Virasoro-type algebras associated with a Penrose tiling

Volodymyr Mazorchuk[1] and Reidun Twarock[2]
[1] Department of Mathematics, Uppsala University
SE-75106, Uppsala, Sweden
[2] Centre for Mathematical Science, City University,
Northampton Square, London EC1V 0HB, England

February 20, 2003

Abstract

A family of infinite dimensional Lie algebras with generators in a one-to-one correspondence with the points of a Penrose tiling is introduced. Central extensions, leading to Virasoro-type algebras, are constructed, and highest weight representations for these algebras are considered. Furthermore, extensions to a super-symmetric setting and thus aperiodic analogs to Virasoro super-algebras are discussed.

1 Introduction

Lie algebras with generators in a one-to-one correspondence with one-dimensional aperiodic point sets or cut-and-project quasicrystals have been studied in a series of papers [5, 6, 8, 3] and their structure is relatively well understood. Moreover, one also has some information about the structure of highest weight modules, in particular, there is a criterion for irreducibility of Verma modules. Results about Lie algebras related to higher-dimensional aperiodic point sets are presently not known.

¿From the point of view of real-life quasicrystals, however, the one-dimensional setting is artificial and it is therefore necessary to generalize the above algebras to higher dimensions. A canonical example of a two-dimensional aperiodic structure is the vertex sets of a Penrose tiling [7]. We address here the question, asked by Prof. P. Kramer at the conference "Symmetries in Science" in Bregenz in 1999, if one can construct Lie algebras in a one-to-one correspondence with the vertex set of a Penrose tiling. We introduce a corresponding family of Lie algebras and discuss structural properties, central extensions and Verma modules.

For our construction it is crucial that the vertex set of a Penrose tiling may be considered as a two-dimensional cut-and-project or model set [7, 9], that is that it can be obtained via a projection from a four-dimensional regular lattice onto a suitably chosen two-dimensional hyperplane. We therefore describe Penrose tilings as cut-and-project sets in section 2, where we also set up notations related to the aperiodic point sets.

The family of algebras is then introduced in section 3. Here, particular types of Penrose tilings are identified, which may be used to construct Lie algebras similarly as in the one-dimensional case. The corresponding family of Lie algebras is introduced and structural properties are discussed.

In section 4, we discuss central extensions of the algebras. The algebras we consider are not perfect and thus they do not have universal central extensions in the usual sense. Therefore we define a special class of central extensions, which we call geometric, since the definition is based on the geometric properties of the acceptance window. For a certain class of our algebras we construct universal geometrical central extensions.

In section 5, we finally study highest weight representations for the algebras and consider Verma modules and their irreducibility.

By construction, our family of algebras can be viewed as an aperiodic analog to the (higher rank) Virasoro algebra. Since the Virasoro algebra plays a crucial role in various branches of mathematical physics, these results are of independent interest for various applications. In addition to the Virasoro algebra, the super-Virasoro algebra plays an important role in many applications, especially, in string theory. We therefore discuss a possibility of a super-symmetric extension of our algebras in the concluding remarks in section 6.

2 Penrose tilings as cut-and-project sets

For the purpose of this article, it is convenient to view Penrose tilings as cut-and-project quasicrystals or model sets, that is aperiodic point sets obtained via a projection formalism from a higher dimensional regular lattice. In particular, we consider a projection from the root lattice of A_4 onto two hyperplanes each containing a copy of the root system of type H_2 (see e.g. [1]). We introduce this set up in the following:

Let $\tau = \frac{1}{2}(1+\sqrt{5})$, and let $\mathbb{Z}[\tau] = \{a+\tau b | a, b \in \mathbb{Z}\}$ denote the ring of integers in the algebraic extension $\mathbb{Q}[\sqrt{5}]$ of the rational numbers by $\sqrt{5}$. Consider the Galois automorphism $' : \mathbb{Q}[\sqrt{5}] \to \mathbb{Q}[\sqrt{5}]$ defined via $(a+\sqrt{5}b)' = a - \sqrt{5}b$. Restricted to $\mathbb{Z}[\tau]$, this automorphism maps $a+\tau b$ to $a+\tau' b$ and links the two solutions of the equation $x^2 = x+1$, namely the first solution τ and the second solution $\tau' = \frac{1}{2}(1-\sqrt{5})$.

In the cut-and-project picture for Penrose tilings, there are two hyperplanes containing a copy of the root system of type H_2 each. One of them is the so-called model space, that is the space containing the Penrose tiling; the other one is the internal space, and it contains a special bounded set called acceptance window. The vertex set of the Penrose tiling then consists of the projection of all those points from the root lattice of A_4 (i.e. the additive subgroups, generated by roots), which under the projection onto the internal space fall into the acceptance window.

The two copies of H_2 in model and internal space are related as follows:

Let $\xi := \exp(\frac{2\pi i}{5})$. Then the root system $\Delta_2 \subset \mathbb{C}$ of type H_2 contains ten roots in the model space, which are given in terms of the simple roots $\alpha_1 := \xi^0$ and $\alpha_2 := \xi^2$ as follows:

$$\Delta_2 = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \tau\alpha_2), \pm(\tau\alpha_1 + \alpha_2), \pm(\tau\alpha_1 + \tau\alpha_2)\}\,.$$

Denote by $\mathbb{Z}[\tau]\Delta_2$ the $\mathbb{Z}[\tau]$–lattice with basis $\{\alpha_1, \alpha_2\}$. The assignment $\xi^* = \xi^2$ uniquely extends to a $\mathbb{Z}[\tau]$–semilinear bijection * on $\mathbb{Z}[\tau]\Delta_2$. This map is usually called the *star map*, see [1, Section 6]. Moreover, according to [1, Section 6], this map uniquely extends to a $\mathbb{Z}[\tau]$–semilinear bijection on $\mathbb{Q}[\tau]\Delta_2$, which we will also denote by *.

It is immediate that the star-map, defined above, acts like the original Galois automorphism $'$: $\mathbb{Q}[\sqrt{5}] \to \mathbb{Q}[\sqrt{5}]$, when restricted to $\mathbb{Z}[\tau]$. Indeed, $\tau^* =(-\xi^2 - \xi^3)^* = (-\xi^4 - \xi^6) =\tau'$. Obviously, the star map is a permutation on $\mathbb{Z}[\tau]\Delta_2$ (or $\mathbb{Q}[\tau]\Delta_2$) of order 4.

Our main objects of interest, namely cut-and-project quasicrystals are defined as follows:

Definition 1. *Let $\Omega \subset \mathbb{C}$ be a bounded set. Then*

$$\Sigma(\Omega) = \{x \in \mathbb{Z}[\tau]\Delta_2 | x^* \in \Omega\}$$

is a planar cut-and-project quasicrystal.

It is clear that $\Sigma(\Omega)$ is an aperiodic point set in $\mathbb{C} = \mathbb{R}^2$.

To obtain the vertex set of a Penrose tiling, we choose Ω to be the solid pentagon Ω_P with corner points ξ^j, $j = 0, \ldots, 4$. For such Ω we get the following point set:

$$\begin{aligned}\Sigma(\Omega) \;\; = \;\; & \{x = (x_1 + \tau x_2)\xi^0 + (x_3 + \tau x_4)\xi^2 \, | x_j \in \mathbb{Z}, j = 1, \ldots, 4 \\ & (x_1 + \tau' x_2)\xi^0 + (x_3 + \tau' x_4)\xi^4 \in \Omega_P\}\;.\end{aligned} \qquad (1)$$

A patch of this point set and the corresponding acceptance window Ω_P are depicted in Fig. 1.

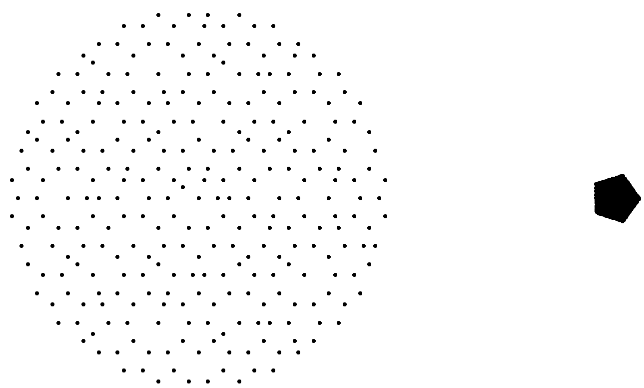

Figure 1: Planar cut-and-project quasicrystal for a pentagonal acceptance window.

Later on we will also need a translated version of the set defined above, which can be obtained in the model space via a translation of the acceptance window in internal space as follows:

Proposition 1. *For $l \in \mathbb{Q}[\tau]\Delta_2$ and arbitrary bounded $\Omega \in \mathbb{R}^2$ one has*

$$\Sigma(\Omega) + l^* = \Sigma(\Omega + l)\,. \tag{2}$$

Proof. It follows from the $\mathbb{Z}[\tau]$–semilinearity of the star-map and the definition of $\Sigma(\Omega)$. ☐

We note that Proposition 1 cannot be extended to arbitrary translations. In fact, one can always choose some Ω and l such that the left hand side of (2) is empty whereas the right hand side is not.

3 The algebras and their basic properties

3.1 Preliminaries

Definition 2. *An acceptance window, Ω, is called* admissible *if for each triple n, m, $k \in \Sigma(\Omega)$ one has the condition*

$$n^* + m^* + k^* \in \Omega \text{ and } n^* + m^* \in \Omega \Rightarrow m^* + k^* \in \Omega \text{ and } n^* + k^* \in \Omega \tag{3}$$

Let Ω_P be as before and let $\Omega_P^T := T\xi^0 + \Omega_P$ be the pentagon Ω_P translated by $T > 0$ into ξ^0 direction. It follows easily from Proposition 1 that for $T \in \mathbb{Q}[\tau]$ the corresponding translated quasicrystal $\Sigma(\Omega_P^T)$ is a translation of $\Sigma(\Omega_P)$ and for $T \notin \mathbb{Q}[\tau]$ the corresponding translated quasicrystal $\Sigma(\Omega_P^T)$ is only a subset of some rescaled translation of $\Sigma(\Omega_P)$. However, for the acceptance window Ω_P^T we have:

Lemma 1. *Ω_P^T is admissible if and only if $T \geq \tau$.*

Proof. Let $h := T - \frac{\tau}{2}$ denote the distance of Ω_P^T from the origin and let $l := 1 + \frac{\tau}{2} - 2h$ denote $(T+1)\mathrm{mod}h$ as in Fig. 2.

Then the admissibility of Ω_P^T is equivalent to the condition $\varphi \leq \psi$, because only in this case we have that whenever x^* and $y^* \in \Omega_P^T$ are such that $x^* + y^* \notin \Omega_P^T$, then also $x^* + y^* + z^* \notin \Omega_P^T$ for any $z^* \in \Omega_P^T$.

Thus we have the condition

$$\frac{\pi}{10} \geq \arctan\left(\frac{l \tan\frac{3\pi}{10}}{h}\right)\,. \tag{4}$$

which implies

$$h \geq \frac{1 + \frac{\tau}{2}}{2 + \frac{\tan\frac{\pi}{10}}{\tan\frac{3\pi}{10}}} \tag{5}$$

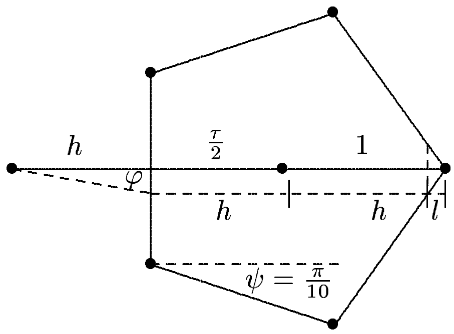

Figure 2: The admissibility condition $\varphi \leq \psi$.

Using $\tau = 2\cos\frac{\pi}{5}$, $\tau' = -2\cos\frac{2\pi}{5}$ and trigonometric identities, one proves that $\frac{\tan\frac{\pi}{10}}{\tan\frac{3\pi}{10}} = 2\tau - 3$, and thus

$$h \geq \frac{\tau}{2}, \tag{6}$$

which implies $T \geq \tau$. $\square$

Definition 3. *An acceptance window Ω is called* trivial *if for each two points n, $m \in \Sigma(\Omega)$ one has that*

$$n^* + m^* \notin \Omega, \tag{7}$$

and called non-trivial *otherwise.*

Lemma 2. *Ω_P^T is non-trivial if and only if $T \leq \tau^2$.*

Proof. Clearly, it is enough to check this condition for the segment $\Omega_P^T \cap \mathbb{R}\xi^0$, which has length $1 + \frac{\tau}{2}$. Thus Ω_P^T is non-trivial as long as its distance h from the origin is smaller than $1 + \frac{\tau}{2}$, which corresponds to a translation $T = h + \frac{\tau}{2} = \tau + 1 = \tau^2$. $\square$

3.2 Definition of the algebras

In the spirit of [5], we consider the following family of algebras, associated with $\Sigma(\Omega_P^T)$:

Lemma 3. *Let $\mathbb{F}$ be any number field such that $\mathbb{F} \supset \mathbb{Q}[\tau]$ and let T be such that Ω_P^T is admissible. Let furthermore $\varphi : \Sigma(\Omega_P^T) \to \mathbb{F}$ be a $\mathbb{Z}$–linear map. Then for each such T the $\mathbb{F}$–span of $\{L_n | n \in \Sigma(\Omega_P^T)\}$ with the Lie bracket defined by*

$$\begin{aligned} [L_n, L_m] &= \begin{cases} \varphi(m-n) L_{n+m} & \text{if } \; n+m \in \Sigma(\Omega_P^T) \\ 0 & \text{if } \; n+m \notin \Sigma(\Omega_P^T) \end{cases} \\ &= \chi_{\Omega_P^T}(n^* + m^*)\varphi(m-n) L_{n+m} \,. \end{aligned} \tag{8}$$

is a Lie algebra.

Proof. The Jacobi identity follows from the fact that Ω_P^T is admissible (condition (3)), and the Jacoby identity for the generalized Witt algebras, see e.g. [10]. Antisymmetry is enforced by the structure constants and $\mathbb{Z}$–linearity of φ. □

The algebras, obtained in Lemma 3 will be called *Witt-type algebras associated with the Penrose tiling* $\Sigma(\Omega_P^T)$ and will be denoted by $\mathcal{W}(\Omega_P^T,\varphi)$.

If $\Omega\subset\Omega_P^T$ is such that the linear span of the elements L_n, $n\in\Sigma(\Omega)$, is a Lie subalgebra in $\mathcal{W}(\Omega_P^T,\varphi)$, we will denote this subalgebra by $\mathcal{W}(\Omega,\varphi)$.

Lemma 4. *If Ω_P^T is trivial, then $\mathcal{W}(\Omega_P^T,\varphi)$ is abelian for any φ. If Ω_P^T is non-trivial, then there exists φ such that $\mathcal{W}(\Omega_P^T,\varphi)$ is not abelian.*

Proof. If Ω_P^T is trivial, then in the Lie bracket, defined by (8), we always have the second option, namely $n+m\notin\Sigma(\Omega_P^T)$, and hence $\mathcal{W}(\Omega_P^T,\varphi)$ is trivial.

Ω_P^T non-trivial ensures that there exist n, $m\in\Sigma(\Omega_P^T)$ such that $\chi_{\Omega_P^T}(n^*+m^*)$ is non vanishing. Taking φ such that $\varphi(m-n)$ does not vanish (it is obvious that such φ exists) we get a non-abelian Lie algebra, completing the proof. □

3.3 Properties

It is clear that the structural properties of $\mathcal{W}(\Omega_P^T)$ have to depend on the translation parameter T.

Lemma 5. 1. *If $T\geq\tau$ then*

$$[[[\mathcal{W}(\Omega_P^T,\varphi),\mathcal{W}(\Omega_P^T,\varphi)],\mathcal{W}(\Omega_P^T,\varphi)],\mathcal{W}(\Omega_P^T,\varphi)]=0$$

for every φ.

2. *If $T>\frac{2+\tau}{4}$ then $[[\mathcal{W}(\Omega_P^T,\varphi),\mathcal{W}(\Omega_P^T,\varphi)],\mathcal{W}(\Omega_P^T,\varphi)]=0$ for every φ.*

3. *If $\tau\leq T\leq\frac{2+\tau}{4}$ then there exists φ such that $[[\mathcal{W}(\Omega_P^T,\varphi),\mathcal{W}(\Omega_P^T,\varphi)],\mathcal{W}(\Omega_P^T,\varphi)]\neq 0$.*

Proof. The first follows from the easy observation that in the case $\tau\leq T$ one has that $x+y+z+u\notin\Omega_P^T$ for every $x,y,z,u\in\Omega_P^T$. The second one follows from the fact that in the case $T>\frac{2+\tau}{4}$ one has that $x+y+z\notin\Omega_P^T$ for every $x,y,z\in\Omega_P^T$. If $\tau\leq T\leq\frac{2+\tau}{4}$ then we can always find $x,y,z\in\Sigma(\Omega_P^T)$ such that $x+y+z\in\Sigma(\Omega_P^T)$ and then one easily chooses a corresponding φ. □

In particular, in summary we have the following three critical values of T, at which we have the following structural changes for the (generic) algebras of the form $\mathcal{W}(\Omega_P^T,\varphi)$:

1. $T=\tau$: It divides the range of T for which no Lie algebras exist ($T<\tau$) from the range for which Lie algebras always exist.

2. $T = \frac{2+\tau}{4}$: It divides the range of T for which $[[\mathcal{W}(\Omega_P^T,\varphi),\mathcal{W}(\Omega_P^T,\varphi)],\mathcal{W}(\Omega_P^T,\varphi)]$ is not necessarily 0 ($T < \frac{2+\tau}{4}$) from the range where one has

$$[[\mathcal{W}(\Omega_P^T,\varphi),\mathcal{W}(\Omega_P^T,\varphi)],\mathcal{W}(\Omega_P^T,\varphi)] = 0$$

independently of φ ($T > \frac{2+\tau}{4}$), that is it marks a change in the degree of nilpotency.

3. $T = \tau^2$: It divides the range of T for which the Lie algebras are not always abelian ($T < \tau^2$) from the range for which Lie algebras are always abelian.

In what follows, we are going to study the family $\mathcal{W}(\Omega_P^T,\varphi)$ with $T \in [\tau,\tau^2]$, which is the only non-trivial case according to Lemma 5. The properties of the algebras are encoded in the geometry of Ω_P^T.

Denote by $\widetilde{\Omega_P}$ the regular pentagon with the center at $\frac{2+\tau}{4}\xi^0$ and one vertex at the origin.

Proposition 2. *For arbitrary φ the subalgebra $\mathcal{W}(\Omega_P^T \setminus \widetilde{\Omega_P},\varphi)$ of $\mathcal{W}(\Omega_P^T,\varphi)$ is central.*

Proof. Indeed, if $L_n \in \mathcal{W}(\Omega_P^T \setminus \widetilde{\Omega_P},\varphi)$ then, from the definition of $\widetilde{\Omega_P}$, we get that $n+m \notin \Sigma(\Omega_P^T)$ for every $m \in \Sigma(\Omega_P^T)$, which completes the proof. □

4 Central extensions and triangular decomposition

4.1 Central extensions

For $v = v_\lambda\xi^0 \in \Omega_P^T$ we denote by $H_{v_\lambda}^T \subset \Omega_P^T$ the maximal subset, invariant with respect to the involution $\omega_\lambda : \mathbb{C} \to \mathbb{C}$, defined by $\omega_\lambda(x) = v_\lambda\xi^0 - x$. Geometrically, $H_{v_\lambda}^T$ is given as the intersection of the two pentagons in Fig. 3.

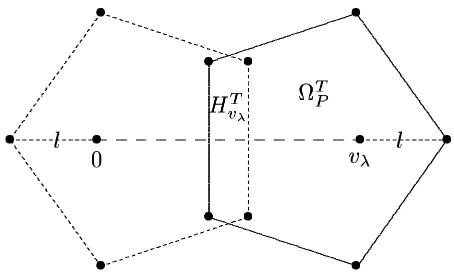

Figure 3: The set $H_{v_\lambda}^T$.

For $n \in \Sigma(\Omega_P^T)$ we introduce the notation $n = n_1\xi^0 + n_2\xi^2$ with $n_j = n_j^1 + \tau n_j^2$ and n_j^k, j, $k \in \{1,2\}$.

Lemma 6. *The vector space with the basis* $\{L_n | n \in \Sigma(\Omega_P^T) \cup \{0\}\}$ *and* $\hat{c}_{v_\lambda}$ *with* $v_\lambda \in \mathbb{Z}[\tau] \cap [2T - \tau, T+1]$*, and the Lie bracket defined by*

$$\begin{aligned} [L_n, L_m] &= (m_2 - n_2)\chi_{\Omega_P^T}(n^* + m^*)L_{n+m} \\ &\quad + \tfrac{\hat{c}_{v_\lambda}}{12}(n_2^3 - n_2)\delta_{n, v_\lambda \xi^0 - m} \\ [L_n, \hat{c}_{v_\lambda}] &= 0 \text{ for all } L_n \in \mathcal{V}(\Omega_P^T) \end{aligned} \tag{9}$$

is a Lie algebra.

Proof. Direct computation. □

The Lie algebras constructed in Lemma 6 will be called *Virasoro-type algebras associated with the Penrose tiling* $\Sigma(\Omega_P^T)$ and will be denoted by $\mathcal{V}(\Omega_P^T)$.

The algebras $\mathcal{W}(\Omega_P^T, \varphi)$ are not perfect by construction, hence it is impossible to hope for an existence and nice properties of the universal central extension of these algebras. However, under some natural conditions, related to the geometry of the acceptance window Ω_P^T, the algebras $\mathcal{V}(\Omega_P^T)$ indeed result as a universal (in some class of central extensions) central extension of $\mathcal{W}(\Omega_P^T, \varphi)$. We call a central extension of $\mathcal{W}(\Omega_P^T, \varphi)$, defined by

$$[L_n, L_m] = (m_2 - n_2)\chi_{\Omega_P^T}(n^* + m^*)L_{n+m} + c(n, m)\,,$$

geometric provided that $c(0, a\xi^0) = 0$ for $T - \tau/2 \le a \le 2T - \tau$, $c(n, m) = 0$ if $n + m \notin \Sigma(\Omega_P^T)$, and for arbitrary $m, n \in H_{v_\lambda}^T$, $m_2 \neq 0$, the cocycles $c(n, v'_\lambda \xi^0 - n)$ and $c(m, v'_\lambda \xi^0 - m)$ are linearly dependent with coefficient $\frac{n_2^3 - n_2}{m_2^3 - m_2}$. It follows by a direct calculation that the algebra $\mathcal{V}(\Omega_P^T)$ is a geometric central extension of $\mathcal{W}(\Omega_P^T, \varphi)$.

For the Virasoro-type algebras $\mathcal{V}(\Omega_P^T)$ we pick a very special map φ, which is justified by the following statement, which is an analogue of the one-dimensional statement [8, Theorem III.4]. In the proof we will crucially use the fact that the map φ, which we are considering now, is such that for given $x = x_1\xi^0 + x_2\xi^2$ it annihilates $x_1\xi^0$ for all $x_1 \in \mathbb{Z}[\tau]$ (thus it annihilates both ξ^0 and $\tau\xi^0$), and the intersection of $x_2\xi^2$ with the kernel of φ is zero.

Theorem 1. *The algebra* $\mathcal{V}(\Omega_P^T)$ *is the universal geometric central extension of the algebra* $\mathcal{W}(\Omega_P^T, \varphi)$*, where* $\varphi(x_1\xi^0 + x_2\xi^2) = x_2$*.*

Proof of Theorem 1. Let n, $m \in \Sigma(\Omega_P^T)$. As conditions on a cocycle $c(n, m)$ with

$$[L_n, L_m] = (m_2 - n_2)\chi_{\Omega_P^T}(n^* + m^*)L_{n+m} + c(n, m)\,,$$

we have from the Jacobi identity the following equation:

$$(m_2 - n_2)\chi_{\Omega_P^T}(n^* + m^*)c(n + m, k) + \text{ cyclic permutations} = 0 \tag{10}$$

with n, m, $k \in \Sigma(\Omega_P^T)$.

Since $L_0 \in \mathcal{V}(\Omega_P^T)$ by definition, the choice $m = 0$ is possible and one obtains

$$(k_2 + n_2)c(k,n) + (n_2 - k_2)\chi_{\Omega_P^T}(n^* + k^*)c(n+k,0) = 0\,. \tag{11}$$

Suppose first that $n + k \neq v_\lambda \xi^0$ for any $v_\lambda \in \mathbb{R}$, that is $k_2 + n_2 \neq 0$. Since we have $[L_0, L_r] = r_2 L_r + c(0,r)$ we may choose (in the case $r_2 \neq 0$) a gauge where $c(0,r) = 0$, i.e. $\hat{L}_r = L_r + \frac{1}{r_2}c(0,r)$ and rename it again as L_r by an abuse of notation. It thus follows from (11), since $n_2 + k_2 \neq 0$, that in this case we have $c(k,n) = 0$. Moreover, $c(k,n) = 0$, as soon as $k + n \notin \Sigma(\Omega_P^T)$ by the definition of the geometric central extension.

Suppose now that $n + k = v_\lambda \xi^0$ for some $v_\lambda \xi^0 \in \Omega_P^T$ and $n_2 - k_2 \neq 0$. We have $n_2 + k_2 = 0$ and since $(n_2 - k_2)\chi_{\Omega_P^T}(n^* + k^*) = (n_2 - k_2) \neq 0$ it follows, again from (11), that $c(v_\lambda \xi^0, 0) = 0$. Now, from the definition of the geometric central extension we immediately deduce $c(k,n) \sim \delta_{k, v_\lambda \xi^0 - n}$.

The existence of the universal geometric central extension now follows from the condition that the cocycles $c(n, v'_\lambda \xi^0 - n)$ and $c(m, v'_\lambda \xi^0 - m)$ are linearly dependent with coefficient $\frac{n_2^3 - n_2}{m_2^3 - m_2}$. Since the algebra $\mathcal{V}(\Omega_P^T)$ satisfies all the conditions above, we get that it is isomorphic to the universal geometric central extension of $\mathcal{W}(\Omega_P^T, \varphi)$. □

4.2 A triangular decomposition

For the algebra $\mathcal{V}(\Omega_P^T)$ let us denote by $\mathcal{V}(\Omega_P^T)^+$ and $\mathcal{V}(\Omega_P^T)^-$ the subalgebras generated by elements L_n, where $\Sigma(\Omega_P^T) \ni n = n_1\xi^0 + n_2\xi^2$ and $n'_2 < 0$ or $n'_2 > 0$, respectively. Denote by $\mathfrak{h}_T$ the subalgebra generated by L_n, where $\Sigma(\Omega_P^T) \cup \{0\} \ni n = n_1\xi^0$, and by all $\hat{c}_{v_\lambda}$ with $v_\lambda \in \mathbb{Z}[\tau] \cap [2T - \tau, T+1]$. Our choice is motivated by the fact whether the indices of the generators are located in the positive or in the negative half plane in the internal space. In particular, $n'_2 > 0$ is equivalent to n^* located in the negative half plane, and we thus choose for generators related to such points the superscript $-$. Similarly, $n'_2 < 0$ is equivalent to n^* located in the positive half plane, which justifies the superscript $+$. The generators of $\mathfrak{h}_T$ can then be viewed as located on the line $n'_2 = 0$.

Lemma 7. *The algebra $\mathfrak{h}_T$ is abelian and we have the following decomposition of $\mathcal{V}(\Omega_P^T)$:*

$$\mathcal{V}(\Omega_P^T) = \mathcal{V}(\Omega_P^T)^- \oplus \mathfrak{h}_T \oplus \mathcal{V}(\Omega_P^T)^+.$$

Proof. The first statement follows from our choice of φ. The rest follows from the fact that the conditions $n'_2 > 0$, $n'_2 < 0$ and $n'_2 = 0$ are preserved by the addition of vectors. □

It is natural to call the decomposition of $\mathcal{V}(\Omega_P^T)$ given by Lemma 7 a *triangular decomposition.* However, we remark that the adjoint action of the "Cartan subalgebra" $\mathfrak{h}_T$ on $\mathcal{V}(\Omega_P^T)$ is not diagonalizable in general.

5 Verma modules

¿From now on assume that $\mathbb{F}$ is algebraically closed. Then all simple $\mathfrak{h}_T$-modules are one-dimensional and have the following form: for $\lambda \in \mathfrak{h}_T^*$ we consider $\mathbb{F}_\lambda = \mathbb{F}$ as a $\mathfrak{h}_T$-module with the action $hz = \lambda(h)z$ for all $h \in \mathfrak{h}_T$ and $z \in \mathbb{F}$. Setting $\mathcal{V}(\Omega_P^T)^+\mathbb{F}_\lambda = 0$ we turn $\mathbb{F}_\lambda$ into a $\mathfrak{b} = \mathfrak{h}_T \oplus \mathcal{V}(\Omega_P^T)^+$-module. Now we can consider the induced module

$$M(\lambda) = U(\mathcal{V}(\Omega_P^T)) \otimes_{U(\mathfrak{b})} \mathbb{F}_\lambda,$$

which it is natural to call the Verma module associated with the triangular decomposition above. This module is a universal highest weight module for the algebra $\mathcal{V}(\Omega_P^T)$. It is obvious that $M(\lambda)$ is indecomposable and has a unique simple quotient, which we will denote by $L(\lambda)$.

One of the principal question in the representation theory, especially important for physical applications, is the description of the simple highest weight module $L(\lambda)$, in particular, the irreducibility of $M(\lambda)$ (i.e. when $M(\lambda) = L(\lambda)$).

Theorem 2. *1. $M(\lambda)$ is always reducible.*

2. The module $L(\lambda)$ is one dimensional if and only if λ is zero on $\mathfrak{h}_T^{ess} := \langle \hat{c}_{v_\alpha}, L_{v'_\alpha} | v_\alpha \in \mathbb{Z}[\tau] \cap [2T - \tau, T+1] \rangle$. Otherwise, $L(\lambda)$ is infinite-dimensional.

Later on we will need the following notion: For the element L_n, $n = n_1\xi^0 + n_2\xi^2$, let us call n_2 the *degree* of L_n (this also can be regarded as the *weight* with respect to L_0, however, all the arguments below remain valid if we take away the element L_0 from our algebra). Moreover, we assume that the degree of all elements $\hat{c}_{v_\alpha}$ is zero. In this way $\mathcal{V}(\Omega_P^T)$ and $U(\mathcal{V}(\Omega_P^T))$ become graded (the latter one by the abelian subgroup generated by all degrees of $\mathcal{V}(\Omega_P^T)$). Moreover, all the components of the triangular decomposition then can be written as a direct sum of homogeneous components with respect to the notion of degree introduced above. Furthermore, the positive and negative parts $\mathcal{V}(\Omega_P^T)^+$ and $\mathcal{V}(\Omega_P^T)^-$ (and their enveloping algebras) contain only elements of positive and negative degree respectively, and the Cartan part $\mathfrak{h}_T$ (and its enveloping algebra) consists of elements of degree 0.

Proof. The algebra $\mathcal{V}(\Omega_P^T)$, being nilpotent, has a large center. Moreover, this center intersects both $\mathcal{V}(\Omega_P^T)^+$ and $\mathcal{V}(\Omega_P^T)^-$. If $c \in \mathcal{V}(\Omega_P^T)^-$ is an arbitrary central element and v be a canonical generator of $M(\lambda)$, then $U(\mathcal{V}(\Omega_P^T))cv$ is certainly a proper submodule of $M(\lambda)$ and hence $M(\lambda)$ is reducible. This proves the first statement.

Assume now that the restriction of λ to $\mathfrak{h}_T^{ess}$ is zero. To prove that $L(\lambda)$ is one dimensional in this case, it is enough to show that for every $u \in U(\mathcal{V}(\Omega_P^T)^-)$ the submodule $U(\mathcal{V}(\Omega_P^T))uv \subset M(\lambda)$ is proper, which is equivalent to the fact that it does not intersect $\mathbb{F}v$.

Since $\mathcal{V}(\Omega_P^T)$ is graded, it follows immediately that, without loss of generality, we can assume that u is graded, and it is enough to show that for arbitrary graded u' such that $\deg(u') = -\deg(u)$ the element $u'uv = 0$. But this can be written as $u'uv = [u', u]v$, and the element $[u', u]$ belongs to $U(\mathfrak{h}_T^{ess})$ by the definition of $\mathfrak{h}_T^{ess}$. Hence $[u', u]v = 0$ as required.

Finally, let us assume that the restriction of λ to $\mathfrak{h}_T^{ess}$ is not zero, and let L_m or $\hat{c}_{v_\alpha}$ be the element of $\mathfrak{h}_T^{ess}$ such that $\lambda(L_m) \neq 0$ for $m = v'_\alpha \xi_0$ or $\lambda(c_{v_\alpha}) \neq 0$. From (9) we immediately get that there exist infinitely many $n = n_1\xi^0 + n_2\xi^2$ (essentially, infinitely many n_2) such that

$$L_{\omega_\alpha(n)} L_n v = [L_{\omega_\alpha(n)}, L_n] v = \left(2n_2 \lambda(L_m) + \frac{n_2^3 - n_2}{12} \lambda(\hat{c}_{v_\alpha}) \right) v \neq 0.$$

The module $L(\lambda)$ is obviously graded, and the arguments above imply that infinitely many graded components of $L(\lambda)$ have positive dimension. Hence $L(\lambda)$ is infinite-dimensional, which completes the proof. □

One can say even more about the structure of $L(\lambda)$. In fact the following holds:

Proposition 3. *Assume that* $\dim(L(\lambda)) = \infty$*. Then all homogeneous graded components of* $L(\lambda)$*, except the component containing the canonical generator, are infinite-dimensional.*

Proof. The shortest way to prove this theorem is to use the technique of the Shapovalov form, see [2] (or [8, 3] for an application to quasicrystal Lie algebras), which we are not going to describe here. We refer the reader to the original papers. We proceed with the notation from Theorem 2.

One starts from the easy observation that the dimensions of the graded components of a non-trivial $L(\lambda)$ cannot decrease if we increase the absolute value of the degree $n_2 < 0$. Hence it is enough to prove this statement for arbitrarily small absolute values of n_2. In particular, it is enough to prove that this dimension is greater or equal than an arbitrary but fixed positive integer N.

Let n be such that the absolute value of n_2 is small enough to ensure the existence of elements $L_{n(1)}, \dots, L_{n(N)} \in \mathcal{V}(\Omega_P^T)^-$ such that $n(i) = n(i)_1\xi^0 + n(i)_2\xi^2$ and $n(i)_2 = in_2$, $i = 1, \dots, N$. The arguments from [3, Theorem 2] immediately imply that the determinant of the Shapovalov form on the space with the basis $\{L_{n(1)}^N, L_{n(1)}^{N-2} L_{n(2)}, \dots, L_{n(N)}\}$ is a product of diagonal factors, which is a polynomial of a fixed degree in n_2. Moreover, this polynomial is non-trivial as either $\lambda(L_m)$ or $\lambda(c_{v_\alpha})$ is not zero. But then there exist infinitely many small n_2 such that this polynomial takes non-zero values for these n_2, which implies that the dimension of the graded component of degree Nn_2 is at least N. Since n_2 can be arbitrarily small, the proof is complete. □

The same phenomenon occurs also for higher rank Witt and Virasoro algebras, [4].

6 Concluding remarks

By construction, our Witt- and Virasoro-type algebras may be viewed as aperiodic analogs to the Witt- and Virasoro algebras and as such are of independent interest for various applications in mathematical physics. It has already been demonstrated that the Aperiodic Virasoro algebra, which is the one-dimensional analog to the construction presented here, can be used to construct a Calogero-Sutherland model [11] and we expect that new models can be constructed based on the algebraic structures introduced and discussed here. For such applications highest weight representations are crucial, and this paper thus provides the necessary tools for a generalisation of [11]. Furthermore, algebras of the tpye considered here are of interest as they can be used to model the breaking of Virasoro symmetry, as has been demonstrated for the case of the Aperiodic Virasoro algebra in [12], and various applications of the new algebraic structures in quantum mechanics are possible along the lines of [13].

Since super-Virasoro algebras (with or without center) are of crucial importance in string theory, we finally address the question in how far our algebras allow for a generalization to a super-symmetric setting. We start by introducing super-symmetric generalizations of the one-dimensional setting, which can be considered as an aperiodic analog to the super-Witt algebras (see e.g. [14]):

Lemma 8. *Let $\mathbb{F}$ be any number field such that $\mathbb{F} \supset \mathbb{Q}[\tau]$ and let $I(a,b)$ be any interval with boundary points $a < b$ such that $ab \geq 0$. Let furthermore $\varphi : \Sigma(I(a,b)) \to \mathbb{F}$ be a $\mathbb{Z}$–linear map and let x be any of $\frac{1}{2}$, $\frac{\tau}{2}$ or $\frac{1}{2}\tau^2$. Then the $\mathbb{F}$–span of $\{L_n | n \in \Sigma(I(a,b)), G_r | r \in \Sigma(I(a,b)) + x\}$ with the super Lie brackets defined by*

$$\begin{array}{rcl} [L_n, L_m] & = & \chi_{I(a,b)}(n'+m')\varphi(m-n)L_{n+m} \\ [L_n, G_r] & = & \chi_{I(a,b)}(n'+r')\varphi(\frac{1}{2}n-r)G_{n+r} \\ \{G_r, G_s\} & = & \chi_{I(a,b)}(r'+s')2L_{r+s} \end{array} \tag{12}$$

is a Lie super-algebra.

Proof. The super Jacobi identity follows from the Jacobi identity for the super-Witt algebras (centerless super-Virasoro algebras) together with the fact that the condition $ab \geq 0$ on $I(a,b)$ ensures that if a commutator of three elements is non-vanishing, then also all commutators of any two elements in this triple are non-vanishing. All other properties follow from the corresponding results on super-Witt algebras. □

It is natural to call the algebras obtained in Lemma 8 *(centerless) aperiodic super-Virasoro algebras* or *aperiodic super-Witt algebras.* Along the same lines, one obtains super-algebras related to the Witt-type algebras associated with the Penrose tiling, which have been introduced and studied in this paper. Indeed, for this we have to replace in Lemma 8 the interval $I(a,b)$ by an admissible Ω_P^T, use the star-map in the definition of (12) (as in (8)) and use for x any of $\frac{1}{2}\xi^0$, $\frac{\tau}{2}\xi^0$, $\frac{1}{2}\tau^2\xi^0$, $\frac{1}{2}\xi^2$, $\frac{\tau}{2}\xi^2$, $\frac{1}{2}\tau^2\xi^0$, or a combination of any of the first three with any of the latter three.

We remark that our choice of the acceptance window was motivated by the importance of the Penrose tiling for the study of quasicrystals. However, this led to very strong restrictions on the displacement of the pentagonal window with respect to the origin, which resulted in the fact that the algebras we obtained are not perfect. In particular, this implies that no universal central extension exists and we were thus forced to consider only a subclass of special central extensions, which we called geometric. The existence of a universal central extension for analogues to the Witt algebra can be restored for other choices of the acceptance window. The necessary condition to get a perfect algebra is that 0 must belong to the closure of the acceptance window. Some examples, which are compatible with the Jacobi identity, are (truncated) open cones with the vertex at zero or slight variations given by parallelograms etc. For quasicrystals in all dimensions these examples lead to perfect Witt-type Lie algebras with universal central extensions, which can be considered as Virasoro-type algebras associated with corresponding quasicrystals.

We finally point out that in an analogous way Witt- and Virasoro-type algebras may be assigned to any planar tiling which can be described via the cut-and-project formalism. The quantitative picture however changes crucially with the geometric properties of the acceptance window so that it is not convenient to treat these cases uniformly. We therefore have concentrated on an example which is of primary importance in tiling theory and in applications to quasicrystals.

Acknowledgements:

The research was done during the visit of the second author to Uppsala University, Sweden, which was supported by The Swedish Foundation for International Cooperation in Research and Higher Education (STINT). The financial support of STINT is gratefully acknowledged. The second author thanks Uppsala University for the warm hospitality during the stay. The first author was also partially supported by The Swedish Research Council.

References

[1] Chen L, Moody R V, Patera J, Non-crystallographic root systems. *Quasicrystals and discrete geometry* (Toronto, ON, 1995), 135–178, Fields Inst. Monogr., 10, Amer. Math. Soc., Providence, RI, 1998.

[2] Kac V, Kazhdan D 1979, Structure of representations with highest weight of infinite-dimensional Lie algebras. *Adv. in Math.* **34** 97–108

[3] Mazorchuk V 2002, On quasicrystal Lie algebras. *J. Math. Phys.* **43** 2791–2801

[4] Mazorchuk V 1999, Verma modules over generalized Witt algebras. Compositio Math. 115 (1999), no. 1, 21–35.

[5] Patera J, Pelantova E and Twarock R 1998, Quasicrystal Lie algebras *Phys. Lett. A* **246** 209–213

[6] Patera J and Twarock R 1999, Quasicrystal Lie algebras and their generalizations *Phys. Rep.* **315** 241–256

[7] Senechal M 1996, Quasicrystals and Geometry, Cam. Univ. Press

[8] Twarock R 1999, Aperiodic Virasoro algebra. *J. Math. Phys.* **41** 5088–5106.

[9] Patera J 1997, Noncrystallographic root systems and quasicrystals *Mathematics of Long Range Aperiodic Order*, ed R V Moody (Kluwer) 443–465.

[10] Patera J, Zassenhaus H 1991, The higher rank Virasoro algebras. *Comm. Math. Phys.* **136** 1–14.

[11] Twarock R 2000, A Calogero-Sutherland model based on the Aperiodic Virasoro algebra. *Int. J. Mod. Phys. A* **15**, 4179–4189.

[12] Twarock R 1999, Breaking Virasoro symmetry, *Quantum Theory and Symmetries*, eds H-D Doebner, V K Dobrev, J-D Hennig, and W Lücke (World Scientific, Singapore) 530–534.

[13] Twarock R 2002, A new Virasoro type symmetry and its application in quantum physics, *Proceedings of the Sixth Wigner Symposium*, ed M Arik (Bogazici Univ. Press, Istanbul), 425–432.

[14] Yucai Su 2000, Simple Modules over the Higher-Rank super-Virasoro Algebras. *Lett. Math. Phys.* **53** 263–272