between simple weight $\mathfrak{A}$-modules $M \in \mathcal{M}(\gamma)$ and subsets $P_{\mathbf{m}} \in \gamma$.*

Proof. By [6], Theorem 18, for any $\mathbf{m} \in \gamma$ there is the unique simple weight $\mathfrak{A}$-module M with $M_{\mathbf{m}} \neq 0$. Thus to prove the theorem we need only to construct an $\mathfrak{A}$-module M with $\operatorname{Supp} M = P_{\mathbf{m}}$ for each $\mathbf{m} \in \gamma$.

Set $M_{\mathbf{n}} = R/\mathbf{n}$, $\mathbf{n} \in P_{\mathbf{m}}$. Since Γ is a tree it is a biserial graph. Let $\Gamma_0 = \Gamma_0^1 \cup \Gamma_0^2$ be a disjoint union such that $i \in \Gamma_0^a$, $j \in \Gamma_0^b$, $a \neq b$ for any $(i, j) \in \Gamma_1$.

Consider $\mathbf{n} \in \operatorname{Max} R$. Clearly, any σ_i defines a homomorphism from $R/\mathbf{n}$ to $R/\sigma_i(\mathbf{n})$ which we will denote by the same symbol σ_i.

For $\mathbf{n} \in P_{\mathbf{m}}$ and $v \in M_{\mathbf{n}}$ define

$$X_i v = \begin{cases} \sigma_i(t_i v), & \sigma_i(\mathbf{n}) \in P_{\mathbf{m}}, \\ 0, & \text{otherwise}, \end{cases}$$

$$Y_i v = \begin{cases} \sigma_i^{-1}(v), & \sigma_i^{-1}(\mathbf{n}) \in P_{\mathbf{m}}, \\ 0, & \text{otherwise}, \end{cases}$$

if $i \in \Gamma_0^1$ and

$$X_i v = \begin{cases} \sigma_i(v), & \sigma_i(\mathbf{n}) \in P_{\mathbf{m}}, \\ 0, & \text{otherwise}, \end{cases}$$

$$Y_i v = \begin{cases} t_i \sigma_i^{-1}(v), & \sigma_i^{-1}(\mathbf{n}) \in P_{\mathbf{m}}, \\ 0, & \text{otherwise}, \end{cases}$$

if $i \in \Gamma_0^2$.

It follows from regularity of $P_{\mathbf{m}}$ and direct calculation that these formulae define a structure of an $\mathfrak{A}'$-module on M. From Lemma 8 and Lemma 6 we obtain that M is a simple $\mathfrak{A}$-module. This completes the proof of the theorem. □

Suppose now that W is not a free abelian group which is equivalent to the condition $\ker \varphi \neq 0$. Consider a map ψ from the set of all monomials in X_i, Y_i, $i \in \Gamma_0$ into W defined in the following way: $\psi(X_i) = \sigma_i$, $\psi(Y_i) = \sigma_i^{-1}$ and $\psi(AB) = \psi(A)\psi(B)$ for any monomials A and B.

Fix $\mathbf{m} \in \gamma$ and denote by $A(\mathbf{m})$ the subalgebra of A generated by all monomials $Z_1 Z_2 \dots Z_k$ having the following property: $\psi(Z_l \dots Z_k)(\mathbf{m}) \in P_{\mathbf{m}}$ for all $1 \leq l \leq k$.

Theorem 2. *There is a* $1 - 1$ *correspondence between canonical simple weight* $\mathfrak{A}$*-modules* $M \in \mathcal{M}(\gamma)$ *such that* $\operatorname{Supp} M = P_{\mathbf{m}}$ *and elements* $\chi \in (A(\mathbf{m})/(\mathbf{m}))^*$*, where* $(\mathbf{m})$ *is an ideal of* $A(\mathbf{m})$ *generated by* $\mathbf{m}$ *and* $(A(\mathbf{m})/(\mathbf{m}))^*$ *denotes the space of all* $R/(\mathbf{m})$*-linear homomorphisms from* $A(\mathbf{m})/(\mathbf{m})$ *to* $R/(\mathbf{m})$.

Proof. The statement on 1-1 correspondence follows from [6], Theorem 18. Hence we will only give a construction of simple weight $\mathfrak{A}$-module M associated with $\chi \in (A(\mathbf{m})/(\mathbf{m}))^*$.

Let $\Gamma_0 = \Gamma_0^1 \cup \Gamma_0^2$ be the decomposition of the tree Γ_0 given in Theorem 1. Consider the subset $S \subset \mathbb{Z}^n$ satisfying the following conditions:

- $0 \in S$;
- for any $\mathbf{n} \in P_\mathbf{m}$ there is a unique $s \in S$ such that $\varphi(s)(\mathbf{m}) = \mathbf{n}$;
- $\varphi(s)(\mathbf{m}) \in P_\mathbf{m}$ for any $s \in S$;
- for any $s \in S$ there exists a decomposition $s = (-1)^{\varepsilon_1} e_{i_1} + \ldots + (-1)^{\varepsilon_{i_l}} e_{i_l}$ such that $(-1)^{\varepsilon_l} e_{i_l} + \ldots + (-1)^{\varepsilon_{i_k}} e_{i_k} \in S$ for any $l = 1, \ldots, k$.

The existence of S is trivial.

For $s \in S$ we fix a decomposition $s = (-1)^{\varepsilon_1} e_{i_1} + \ldots + (-1)^{\varepsilon_{i_l}} e_{i_l}$ such that $(-1)^{\varepsilon_l} e_{i_l} + \ldots + (-1)^{\varepsilon_{i_k}} e_{i_k} \in S$ for any $l = 1, \ldots, k$ and define $X(s) = S_1 S_2 \ldots S_k$, where $S_l = X_{i_l}$ if ε_{i_l} is even and $S_l = Y_{i_l}$ if ε_{i_l} is odd. We also set $Z(s) = X(s)\mathfrak{i}(X(s)) \in R$ for $s \in S$.

Clearly, $\varphi(s)$ is an isomorphism from $R/(\mathbf{m})$ to $R/(\varphi(s)(\mathbf{m}))$. Thus $R/(\varphi(s)(\mathbf{m})) = \{\varphi(s)(v) \mid v \in R/(\mathbf{m})\}$.

Let $M_\mathbf{n} = R/(\mathbf{n})$, $\mathbf{n} \in P_\mathbf{m}$. For $i \in \Gamma_0$, $\mathbf{n} \in P_\mathbf{m}$ and $w \in R/(\mathbf{n})$ we define $X_i w = 0$ ($Y_i w = 0$) if $\sigma_i(\mathbf{n}) \notin P_\mathbf{m}$ ($\sigma_i^{-1}(\mathbf{n}) \notin P_\mathbf{m}$).

Let $\mathbf{n} \in P_\mathbf{m}$ and $\mathbf{n} = \varphi(s)(\mathbf{m})$ for some $s \in S$. Suppose that $e_i + s \in S$ ($-e_i + s \in S$). Then for $v \in R_\mathbf{m}$ we define $X_i \varphi(s)(v) = \sigma_i(t_i \varphi(s)(v))$ ($Y_i \varphi(s)(v) = \sigma_i^{-1}(\varphi(s)(v))$) if $i \in \Gamma_0^1$ and $X_i \varphi(s)(v) = \sigma_i(\varphi(s)(v))$ ($Y_i \varphi(s)(v) = t_i \sigma_i^{-1}(\varphi(s)(v))$) if $i \in \Gamma_0^2$ similar to that in Theorem 1.

This implies that $X(s)v = \alpha(s)\varphi(s)(v)$ for some $\alpha(s) \in R/(\varphi(s)(\mathbf{m}))$ which does not depend on v. Consider $Z = X_i$ or $Z = Y_i$ for $i \in \Gamma_0$ and set $e = e_i$ if $Z = X_i$ and $e = -e_i$ if $Z = Y_i$. Suppose that $\varphi(e + s)(\mathbf{m}) \in P_m$ but $e + s \notin S$. Then there is an element $s' \in S$ such that $\varphi(e + s)(\mathbf{m}) = \varphi(s')(\mathbf{m})$. It is left to define $Z\varphi(s)(v)$ for $v \in R/(\mathbf{m})$. We begin with the equality $\alpha(s)\varphi(s)(v) = X(s)v$ and multiply it by Z obtaining $Z\alpha(s)\varphi(s)(v) = ZX(s)v$. Since $\alpha(s) \in R/(\varphi(s)(\mathbf{m}))$ is a non-zero scalar we conclude that $\varphi(e)(\alpha(s)) \in R/(\varphi(s + e)(\mathbf{m}))$ is non-zero and $Z\varphi(s)(v) = (\varphi(e)(\alpha(s)))^{-1} ZX(s)v$. Applying $\mathfrak{i}(X(s'))$ to the both sides of this equality, we have

$$\mathfrak{i}(X(s'))Z\varphi(s)(v) = \mathfrak{i}(X(s'))(\varphi(e)(\alpha(s)))^{-1} ZX(s)v$$

and hence

$$\mathfrak{i}(X(s'))Z\varphi(s)(v) = (\varphi(-s')(\varphi(e)(\alpha(s))))^{-1}\mathfrak{i}(X(s'))ZX(s)v.$$

Since $\mathfrak{i}(X(s'))ZX(s) \in A(\mathbf{m})$, we have $\mathfrak{i}(X(s'))ZX(s)v = \chi(\mathfrak{i}(X(s'))ZX(s))v$. Let $u = \chi(\mathfrak{i}(X(s'))ZX(s))$. Now we can apply $X(s')$ to equality above obtaining

$$X(s')\mathfrak{i}(X(s'))Z\varphi(s)(v) = X(s')(\varphi(-s')(\varphi(e)(\alpha(s))))^{-1}uv,$$

$$Z(s')Z\varphi(s)(v) = X(s')(\varphi(-s')(\varphi(e)(\alpha(s))))^{-1}uv$$

and since $Z(s')$ is non-zero we obtain

$$Z\varphi(s)(v) = (Z(s'))^{-1}X(s')(\varphi(-s')(\varphi(e)(\alpha(s))))^{-1}uv =$$

and finally

$$= (Z(s'))^{-1}((\varphi(e)(\alpha(s))))^{-1}\varphi(s')(\chi(\mathfrak{i}(X(s'))ZX(s)))\alpha(s')\varphi(s')(v).$$

Since the order of each σ_i is infinite all the operators above are well-defined. Direct calculation based on the above arguments shows that the obtained formulae define an action of an $\mathfrak{A}'$-module on M. By Lemma 8, M is an $\mathfrak{A}$-module. □

Remark 1. *We note that the problem of classification of simple weight modules for $\mathfrak{A}'$ in a torsion-free case is wild in general. One can easily obtain this with $n = 2$, Γ is of type A_2, $R = \mathbb{Z}_{35}[X]$, $\sigma_1(X) = X - 7$, $\sigma_2 = \sigma_1^{-1}$, $t_1 = (X-7)(X-14)(X-21)(X-28)X^2$, $t_2 = (X-7)^2(X-14)(X-21)(X-28)X$.*

Remark 2. *It is possible that there exist simple weight $\mathfrak{A}$-modules which are not canonical under the conditions of Theorem 2. For example, let $R = \mathbb{C}[x,y]$, $\sigma_1 = \sigma_2$, where $\sigma_1(x) = x+1$, $\sigma_1(y) = y$ and $t_1 = t_2 = y$. For the corresponding algebra $\mathfrak{A}$ any orbit γ containing a break contains the breaks only. Fix such γ and put $\mathbb{C}$ in any point of γ. Define that X_1 and Y_2 act as identity operators and X_2 and Y_1 act as zero operators. One can easily see that we obtain a simple weight $\mathfrak{A}$-module which is not canonical.*

4 Examples

In this section we give two examples of twisted generalized Weyl algebras with complete description of simple weight modules. The main goal is to explain geometrical structure of support of simple weight module and provide some analogue with simple finite-dimensional Lie algebras.

4.1 Example 1

Let Γ be a graph with two vertices and one edge between them (the Coxeter graph of type A_2). Let $R = \mathbb{C}[H_1, H_2]$, $t_1 = H_1 H_2$, $t_2 = H_1 + 1$ and the automorphisms σ_1 and σ_2 are defined by:

$$\sigma_1(H_1) = H_1 + 1,\ \sigma_1(H_2) = H_2 + 1, \quad \sigma_2(H_1) = H_1 - 1,\ \sigma_2(H_2) = H_2.$$

One can easily check all necessary relations and conclude that $\mathfrak{A} = \mathfrak{A}(R, \sigma_1, \sigma_2,\ t_1, t_2)$ is a twisted generalized Weyl algebra.

It follows easily from the definition of σ_1, σ_2 that any orbit $\gamma \in \Omega$ is torsion-free and $W \simeq \mathbb{Z}^2$. Thus all simple weight $\mathfrak{A}$-modules can be obtained by Theorem 1. We shall describe supports of such modules. The action of elements of $\mathfrak{A}$ on a module can be obtained directly from the proof of Theorem 1.

We identify $\operatorname{Max} R$ with $\mathbb{C}^2$ in a natural way. Thus W becomes a subgroup in the group of all affine transformations of $\mathbb{C}^2$.

Case 1. Suppose that $\mathbf{m} = (a, b)$, $a, b \notin \mathbb{Z}$ and consider $\gamma = W\mathbf{m}$. One can easily see that γ does not contain breaks and thus there exists the unique simple $\mathfrak{A}$-module $M \in \mathcal{M}(\gamma)$ with $\operatorname{Supp} M = W\mathbf{m}$.

Case 2. Suppose that $\mathbf{m} = (a, b)$, $a, b \in \mathbb{Z}$ and let $\gamma = W\mathbf{m}$. It follows from the definition of t_i that all forward (backward) 1-breaks are $(0, c)$, $(c, 0)$ $((1, c+1)$, $(c+1, 1))$ $c \in \mathbb{Z}$ and all forward (backward) 2-breaks are $(-1, c)$ $((0, c))$, $c \in \mathbb{Z}$. Thus there are four non-isomorphic simple weight modules M_i, $i = 1, 2, 3, 4$, in $\mathcal{M}(\gamma)$ and

$$\operatorname{Supp} M_1 = \{(c, d) \mid -c \in \mathbb{N}; -d \in \mathbb{N}\};$$

$$\operatorname{Supp} M_2 = \{(c, d) \mid c \in \mathbb{Z}_+; -d \in \mathbb{N}\};$$

$$\operatorname{Supp} M_3 = \{(c, d) \mid c \in \mathbb{Z}_+; d \in \mathbb{Z}_+\};$$

$$\operatorname{Supp} M_4 = \{(c, d) \mid -c \in \mathbb{N}; d \in \mathbb{Z}_+\}.$$

We note that the supports obtained in this case differ from the supports for classical higher rank GWA since the set of 2-breaks is not invariant under the action of σ_1.

Case 3. Suppose that $\gamma = W(a, b)$, $a \in \mathbb{Z}$, $b \notin \mathbb{Z}$. In this case there are exactly 2 simple weight modules M_1 and M_2 in $\mathcal{M}(\gamma)$ with

$$\operatorname{Supp} M_1 = \{(c, d) \mid c \in \mathbb{Z}_+; d \in b + \mathbb{Z}\};$$

$$\mathrm{Supp}\, M_2 = \{(c,d) \mid -c \in \mathbb{N}; d \in b + \mathbb{Z}\}.$$

Case 4. Suppose $\gamma = W(a,b)$, $a \notin \mathbb{Z}$, $b \in \mathbb{Z}$. In this case there are exactly 2 simple weight modules M_1 and M_2 in $\mathcal{M}(\gamma)$ with

$$\mathrm{Supp}\, M_1 = \{(c,d) \mid c \in a + \mathbb{Z}; -d \in \mathbb{N}\};$$

$$\mathrm{Supp}\, M_2 = \{(c,d) \mid c \in a + \mathbb{Z}; d \in \mathbb{Z}_+\}.$$

As a corollary we obtain that $\mathfrak{A}$ does not have any finite-dimensional simple weight module. Next example presents an algebra which possesses finite-dimensional weight simple modules.

4.2 Example 2

Let Γ, R, σ_1, σ_2 be as in the previous example. Set $t_1 = H_2(H_2 - 3)(H_1 - 1)(H_1 - 4)$, $t_2 = H_1(H_1-3)(H_1-H_2+1)(H_1-H_2-2)$. One can easily check that $\mathfrak{A} = \mathfrak{A}(R, \sigma_1, \sigma_2, t_1, t_2)$ is a twisted generalized Weyl algebra. We consider only the most interesting case of $\gamma = \{(a,b) \mid a, b \in \mathbb{Z}\}$. It is easy to see that in this case there are exactly 18 simple weight modules in $\mathcal{M}(\gamma)$, and 6 from them are finite-dimensional. Among these modules there is one module M with $\mathrm{Supp}\, M = \{(1,0), (1,1), (2,0), (2,1), (2,2), (3,1), (3,2)\}$.

Geometrically $\mathrm{Supp}\, M$ is a hexagon which is analogous to the general form of the support of a finite-dimensional $sl(3)$-module. One can easily construct analogous examples for an arbitrary simple Lie algebra with a simply-laced Dynkin diagram. Moreover, it can be easily shown that the category of finite-dimensional $\mathfrak{A}$-modules in the second example is not semi-simple. This property is analogous to that of algebras studied in [10].

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