# The coinvariant algebra and representation types of blocks of category $\mathcal{O}$ *

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**Abstract**

Let $\mathfrak{G}$ be a finite dimensional semisimple Lie algebra over the complex numbers. Let $A$ be the finite dimensional algebra of a (regular or singular) block of the BGG–category $\mathcal{O}$. By results of Soergel [20], $A$ has a combinatorial description in terms of a subalgebra $C_0$ of the coinvariant algebra $C$. In [18], an embeddding has been constructed from $C_0$–mod into the category $\mathcal{F}(\Delta)$ of $A$–modules having a Verma flag. This is the main tool for our classification of $\mathcal{F}(\Delta)$ into finite, tame and wild representation type. As a consequence we also obtain a classification of $A$–mod into finite, tame and wild representation type, thus reproving a recent result of Futorny, Nakano and Pollack [10].

## 1 Introduction and statement of results

Let $\mathfrak{G}$ be a finite dimensional semisimple Lie algebra over the complex numbers. Fix a triangular decomposition, $\mathfrak{G} = \mathfrak{N}_- \oplus \mathfrak{H} \oplus \mathfrak{N}_+$. Then the BGG-category $\mathcal{O}$, which has been defined in [4], decomposes into a direct sum of indecomposable subcategories, called *blocks*. By a result of Soergel ([20]) we can restrict our attention from now on to blocks with integral support, since any other block is equivalent to one with integral support (note, however, that applying such an equivalence may involve changing $\mathfrak{G}$ and thus also changing the root system). Each block is equivalent to the module category of a finite dimensional associative algebra $A$ which is unique up to Morita equivalence. The full subcategory of $A$–mod consisting of modules having a Verma flag is denoted by $\mathcal{F}(\Delta)$. Associated with a block having integral support are the following combinatorial data: an antidominant integral weight $\lambda$, the Weyl group $W$ and the stabilizer subgroup $W_0$ of $W$ fixing $\lambda$. Denote the root systems corresponding to $W$ and to $W_0$ by $\Phi$ and $\Phi_0$, respectively. Our classifications will depend on these root systems only.

*AMS Subject Classification: 16G60, 17B10

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**Theorem 1.** *Fix a block and its finite dimensional algebra $A$. Then the representation type of the category $\mathcal{F}(\Delta)$ is given by the following table. All cases not listed are wild.*

| *representation type* | $\Phi$ | $\Phi_0$ |
|---|---|---|
| *finite* | *any* $\Phi$ | $\Phi_0 = \Phi$ |
| | $A_1$ | $\emptyset$ |
| | $A_2$ | $A_1$ |
| | $A_3$ | $A_2$ |
| | $B_2$ | $A_1$ |
| | $A_4$ | $A_3$ |
| *tame* | $A_1 \times A_1$ | $\emptyset$ |
| | $A_5$ | $A_4$ |
| | $B_3$ | $B_2$ |

Of course, an equality $\Phi = \Phi_0$ occurs precisely for $A$ a simple algebra.

As a consequence of the proof of Theorem 1 we also obtain:

**Theorem 2.** *Fix a block and its finite dimensional algebra $A$. Then the representation type of the category $A$–mod is given by the following table. All cases not listed are wild.*

| *representation type* | $\Phi$ | $\Phi_0$ |
|---|---|---|
| *finite* | *any* $\Phi$ | $\Phi_0 = \Phi$ |
| | $A_1$ | $\emptyset$ |
| | $A_2$ | $A_1$ |
| *tame* | $A_3$ | $A_2$ |
| | $B_2$ | $A_1$ |

Theorem 2 has been proved in [10] by completely different methods. The formulation there is for simple Lie algebras. However, our more general formulation follows easily from the results in [10].

Our method is based on Soergel's [20] description of blocks of $\mathcal{O}$ as endomorphism rings over certain subalgebras $C_0$ of coinvariant algebras and on a result in [18] which states that the module category of $C_0$ embeds into $\mathcal{F}(\Delta)$ (we reprove this result here for the sake of completeness). The algebra $C_0$ is local and commutative and explicit computations therein can be easily performed using a basis coming from Schubert calculus. Therefore, the category $C_0$–mod is seen to be wild unless we are in one of a small number of limited situations where we really have to study the categories $A$-mod and $\mathcal{F}(\Delta)$. These situations have to be dealt with case by case. (A full classification of $C_0$-mod into finite, tame and wild type has been obtained by I.Gordon and A.Premet [12].)

In Section 2 we collect some information on coinvariant algebras and their relation to blocks of $\mathcal{O}$. In Section 3 we consider some special cases, starting with the regular block for $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. In Section 4 a proof of Theorem 1 and of Theorem 2 is given.

## 2 Coinvariant algebras

We fix $\mathfrak{G}$, $\mathcal{O}$, the root system $\Phi$ and the Weyl group $W$ as before. The Weyl group $W$ is a Coxeter group with length function $l$. Moreover, we fix a subset, $\Theta$, of the set of simple roots and denote by $W_0$ the "parabolic" subgroup of $W$ generated by the corresponding simple reflections. Its root system is called $\Phi_0$. The Weyl group acts as a reflection group on the complex vector space $V$ and thus on the symmetric algebra $S(V)$. By definition, the *coinvariant algebra* $C$ is the quotient of $S(V)$ modulo the two-sided ideal generated by homogenous $W$-invariants of degree at least one. By $C_0$ we denote the (graded) subalgebra of $C$ consisting of $W_0$-invariant elements.

It is well-known that $C$ is isomorphic to the cohomology algebra of the flag manifold $\mathcal{G}/\mathcal{B}$. Therefore, Schubert calculus provides us with a special basis of $C$, which contains a basis of $C_0$ as well. This has been worked out in Hiller's book [14], from which we take the following information (Chapter IV, in particular Sections 3 and 4).

There is a complete set of left coset representatives of $W_0$ in $W$ consisting of elements $w$ such that each $w$ has minimal length in its coset. These are precisely those $w$ which satisfy $l(ws) = l(w) + 1$ for all simple reflections $s \in \Theta$. The algebra $C_0$ has a basis consisting of elements $X_w$ indexed by these coset representatives, such that the following properties are satisfied: The element $X_w$ is homogenous of degree $l$. For a simple reflection, $s = s_\alpha$, and any element, $w \in W$, the product $X_s X_w$ in $C$ is given by the *Pieri formula* (where the $\omega_\alpha$ are fundamental weights):

$$X_s X_w = \sum_{\beta \in \Phi^+, l(ws_\beta) = l(w)+1} (\beta^\vee, \omega_\alpha) X_{ws_\beta}. \qquad (1)$$

If $s$ and $w$ are coset representatives, then this is the product formula in $C_0$. In particular, if $w$ is chosen in such a way that the product $ws$ is longer than $w$, then the product $X_s X_w$ is not zero. In fact, in this case $X_{ws}$ occurs with a non–zero coefficient in the right hand term.

Let us now recall the role of $C$ and $C_0$ in describing blocks of $\mathcal{O}$. Fix a block by choosing an antidominant integral weight $\lambda$. Its stabilizer is a subgroup, $W_0$. Then, by Soergel's *Struktursatz 2* and *Endomorphismensatz 3* from [20], the finite dimensional algebra $A$ associated with this block is the endomorphism ring of a certain module, $P$, (the "big projective module") over $C_0$:

**Theorem 3 (Soergel, [20]).** $A \simeq End_{C_0}(P)$.

We remark that $C_0$ is a symmetric algebra. In fact, by Soergel's result it is the endomorphism ring over $A$ of the unique indecomposable projective–injective module $P$. Hence $C_0$ is self–injective. As a subalgebra of $C$, the algebra $C_0$ is commutative, thus it is symmetric.

Clearly, the module category of $C_0$ can be embedded into this block. From [18] it follows that even the following is true:

**Theorem 4.** *The category $C_0$-mod embeds into $\mathcal{F}(\Delta)$.*

*Proof.* This follows by combining Theorem 4 in [18] with Theorem 1 in [18]. Here, we give another - more direct - proof which is built up from arguments already used in [18].

By Soergel's result, the algebra $C_0$ is the endomorphism ring of the projective–injective $A$–module $P$ which we can write as $Ae$ for some primitive idempotent $e \in A$. Thus we can identify $C_0$ with $eAe$. It is well–known (see e.g. [1]) that the category $eAe$–mod is equivalent to the full subcategory of $A$–mod consisting of modules $M$ which are $P$–copresented, i.e. there is an exact sequence $0 \to M \to P_1 \stackrel{\varphi}{\to} P_2$ with $P_1, P_2 \in add(P)$ direct sums of copies of $P$. Fix such a module $M$ and a copresentation. It will be enough to show that $M$ has a Verma flag.

Let $F$ be a minimal submodule of $P_1$ such that it has the following properties: $F$ contains $M$; there exists a $\Delta$-filtration $F_0 \subset F_1 \subset \cdots \subset P_1$ of $P_1$ such that $F$ equals some $F_i$. We are going to show that $F$ equals $M$.

Let us recall some well–known properties of blocks of $\mathcal{O}$ (see e.g. [6, 4]). The projective module $P$ has a Verma flag. There exists a unique simple Verma module, $L$, in this block which is the socle of each Verma module. Moreover, $L$ is both the socle and the top of $P$.

In particular, the image of $\varphi : P_1 \to P_2$ has a socle which is a direct sum of copies of $L$. By the minimality of $F$, the quotient $F/M$ cannot have a composition factor isomorphic to $L$. Since $M$ is sent to zero by $\varphi$, $F$ must be in the kernel as well. This implies equality $F = M$. □

The embedding preserves isomorphisms and indecomposability. As observed in [18], we therefore have the following lower bound for representation types:

**Corollary 1.** *If $C_0$–mod has infinite representation type, then $\mathcal{F}(\Delta)$ and $A$–mod also have infinite type. If $C_0$–mod is wild, then $\mathcal{F}(\Delta)$ and $A$–mod are wild as well.*

Here we use the following definition of wild representation type for $\mathcal{F}(\Delta)$: The module category $A$–mod is wild if there exists an exact functor $F : \mathbb{C}[x, y]$–mod $\to A$–mod that preserves non-isomorphism and indecomposability. Now, the exact subcategory $\mathcal{F}(\Delta)$ of $A$–mod is called wild if there exists an exact functor $F$ as above that factors through $\mathcal{F}(\Delta)$. Drozd's tame and wild theorem [8] applies to modules having a Verma flag, thus each $\mathcal{F}(\Delta)$ is either wild in the sense defined above or else it is tame.

# 3 Some explicit examples

In this section we consider some algebras which will appear in special cases where the lower bound from Corollary 1 cannot be used.

## 3.1 The case $A_1 \times A_1$

If the root system is of type $A_1 \times A_1$, then $C_0$ has two generators and $\mathbb{C}$–dimension four. In fact, $C_0 \simeq \mathbb{C}[x, y]/(x^2, y^2)$. Then $C_0$ is known to be tame [2]; in fact, it is a string algebra ([11, 19]) whose Auslander–Reiten quiver is well–known.

The dimension of $C_0$ coincides with the number of cosets of $W_0$ in $W$. Therefore, this case occurs only for the root system $A_1 \times A_1$ with trivial $W_0$. Picking $(\Phi, \Phi_0) = (B_2, A_1)$ yields the same dimension, but in this case $C_0$ is generated by one element.

The algebra $A$ by definition is a tensor product of two copies of the principal block $B$ of $\mathfrak{sl}(2)$, i.e. of type $A_1$. By quiver and relations the last one is given as follows:

$$B: \qquad \mathrm{a}\ \bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet\ \mathrm{b}$$

modulo the ideal generated by the relation: $\beta \cdot \alpha = 0$.

Therefore, the algebra $A$ has the following quiver and relations:

$$A: \qquad \begin{array}{ccc} \mathrm{a}\ \bullet & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & \bullet\ \mathrm{b} \\ \gamma \downarrow \uparrow \delta & & \gamma \downarrow \uparrow \delta \\ \mathrm{c}\ \bullet & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & \bullet\ \mathrm{d} \end{array}$$

modulo the ideal generated by the following relations:

$\beta \cdot \alpha = 0, \quad \delta \cdot \gamma = 0,$

$\alpha \cdot \gamma = \gamma \cdot \alpha, \quad \alpha \cdot \delta = \delta \cdot \alpha,$

$\beta \cdot \gamma = \gamma \cdot \beta, \quad \beta \cdot \delta = \delta \cdot \beta.$

Here are the composition series of the indecomposable projective modules:

$$P(a) = \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array}, \quad P(b) = \begin{array}{ccc} & b & \\ d & & a \\ b & & c \\ & d & \end{array}, \quad P(c) = \begin{array}{ccc} & c & \\ a & & d \\ b & & c \\ & d & \end{array}, \quad P(d) = \begin{array}{ccccc} & & d & & \\ & b & & c & \\ d & & a & & d \\ & b & & c & \\ & & d & & \end{array}.$$

The projective resolutions of the standard modules look as follows:

$$0 \to P(a) \to \Delta(a) = \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array} \to 0$$

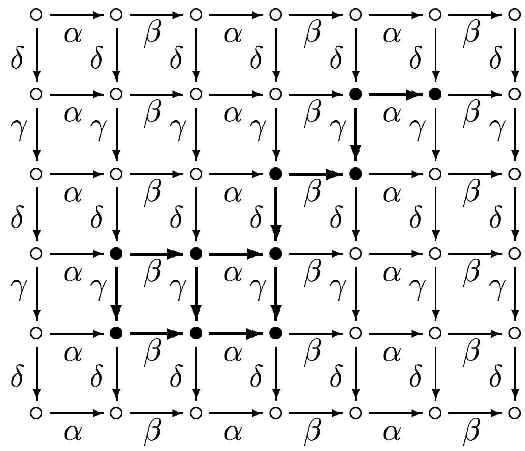
$$0 \to P(a) \to P(b) \to \Delta(b) = \begin{array}{c} b \\ d \end{array} \to 0$$

$$0 \to P(a) \to P(c) \to \Delta(c) = \begin{array}{c} c \\ d \end{array} \to 0$$

$$0 \to P(a) \to P(b) \oplus P(c) \to P(d) \to \Delta(d) = \ d \ \to 0$$

**Lemma 1.** *The algebra $A$ is wild.*

*Proof.* The universal covering $\widetilde{A}$ of the algebra $A$ is given by a quiver, $\widetilde{Q}$, which is a two-dimensional lattice with arrows labeled as in the figure below, and satisfying the same relations as for the algebra $A$. We also marked in the figure a convex subcategory, $E$, (supported by the 10 thick vertices) of the covering $\widetilde{A}$. As $E$ is contained in Unger's list of wild algebras [21], we conclude that $\widetilde{A}$ and therefore also $A$ is wild. □



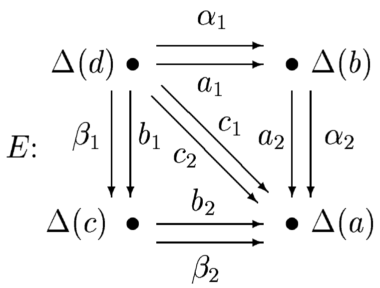

We now turn to investigate the category $\mathcal{F}(\Delta)$. Unfortunately, it is sandwiched between the tame category $C_0$–mod and the wild module category $A$–mod, hence we have to use different methods to determine the representation type of $\mathcal{F}(\Delta)$ in this case. We therefore recall from [15] some facts on $A_\infty$–categories: Let $E = \oplus_{i\geq 0} \operatorname{Ext}^i_A(\oplus\Delta(x), \oplus\Delta(x))$ denote the (graded) Ext–algebra of $\mathcal{F}(\Delta)$. It is endowed with an $A_\infty$–structure, given by maps $m_n : E^{\otimes n} \to E$ for $n \geq 2$, where $m_2 : E \otimes E \to E$ is the Yoneda product. The data $(E, (m_n)_{n\geq 2})$ define a category filt$(E)$, which we refer to as the category of representations of the $A_\infty$–algebra $E$. In case the maps $m_n$ vanish for $n \geq 3$, the algebra $(E, m_2)$ is a graded associative algebra and the category filt$(E)$ is what is known as the category of representations of the differential graded algebra $E$ (with zero differential, of course).

**Proposition 1 (Keller [15],Theorem 7.7).** *The category $\mathcal{F}(\Delta)$ is equivalent to filt$(E)$.*

Thus we can determine the representation type of $\mathcal{F}(\Delta)$ by studying the category filt$(E)$.

**Proposition 2.** *The category $\mathcal{F}(\Delta)$ is tame.*

*Proof.* Of course, we do not verify tameness directly; instead, we reduce this problem to some problem which is known to be tame. First compute the Yoneda product of the Ext–algebra $E$ of $\mathcal{F}(\Delta)$ (e.g. by representing elements in $E$ as shifted morphisms between complexes which are given by the above projective resolutions). It is given by the following (bi–)quiver $Q$ with relations, where the arrows labeled by greek letters describe the degree 0 part of $E$ and the latin letters stand for the degree 1 part.




relations:

$\alpha_2 \cdot \alpha_1 = \beta_2 \cdot \beta_1$

$a_2 \cdot \alpha_1 = c_1 = \beta_2 \cdot b_1$

$\alpha_2 \cdot a_1 = c_2 = b_2 \cdot \beta_1$

$a_2 \cdot a_1 = b_2 \cdot b_1.$

Since $A$ is quasi-hereditary, the algebra $E$ is directed. In fact, there are no paths in the quiver $Q$ of length $> 2$, hence the maps $m_n$ vanish for $n \geq 3$ (c.f. [15], section 3.4). Thus, the category $\mathrm{filt}(E)$ can be described as follows: A representation $V \in \mathrm{filt}(E)$ is given by linear maps $V_x$ for each latin letter $x$ of $Q$ such that the relation $V_{a_2}V_{a_1} = V_{b_2}V_{b_1}$ holds. In other words, the objects of the category $\mathrm{filt}(E)$ are the representations of the quiver (with relations) given by the latin letters.

The arrows indexed by greek letters come into play when we define morphisms: A morphism in $\mathrm{filt}(E)$ from $V$ to $W$ is given by a family of linear maps

$$H = (H_{\Delta(a)}, H_{\Delta(b)}, H_{\Delta(c)}, H_{\Delta(d)}, H_{\alpha_1}, H_{\alpha_2}, H_{\beta_1}, H_{\beta_2})$$

satisfying $H_{\alpha_2}H_{\alpha_1} = H_{\beta_2}H_{\beta_1}$ such that the following equations hold:

$$H_{\Delta(b)}V_{a_1} = W_{a_1}H_{\Delta(d)}, \quad H_{\Delta(a)}V_{a_2} = W_{a_2}H_{\Delta(b)},$$

$$H_{\Delta(c)}V_{b_1} = W_{b_1}H_{\Delta(d)}, \quad H_{\Delta(a)}V_{b_2} = W_{b_2}H_{\Delta(c)},$$

$$H_{\beta_2}V_{b_1} + H_{\Delta(a)}V_{c_1} = W_{c_1}H_{\Delta(d)} + W_{a_2}H_{\alpha_1},$$

$$H_{\alpha_2}V_{a_1} + H_{\Delta(a)}V_{c_2} = W_{c_2}H_{\Delta(d)} + W_{b_2}H_{\beta_1}.$$

In order to show finally that $\mathrm{filt}(E)$ is tame, we use the classical reduction method for differential graded categories ([16, 17]): Consider the (graded) subalgebra $G$ of $E$ generated by the arrows $c_1$ and $c_2$. The reduction of $E$ with respect to $G$ leads to a new differential graded category, $K$, whose category of representations is equivalent to $\mathrm{filt}(E)$.

In our case, the quotient $E/G$ is given by a commutative square, a well-known representation-finite algebra. It is therefore easy to compute the category $K$ and it turns out that $K$ is given by a bundle of chains in the sense of [3], namely by the following two pairs of posets

$$(x \to \bullet\,, \quad y \to \bullet) \quad \text{and} \quad (\tilde{x} \to \bullet\,, \quad \tilde{y} \to \bullet)$$

with involution $x \mapsto \tilde{x}$ and $y \mapsto \tilde{y}$. Thus, by [3] or equivalently [5], the category of representations of $K$ is tame and its indecomposable objects have been completely classified.

□

## 3.2 An enlargement of $A_1 \times A_1$

When proving the theorems we have to consider some enlargements of the case $A_1 \times A_1$. We therefore collect in the next two subsections auxiliary statements about quasi-hereditary algebras that are given by composition series of projective modules and by projective resolutions of standard modules. In the proofs later on, quotients and subalgebras of blocks of category $\mathcal{O}$ will appear which have precisely these properties (this in particular will ensure the existence of these algebras). It will turn out that for proving tameness or wildness of the corresponding categories $\mathcal{F}(\Delta)$ the partial information we have on these algebras is sufficient.

The application we are aiming at allows us to make some assumptions which simplify notation. In particular, we only consider algebras with a duality. In category $\mathcal{O}$ terminology the algebras appearing in this context are multiplicity free. Thus all indecomposable projective modules are contained in a unique 'big' projective module. The quotients or subalgebras to be studied now inherit this property which explains another assumption made below.

Thus, let $A$ be a quasi-hereditary algebra (admitting a duality fixing simples) which has five primitive idempotents $\{a, b, c, d, e\}$ such that the standard modules have the following projective resolutions:

$$0 \to P(a) \to \Delta(a) = \begin{array}{ccc} & a & \\ b & & c \\ & d & \\ & e & \end{array} \to 0$$

$$0 \to P(a) \to P(b) \to \Delta(b) = \begin{array}{c} b \\ d \\ e \end{array} \to 0$$

$$0 \to P(a) \to P(c) \to \Delta(c) = \begin{array}{c} c \\ d \\ e \end{array} \to 0$$

$$0 \to P(a) \to P(b) \oplus P(c) \to P(d) \to \Delta(d) = \begin{array}{c} d \\ e \end{array} \to 0$$

$$0 \to P(d) \to P(e) \to \Delta(e) = e \to 0$$

The composition series of the indecomposable projective modules are:

$$P(a) = \begin{array}{ccc} & a & \\ b & & c \\ & d & \\ & e & \end{array}, \quad P(b) = \begin{array}{cccc} & b & & \\ d & & a & \\ e & b & & c \\ & & d & \\ & & e & \end{array}, \quad P(c) = \begin{array}{cccc} & & c & \\ & a & & d \\ b & & c & e \\ & d & & \\ & e & & \end{array}, \quad P(d) = \begin{array}{ccccc} & & d & & \\ & b & e & c & \\ d & & a & & d \\ e & b & & c & e \\ & & d & & \\ & & e & & \end{array},$$

$$
P(e) = \begin{array}{ccccc}
 & & e & & \\
 & & d & & \\
 & b & e & c & \\
d & & a & & d \\
e & b & & c & e \\
 & & d & & \\
 & & e & &
\end{array}
$$

We further suppose that the projectives admit inclusions $P(a) \subset P(b) \subset P(d)$, $P(a) \subset P(c) \subset P(d)$ and $P(d) \subset P(e)$.

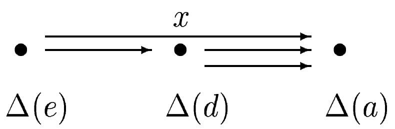
**Proposition 3.** *The category $\mathcal{F}(\Delta)$ is wild.*

*Proof.* As in the proof of Proposition 2, we consider $E = \oplus_{i\geq 0} \operatorname{Ext}_A^i(\oplus\Delta(x), \oplus\Delta(x))$, the Ext–algebra of the standard modules. Let $F$ be the (graded) subalgebra of $E$ generated by extensions of $\Delta(a) \oplus \Delta(d) \oplus \Delta(e)$, i.e.

$$F = \oplus_{i\geq 0} \operatorname{Ext}_A^i(\Delta(a) \oplus \Delta(d) \oplus \Delta(e), \Delta(a) \oplus \Delta(d) \oplus \Delta(e)).$$

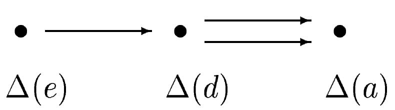
We know that $E$ is endowed with an $A_\infty$–structure such that $\mathcal{F}(\Delta)$ and $\operatorname{filt}(E)$ are equivalent. Our strategy is the following: we show that the $A_\infty$–structure restricted to $F$ is just given by the Yoneda-product (i.e. $m_n : E^{\otimes n} \to E$ vanishes for $n \geq 3$ when restricted to $F^{\otimes n}$). Thus, $\operatorname{filt}(F)$ can be viewed as the category of representations of the (differential) graded algebra $F$ and it is a full subcategory of $\operatorname{filt}(E)$. Therefore it suffices to show that $\operatorname{filt}(F)$ is wild. Note that this proof cannot be obtained on the level of quasi-hereditary algebras: the category $\operatorname{filt}(F)$ consists of those objects in $\mathcal{F}(\Delta)$ whose multiplicity of $\Delta(b)$ and $\Delta(c)$ is zero. The corresponding algebra $fAf$ with $f = a + d + e$ however is not quasi-hereditary.

From the projective resolutions of the standard modules and the given embeddings of the projective modules we can easily compute the dimensions of morphism and extension spaces between standard modules. In particular, the $\operatorname{Ext}^1$–quiver of the graded algebra $F$ looks as follows (here the arrows correspond to basis vectors of the $\operatorname{Ext}^1$–spaces between the various standard modules):




Since $A$ is quasi-hereditary, the (differential) graded algebra $F$ is directed. Moreover, $F$ admits only three primitive idempotent elements, namely the identity morphisms of $\Delta(e)$, $\Delta(d)$ and $\Delta(a)$. Therefore, by [15, section 3.4], the maps $m_n : E^{\otimes n} \to E$ vanish when restricted to $F^{\otimes n}$ for $n \geq 3$.

The $\operatorname{Ext}^1$–quiver of $F$ contains the following wild subquiver $\Gamma$:



One easily sees that $\mathrm{Ext}^2(\Delta(a), \Delta(e)) = 0$. Moreover, the standard modules have only trivial endomorphisms, and all morphisms between different standard modules induce morphisms in $\mathrm{filt}(F)$ that involve the arrow $x$ which is not contained in $\Gamma$. Therfore the obvious functor $\mathrm{rep}\,\Gamma \to \mathrm{filt}(F)$ preserves non-isomorphism and indecomposability. We conclude that the category $\mathrm{filt}(F)$ and hence $\mathcal{F}(\Delta)$ is wild. $\square$

### 3.3 Another enlargement of $A_1 \times A_1$

We consider finally a second enlargement of the case $A_1 \times A_1$: Let $A$ be a quasi-hereditary algebra (admitting a duality fixing simples) with primitive idempotents $\{a, a', b, c, d\}$ whose standard modules have the following projective resolutions:

$$0 \to P(a) \to \Delta(a) = \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array} \to 0$$

$$0 \to P(a') \to \Delta(a') = \begin{array}{ccc} & a' & \\ b & & c \\ & d & \end{array} \to 0$$

$$0 \to P(a) \oplus P(a') \to P(b) \to \Delta(b) = \begin{array}{c} b \\ d \end{array} \to 0$$

$$0 \to P(a) \oplus P(a') \to P(c) \to \Delta(c) = \begin{array}{c} c \\ d \end{array} \to 0$$

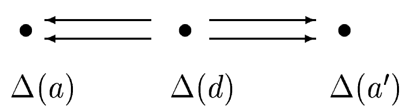
$$0 \to P(a) \oplus P(a') \to P(b) \oplus P(c) \to P(d) \to \Delta(d) = d \to 0$$

The composition series of the indecomposable projective modules are:

$$P(b) = \begin{array}{cccc} & b & & \\ d & a & a' & \\ b & c & c & b \\ d & & & d \end{array}, \quad P(c) = \begin{array}{cccc} & & c & \\ & a & a' & d \\ b & c & c & b \\ d & & & d \end{array}, \quad P(d) = \begin{array}{cccc} & \multicolumn{2}{c}{d} & \\ & b & c & \\ d & a & a' & d \\ b & c & c & b \\ d & & & d \end{array}.$$

**Proposition 4.** *The category $\mathcal{F}(\Delta)$ is wild in this case.*

*Proof.* This works completely analogous to the case we solved before: From the projective resolutions of the standard modules we obtain the following wild subquiver $\Gamma$ of the quiver of $E$ with all arrows of degree 1:




It is also clear that $\mathrm{Hom}_A(\Delta(a), \Delta(a'))$ and $\mathrm{Hom}_A(\Delta(a'), \Delta(a))$ are zero, thus the wildness of the subquiver $\Gamma$ implies that $\mathcal{F}(\Delta)$ is wild. $\square$

# 4 Proof of Theorems

We fix $A$ and $C_0$ as before. Recall that $C_0$ is a graded symmetric algebra. We will go through several cases depending on $\Phi_0$. In each case, the first step of the proof consists of determining the representation type of $C_0$. If $C_0$ is wild, then this case is finished. Only few situations remain where $C_0$ is not wild. In a second step we have to deal with these situations.

In order to decide about the representation type of $C_0$ we will use the following well–known lemma.

**Lemma 2.** *A local algebra is wild if its minimal number of generators is at least three.*

*Proof.* This follows from [13]; see also [9], I.10.10 (a). □

Now let us distinguish four cases according to the number of simple reflections not contained in $\Theta$, i.e. the corank of $\Phi_0$ in $\Phi$. If this number is zero, then $W = W_0$ and the algebra $A$ is simple. If the corank is three or bigger, then the minimal number of generators of $C_0$ is at least three, hence by Lemma 2 the algebra $C_0$ is wild. Thus it remains to consider the two cases of the corank being one or two.

Let us first suppose that $\Phi_0$ has corank two. There are two simple reflections, $s_\alpha$ and $s_\beta$, outside of $\Theta$. Denote by $w$ the product $s_\alpha s_\beta$. The four elements 1, $s_\alpha$, $s_\beta$ and $w$ are minimal length representatives of four different cosets of $W_0$ in $W$, and each of them defines a basis element in $C_0$. If the dimension of $C_0$ is four, then we are in the case $A_1 \times A_1$ which has been discussed in subsection 3.1. Otherwise the dimension of $C_0$ must be at least six since it is symmetric. Passing to quotients or subalgebras of $A$ if necessary we arrive at an algebra $A$ as in subsection 3.3 or at an algebra $A'$ which differs from this $A$ by having some multiplicities (of simple modules in Verma modules) bigger than one, i.e. by some arrows occuring with multiplicities. Clearly, the case of $A'$ also falls into wild type. This automatically implies that the original algebra we started with is wild as well. Note that in this case our lower bound may be too weak, since $C_0$ may be tame (e.g. in type $A_2$, see [12]).

It remains to consider the case of $\Phi_0$ having corank one. Denote by $s$ the simple reflection which is not in $\Theta$. In degree one, the graded algebra $C_0$ consists of the scalar multiples of $X_s$. If all other degrees are one–dimensional as well, then $C_0$ is isomorphic to $\mathbb{C}[x]/(x^m)$ for some $m$. It is obviously of finite representation type, thus our lower bound is trivial.

In this case, we can determine $A$ itself. In fact, by Soergel's double centralizer property (Theorem 3), the algebra $A$ then is the Auslander algebra of $C_0$, i.e. the endomorphism ring of the sum of all indecomposable $C_0$–modules. By [7], Section 7, the category $\mathcal{F}(\Delta)$ has finite representation type if and only if $m \leq 5$; it is tame for $m = 6$ and wild otherwise. The full category $A$–mod is of finite type for $m \leq 3$ only, tame for $m = 4$ and wild otherwise.

The number $m$ is the $\mathbb{C}$–dimension of $C_0$, hence it equals the number of cosets of $W_0$ in $W$. Consequently, we have the following choices for the root systems $(\Phi, \Phi_0)$: For $m = 1$, of course, $\Phi = \Phi_0$ and no further restriction. For $m = 2$ we have $(A_1, \emptyset)$. For $m = 3$ we have $(A_2, A_1)$; $m = 4$ means $(A_3, A_2)$ or $(B_2, A_1)$; $m = 5$ occurs for $(A_4, A_3)$ and $m = 6$ happens precisely for $(A_5, A_4)$ and for $(B_3, B_2)$.

From now on, we may assume that we are not in the situation of Lemma 2, that $\Phi_0$ has corank one and that some graded pieces of $C_0$ have $\mathbb{C}$-dimension bigger than one. Denote the smallest degree where this happens by $l + 1$. Note that the dimension of $C_0$ must be bigger than four in this case. As before, we have a generator $X_s$ associated with a simple reflection, $s \notin \Theta$. If the dimension in degree $l + 1$ is bigger than two, then we are in the situation of Lemma 2. Thus we may assume that the dimension is two. Denote the two standard basis elements by $X_{w_1}$ and $X_{w_2}$.

If $w_1 s$ is longer than $w_1$, then the Pieri formula ((1) in Section 2) implies that the product $X_{w_1} X_s$ is not zero. If also $w_2 s$ is longer than $w_2$, then one of the elements $X_{w_1}$ and $X_{w_2}$ is an additional generator and (passing to quotients or subalgebras if necessary and up to forgetting higher multiplicities) we are in the situation of subsection 3.2.

Suppose now that $w_1 s$ is longer than $w_1$, but $w_2 s$ is shorter than $w_2$. Then the element $w_2 s$ must be the minimal length representative of its $W_0$–coset (if not, write $w_2 s = w_0 v$ for some $w_0 \in W_0$ and $v$ of length at most $l - 1$; then $w_2 = w_0 v s$ cannot be minimal, since its length $l + 1$ is bigger than that of $vs$). Therefore $X_{w_2 s}$ generates the degree $l$–piece of $C_0$. The Pieri formula now tells us that $X_{w_2}$ occurs with a non–zero coefficient in the expansion of the product $X_{w_2 s} X_s$. It follows that $X_{w_1}$ is an additional generator and we are again in the situation of subsection 3.2.

Finally, it cannot happen that both $w_1 s$ and $w_2 s$ are shorter than $w_1$ and $w_2$. If so, by the same arguments as above, the basis elements $X_{w_1 s}$ and $X_{w_2 s}$ would be linearly dependent. Hence $w_1 s$ and $w_2 s$ would have to coincide, a contradiction.

This finishes the proofs of Theorems 1 and 2.

## Acknowledgements

The research was done during the visit of the third author to Bielefeld University as an Alexander von Humboldt fellow. The financial support of Humboldt Foundation and the hospitality of Bielefeld University are gratefully acknowledged. We are indebted to I. Gordon for pointing out some mistakes in an earlier version.

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