Renormalization for Lorenz maps of long monotone combinatorial types

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Abstract. Lorenz maps are maps of the unit interval with one critical point of order $\rho > 1$, and a discontinuity at that point. They appear as return maps of sections of the geometric Lorenz flow.

We construct real *a priori* bounds for renormalizable Lorenz maps with long monotone combinatorics, and use these bounds to show existence of periodic points of renormalization, as well as existence of Cantor attractors for dynamics of infinitely renormalizable Lorenz maps.

1. *Introduction*

E. N. Lorenz in [**8**] demonstrated numerically the existence of certain three-dimensional flows that have a complicated behaviour. The *Lorenz flow* has a saddle fixed point with a one-dimensional unstable manifold and an infinite set of periodic orbits whose closure constitutes a global attractor of the flow.

As it is often done in dynamics, one can attempt to understand the behaviour of a three-dimensional flow by looking at the first return map to an appropriately chosen two-dimensional section. In the case of the Lorenz flow, it is convenient to chose the section as a plane transversal to the local stable manifold, and ,therefore, intersecting it along a curve γ. The first return map is discontinuous at γ.

The *geometric Lorenz flow* has been introduced in [**9**]: a Lorenz flow with an extra condition that the return map preserves a one-dimensional foliation in the section, and contracts distances between points in the leafs of this foliation at a geometric rate. Since the return maps is contracting in the leafs, its dynamics is asymptotically one-dimensional, and can be understood in terms of a map acting on the space of leafs (an interval). This interval map has a discontinuity at the point of the interval corresponding to γ, and is commonly called the *Lorenz map*.

We will start by defining what is known as the standard Lorenz family. Our work is a continuation of the study started in [**11**], and we will, therefore, make a conscientious effort to use the notation of [**11**] so that it would be easier for the reader to compare the approach of this paper with that of [**11**].

Definition 1. *Let $u \in [0,1]$, $v \in [0,1]$, $c \in (0,1)$ and $\rho > 0$. The standard Lorenz family $(u,v,c) \mapsto Q(x)$ is the family of maps $Q : [0,1] \setminus \{c\} \mapsto [0,1]$ with a single critical point at which the map is discontinuous:*

$$Q(x) = \begin{cases} u\left(1 - \left(\frac{c-x}{c}\right)^\rho\right), & x \in [0,c), \\ 1 + v\left(-1 + \left(\frac{x-c}{1-c}\right)^\rho\right), & x \in (c,1], \end{cases}$$

Remark 2. *In the definition above, u is the length of $Q([0,c))$, v is that of $Q((c,1])$, while u and $1-v$ are the critical values. To emphasise that a critical point c corresponds to a map f, we will use the notation $c(f)$. The difference $1-c$ will be denoted as μ:*

$$\mu \equiv 1 - c.$$

More generally,

Definition 3. *A C^k-Lorenz map $f : [0,1] \setminus \{c\} \mapsto [0,1]$ is defined as*

$$f(x) = \begin{cases} f_0(x) \equiv \phi(Q(x)), & x \in [0,c), \\ f_1(x) \equiv \psi(Q(x)), & x \in (c,1], \end{cases}$$

where ϕ and ψ are C^k orientation preserving diffeomorphisms of $[0,1]$ (this space will be denoted by $\mathcal{D}^k$).

We will refer to the diffeomorphisms ϕ and *psi* as *coefficients* of the Lorenz map.

The set of C^k-Lorenz maps will be denoted $\mathcal{L}^k$. Since a Lorenz map (3) can be identified with a quintuple (u,v,c,ϕ,ψ), the space $\mathcal{L}^k$ is isomorphic to $[0,1]^2 \times (0,1) \times \mathcal{D}^k \times \mathcal{D}^k$. $\mathcal{L}^S \subset \mathcal{L}^3$ will denote the subset of maps with the negative Schwarzian derivative S_f,

$$S_f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 = N_f'(x) - \frac{1}{2}N_f(x)^2. \tag{1.1}$$

We will denote the C^k-norm by $|\cdot|_k$. The subsets of $\mathcal{D}^3$ of diffeomorphisms with a negative Schwarzian will be denoted $\mathcal{D}^S$.

Guckenheimer and Williams have proved in [**4**] that there is an open set of three-dimensional vector fields, that generate a geometric Lorenz flow with a smooth Lorenz map of $\rho < 1$. However, one can use the arguments of [**4**] to construct open sets of vector fields with Lorenz maps of $\rho \geq 1$. Similarly to the unimodal family, Lorenz maps with $\rho > 1$ have a richer dynamics that combines contraction with expansion.

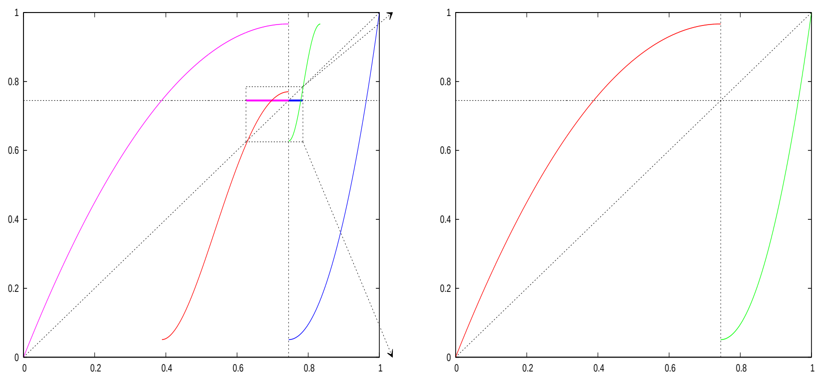
For any $x \in [0,1] \setminus \{c\}$ such that $f^n(x) \neq c$ for all $n \in \mathbb{N}$, define the itinerary $\omega(x) \in \{0,1\}^{\mathbb{N}}$ of x as the sequence $\{\omega^0(x), \omega^1(x), \ldots\}$, such that

$$\omega^i = \begin{cases} 0, & f^i(x) < c, \\ 1, & f^i(x) > c. \end{cases} \tag{1.2}$$

If one imposes the usual order $0 < 1$, then for any two ω and $\tilde{\omega}$ in $\{0,1\}^{\mathbb{N}}$, we say that $\omega < \tilde{\omega}$ iff there exists $r \geq 0$ such that $\omega^i = \tilde{\omega}^i$ for all $i < r$ and $\omega^r < \tilde{\omega}^r$.

The limits

$$\omega(x^+) \equiv \lim_{y \downarrow x} \omega(y), \quad \omega(x^-) \equiv \lim_{y \uparrow x^-} \omega(y)$$



a) b)

FIGURE 1. a) A Lorenz map f of renormalization type $(01, 1000)$ with the critical exponent $\rho = 2$; b) $\mathcal{R}[f]$

exists for all $x \in [0, 1]$.

The kneading invariant $K(f)$ of f is the pair $(K^-(f), K^+(f)) = (\omega(c^-), \omega(c^+))$. Hubbard and Sparrow have found in [**5**] a condition on the kneading invariant of topologically expansive Lorenz maps. Kneading invariants for a general Lorenz map, not necessarily expansive, satisfy the following condition:

$$K_0^- = 0, \quad K_0^+ = 1, \quad \sigma(K^+) \le \sigma^n(K^\pm) \le \sigma(K^-), \quad n \in \mathbb{N},$$

here σ is the shift in $\{0, 1\}^{\mathbb{N}}$. Conversely, any sequence as above is a kneading sequence for some Lorenz map.

A Lorenz map has two critical values

$$c_1^- = \lim_{x \uparrow c} f(x), \quad c_1^+ = \lim_{x \downarrow c} f(x).$$

We will use the notation $c_1^\pm(f)$ whenever we want to emphasise that that critical value corresponds to a function f.

A Lorenz map f with $c_1^+ < c < c_1^-$ is called nontrivial, otherwise f has a globally attracting fixed point. In general, $c_k^\pm$ will denote points in the orbit of the critical values:

$$c_i^\pm = f^{i-1}(c_1^\pm), \quad i \ge 1.$$

DEFINITION 4. *A Lorenz map f is called renormalizable if there exist p and q, $0 < p < c < q < 1$, such that the first return map (f^n, f^m), $n > 1, m > 1$, of $C = [p, q]$ is affinely conjugate to a nontrivial Lorenz map. Choose C such that it is maximal. The rescaled first return map of such $C \setminus \{c\}$ is called the renormalization of f and denoted $\mathcal{R}[f]$.*

We will denote

$$L = [p, c), \quad R = (c, q],$$

while the first return map will be denoted $\mathcal{P}[f]$ and referred to as the prerenormalization. If f is renormalizable, then there exist minimal positive integers n and m such that

$$\mathcal{P}[f](x) = \begin{cases} f^{n+1}(x), & x \in L, \\ f^{m+1}(x), & x \in R, \end{cases}$$

Then, explicitly,

$$\mathcal{R}[f] = A^{-1} \circ \mathcal{P}[f] \circ A, \tag{1.3}$$

where A is the affine orientation preserving rescaling of $[0,1]$ onto C. We will also use the notation $\tilde{f}$ for the renormalization of f.

The intervals $f^i(L)$, $1 \le i \le n$, are pairwise disjoint, and disjoint from C. So are the intervals, $f^i(R)$, $1 \le i \le m$. Since these intervals do not contain c, we can associate a finite sequence of 0 and 1 to each of these two sequences of intervals:

$$\omega^- = \{K_0^-, \ldots, K_n^-\},\ \omega^+ = \{K_0^+, \ldots, K_m^+\},\ \omega = (\omega^-, \omega^+) \in \{0,1\}^{n+1} \times \{0,1\}^{m+1},$$

which will be called the type of renormalization. The subset of maps (3) which are renormalizable of type (ω^-, ω^+) is referred to as the domain of renormalization $\mathcal{D}_\omega$ (cf. [**7**]).

Let

$$\bar{\omega} = (\omega_0, \omega_1, \ldots) \in \prod_{i \in \mathbb{N}} \bigotimes \left(\{0,1\}^{n_i+1} \times \{0,1\}^{m_i+1} \right). \tag{1.4}$$

If $\mathcal{R}^i[f]$ is ω_i-renormalizable for all $i \in \mathbb{N}$, then f is called infinitely renormalizable of combinatorial type $\bar{\omega}$. The set of ω-renormalizable maps will be denoted $\mathcal{L}_\omega$, the set of maps f such that $\mathcal{R}^i[f]$ is ω_i-renormalizable will be called $\mathcal{L}_{\bar{\omega}}$, $\bar{\omega} = (\omega_0, \omega_1, ..., \omega_n)$, with n finite or infinite. If $\bar{\omega}$ is such that $|\omega_i^\pm| < B$, $i = 0, 1, \ldots$, for some $0 < B < \infty$, we say that $\bar{\omega}$ is of *bounded type*.

We would like to draw the attention of the reader to position of the indices in our notation: $\omega_i \in \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ is a pair of two words, while ω^i is an integer 0 or 1 in a single word (cf. (1.2)).

The combinatorics

$$\omega = (0\overbrace{1\ldots1}^{n}, 1\overbrace{0\ldots0}^{m}) \tag{1.5}$$

will be called *monotone*. The set of all monotone combinatorial types will be denoted $\mathcal{M}$, while $\mathcal{L}_\mathcal{M}$ will denote all Lorenz maps which are ω-renormalizable with $\omega \in \mathcal{M}$.

Given an integer $N > 1$, the subset of $\mathcal{M}$ of all ω's such that the length of words in ω satisfies $N \le |\omega^-|$ and $N \le |\omega^+|$, will be denoted $\mathcal{M}_N$.

Given a subset $\mathcal{A} \subseteq \mathcal{M}$, $\mathcal{L}_\mathcal{M}$ will denote all Lorenz maps which are ω-renormalizable with $\omega \in \mathcal{A}$. We will also use the notation $\mathcal{L}^S_\mathcal{A} = \mathcal{L}^S \cap \mathcal{L}_\mathcal{A}$.

The main results of our paper are the following proposition and theorems.

Main Proposition 1. *(A priori bounds). For every $\rho > 2$ there exist an integer $N > 1$, and a relatively compact subset $\mathcal{K}$ of $\mathcal{L}^0$ such that $\mathcal{R}[\mathcal{L}^S_{\mathcal{M}_N} \cap \mathcal{K}] \subset \mathcal{K}$.*

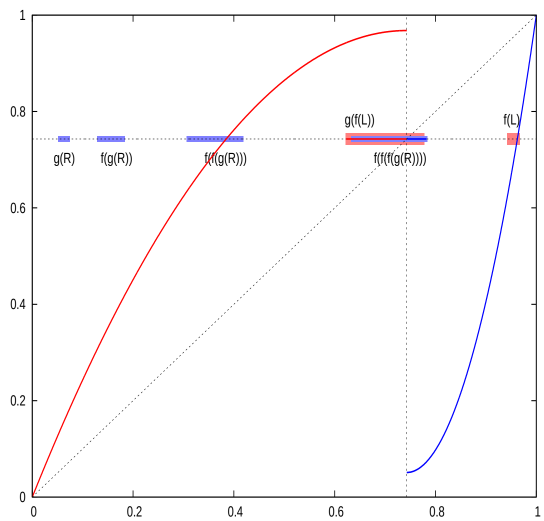


FIGURE 2. Monotone combinatorics $(01, 1000)$ for a map with the critical exponent $\rho = 2$. The two halves of the central interval are given in red and blue, their images under the map in semi-transparent red and blue.

At this point we were able to prove *a priori* bounds only for $\rho > 2$. The somewhat technical reasons for that will become clear in the proof of the invariance of bounds on the critical point in Proposition 20. Proposition 1 is used to obtain the existence of the periodic points of renormalization:

MAIN THEOREM 1. *(Renormalization periodic points). For every $\rho > 2$ and every $\bar{\omega} = (\omega_0, \ldots, \omega_{k-1}) \in \mathcal{M}_N^k$, where N is as in in the Main Proposition 1, the renormalization operator (1.3) has a periodic point in $\mathcal{L}^S_{\mathcal{M}_N} \cap \mathcal{K}$ of type $\bar{\omega}$.*

The proof of the next Main Theorem 2 will not be given here since it practically identical to the proof of a similar result in [**11**], after on establishes *a priori bounds*. We, however, chose to state this as separate main result since the existence of a Cantor attractor for the dynamics merits a special emphasis.

MAIN THEOREM 2. *(Cantor attractors). Let $\rho > 2$, and suppose that $\bar{\omega} = (\omega_0, \ldots, \omega_{k-1} \ldots) \in \mathcal{M}_N^{\mathbb{N}}$, where N is as in in the Main Proportion 1, is of bounded type.*

Consider $f \in \mathcal{L}^S_{\bar{\omega}} \cap \mathcal{K}$, and let Λ be the closure of the orbits of the critical values.

Then,

1) Λ is a Cantor set of a Hausdorff dimension strictly inside $(0, 1)$;

2) Λ is uniquely ergodic;

3) the complement of the basin of attraction of Λ in $[0, 1]$ has zero Lebesgue measure.

The study of renormalizable Lorenz maps was initiated by Tresser et al. (see e.g. [**1**]). A more recent work of Martens and de Melo [**7**] produced a series of important results, specifically about the domains of renormalization and the structure of the parameter plane for two-dimensional Lorenz families.

The work [**10**] presented a computer assisted proof of existence of a renormalization fixed point for the renormalization operator of type $(\{0,1\},\{1,0,0\})$. The renormalization operator of this particular type has been later shown to have a fixed point in the class of maps analytic on a neighbourhood of the unit interval using only complex analytic techniques in [**3**].

In a more general setting, issues of existence of renormalization periodic points and hyperbolicity have been addressed in [**11**], where it is proved that the limit set of renormalization, restricted to monotone combinatorics with the return time of one branch being large and much larger than the return time for the other branch, is a Cantor set, and that each point in the limit set has a two-dimensional unstable manifold. Specifically, [**11**] proves equivalents of our Main Proposition 1 and Main Theorem 1 for monotone combinatorial types with the following return times:

$$[\rho] \le |\omega^-| - 1 \le [2\rho - 1], \quad n_- \le |\omega^+| - 1 \le n_+, \tag{1.6}$$

where n_- is sufficiently large, and n_+ depends on the choice of n_-.

In comparison, our approach allows us to prove the *a priori* bounds for a different class of combinatorial types. We are able to avoid the disparity of return times evident in (1.6), as well as boundedness of return times from above. Nevertheless, we could not avoid a condition of largeness of return times. We would like to emphasise, however, that the lower bounds on the return times for which our results are valid can be expressed in terms of explicit but very cumbersome functions of ρ. A careful computation of these bounds will result in definite (and, likely, not too large) values of N. However, we have not performed these estimates in the present paper.

2. *Preliminaries*

2.1. *The Koebe Principle* We will start by quoting the Koebe Principle which is of a fundamental importance in real dynamics (see, ex. [**6**]). We will say that an interval V is a τ-scaled neighbourhood of $U \subset V$, if both components of $V \setminus U$ have length at least $\tau \cdot U$.

Theorem (Koebe Priniciple) 1. *Let $J \subset T$ be intervals, and $f : T \mapsto f(T)$ be a C^3-diffeomorphism with $S_f < 0$. If $f(T)$ contains a τ-scaled neighbourhood of $f(J)$, then*

$$\left(\frac{\tau}{1+\tau}\right)^2 \le \frac{Df(x)}{Df(y)} \le \left(\frac{1+\tau}{\tau}\right)^2, \quad x, y \in J.$$

2.2. *Distortion and nonlinearity* Let $C^k(A;B)$ be the set of k-continuously differentiable maps from A to B. We denote $\mathcal{D}^k(A;B) \subset C^k(A;B)$ the subset of orientation preserving homeomorphisms whose inverse lie in $C^k(A;B)$. We will use the notation $\mathcal{D}^k$ and C^k whenever $A = B = [0,1]$.

Definition 5. *The nonlinearity operator* $N : \mathcal{D}^2(A;B) \mapsto C^0(A;\mathbb{R})$ *is defined as*

$$N_\phi = D \log D\phi,$$

while

$$N_\phi(x) = \frac{\phi''(x)}{\phi'(x)}$$

is the nonlinearity of ϕ *at point* x.

Definition 6. *Given* $\phi \in \mathcal{D}^1(A;B)$, *the quantity*

$$\operatorname{dist}[\phi] = \max_{x,y\in A} \ln\left(\frac{D\phi(y)}{D\phi(x)}\right)$$

is called the distortion of ϕ.

Notice, that

$$\int_x^y N_\phi(t)dt = \ln \frac{D\phi(y)}{D\phi(x)}.$$

The following Lemma results from a straightforward computation.

Lemma 7. *The nonlinearity operator* $N : \mathcal{D}^2(A;B) \mapsto C^0(A;\mathbb{R})$ *is a bijection. In the case* $A = B = [0,1]$, *the inverse is defined as*

$$N_\phi^{-1}(x) = \frac{\int_0^x \exp\left\{\int_0^r \phi(t)dt\right\} dr}{\int_0^1 \exp\left\{\int_0^r \phi(t)dt\right\} dr}. \tag{2.7}$$

One can turn $\mathcal{D}^2(A;B)$ into a Banach space using the nonlinearity operator. Specifically, for ϕ, ψ in $\mathcal{D}^2(A;B)$ and $a, b \in \mathbb{R}$, the linear structure and the norm are defined via

$$a\phi + b\psi = N^{-1}_{aN_\phi + bN_\psi}, \tag{2.8}$$

$$\|\phi\| = \sup_{x\in A} |N_\phi(x)|. \tag{2.9}$$

Finally, we give a list of useful bounds on derivatives and distortion in $\mathcal{D}^2(A;B)$ in terms on the nonlinearity (see [**11**] or [**6**] for the proofs).

Lemma 8. *If* $\phi, \psi \in \mathcal{D}^2(A;B)$ *then, for all* $x, y \in A$,

$$e^{-|y-x|\|\phi\|} \le \frac{D\phi(y)}{D\phi(x)} \le e^{|y-x|\|\phi\|}, \tag{2.10}$$

$$\frac{|B|}{|A|}e^{-\|\phi\|} \le D\phi(x) \le \frac{|B|}{|A|}e^{\|\phi\|}, \tag{2.11}$$

$$e^{-\|\phi-\psi\|} \le \frac{D\phi(x)}{D\psi(x)} \le e^{\|\phi-\psi\|}. \tag{2.12}$$

Notice, that the set of uniformly bounded homeomorphisms in $\mathcal{D}^2$ is relatively compact. Indeed, consider $\mathcal{U} = \{\phi \in \mathcal{D}^2 : \|\phi\| \le K\}$. Because of (2.11) such maps have a uniformly bounded derivative, and therefore are equicontinous. By Arzelà-Ascoli theorem, $\mathcal{U}$ is relatively compact in C^0.

We will introduce two subsets of Lorenz maps, defined via conditions on their distortion and critical points.

Definition 9. *Given a real constants $\pi > 0$, we set*

$$\mathcal{K}^\pi \equiv \left\{ f \in \mathcal{L}^S : \mathrm{dist}[\psi] \le \pi, \mathrm{dist}[\phi] \le \pi \right\}. \tag{2.13}$$

Given real constants $\pi > 0$, $\varepsilon > 0$, set

$$\mathcal{K}^\pi_\varepsilon \equiv \left\{ f \in \mathcal{K}^\pi \subset \mathcal{L}^S : c(f) \in [\varepsilon, 1-\varepsilon] \right\}. \tag{2.14}$$

The reason for the introduction of these sets is the following compactness result.

Corollary 10. *Let $\pi > 0$ and $\varepsilon > 0$. Then the set $\mathcal{K}^\pi_\varepsilon$ is relatively compact in $\mathcal{L}^0$.*

Proof. Recall that $\mathcal{L}^2$ is isomorphic to $[0,1]^2 \times (0,1) \times \mathcal{D}^2 \times \mathcal{D}^2$. Since c is bounded away from 0 and 1 by a constant, it is, therefore, contained in a compact subset of $(0,1)$. Consider the set

$$\mathcal{B} = \left\{ (\phi, \psi) \in \mathcal{D}^2 \times \mathcal{D}^2 : \mathrm{dist}[\phi] \le \pi, \mathrm{dist}[\psi] \le \pi \right\}.$$

Any sequence from $\mathcal{B}$ is equicontinous since $|\phi(y) - \phi(x)| \le e^\pi |y - x|$, and clearly, uniformly bounded, therefore it has a convergent subsequence by the Arzelà-Ascoli theorem. □

2.3. *Monotone combinatorics* We will quote a lemma from [**11**] (Lemma 2.1.11) which gives the formulae for the factors of a renormalization of a Lorenz map in $\mathcal{L}_\mathcal{M}$. Let I be an interval and g_I be an orientation preserving diffeomorphism. We denote the affine transformation that takes $[0,1]$ onto I as ξ_I. Define the *zoom operator*:

$$Z(g; I) = \xi^{-1}_{g(I)} \circ g \circ \xi_I. \tag{2.15}$$

Lemma 11. *If $f = (u, v, c, \phi, \psi)$ is renormalizable of monotone combinatorics, then*

$$\mathcal{R}[f] = (\tilde{u}, \tilde{v}, \tilde{c}, \tilde{\phi}, \tilde{\psi})$$

is given by

$$\tilde{u} = \frac{|Q(L)|}{|U|}, \quad \tilde{v} = \frac{|Q(L)|}{|V|}, \quad \tilde{c} = \frac{|L|}{|C|}, \tag{2.16}$$

$$\tilde{\phi} = Z(\bar{\phi}; U), \quad \tilde{\psi} = Z(\bar{\psi}; V), \quad \bar{\phi} = f_1^n \circ \phi, \quad \bar{\psi} = f_0^m \circ \psi, \tag{2.17}$$

where $U = \phi^{-1} \circ f_1^{-n}(C)$, $V = \psi^{-1} \circ f_0^{-m}(C)$.

3. *Estimates for Lorenz maps with monotone combinatorics*

In this Section we will obtain bounds on the critical points, critical values and lengths of the central subintervals L and R for Lorens maps with monotone combinatorics whose diffeomorphic coefficients have bounded distortion.

Suppose that $f \in \mathcal{L}^3$ is ω-renormalizable, where

$$\omega = (0\ \omega^1\ \omega^2 \dots \omega^n, 1\ v^1\ v^2 \dots v^m).$$

Given, $l \geq k$, denote

$$\Psi^{-1}_{\omega^k \dots \omega^l} \equiv f_{\omega^l} \circ \dots \circ f_{\omega^k}, \quad \Psi_{\omega^k \dots \omega^l} = f^{-1}_{\omega^k} \circ \dots \circ f^{-1}_{\omega^l},$$

and

$$\omega(k) \equiv \omega^1 \dots \omega^k, \quad \omega^+(k) \equiv v^1 \dots v^k,$$

(note different notations here and in (1.4) and (1.2)), then the prerenormalization can be written as

$$\mathcal{P}[f] = (\Psi^{-1}_{\omega^-(n)} \circ f_0, \Psi^{-1}_{\omega^+(m)} \circ f_1).$$

Furthermore, we denote

$$\begin{aligned} \Psi^{-1}_{\omega^-(n)} \circ f_0(L) &\equiv I \equiv [p, c^-_{n+1}), \quad \Psi_{\omega^k \dots \omega^n}(I) = I_k, \\ \Psi^{-1}_{\omega^+(m)} \circ f_1(R) &\equiv J \equiv (c^+_{m+1}, q], \quad \Psi_{v^k \dots v^m}(J) = J_k. \end{aligned}$$

Notice, that $I_1 = f_0(L)$ and $J_1 = f_1(R)$.

We will mention the following simple lemma (cf. [**11**] for a proof).

Lemma 12. *$\Psi_{\omega^-(n)}$ and $\Psi_{\omega^+(m)}$ extend to neighbourhoods of $(c^+_1, 1)$ and $(0, c^-_1)$, respectively, as analytic maps.*

We will continue with a sequence of lemmas which will prepare us for a construction of *a priori* bounds — construction of a relatively compact set invariant under renormalization.

First of all, we will need simple bounds on the difference of f_0 and f_1 at two points of the domain.

Lemma 13. *Suppose that* $\text{dist}[\phi] \leq \pi$, $\text{dist}[\psi] \leq \pi$, *then*

$$\frac{e^{-\pi} \rho c^-_1}{c} (x-y) \left(\frac{c-x}{c} \right)^{\rho-1} \leq f_0(x) - f_0(y) \leq \frac{e^{\pi} \rho c^-_1}{c} (x-y) \left(\frac{c-y}{c} \right)^{\rho-1}, \tag{3.18}$$

for any $x > y$ in $[0, c)$, and

$$\frac{e^{-\pi} \rho (1-c^+_1)}{\mu} (x-y) \left(\frac{y-c}{\mu} \right)^{\rho-1} \leq f_1(x) - f_1(y) \leq \frac{e^{\pi} \rho (1-c^+_1)}{\mu} (x-y) \left(\frac{x-c}{\mu} \right)^{\rho-1}, \tag{3.19}$$

for any $x > y$ in $(c, 1]$.

Proof. Notice, that the average derivative of ϕ on $(0,u)$ is c_1^+/u, therefore, the derivative $\phi'(x)$ at any point in $(0,u)$ is bounded as

$$\frac{c_1^-}{u}e^{-\pi} \le \phi'(x) \le \frac{c_1^-}{u}e^{\pi}. \tag{3.20}$$

Similarly, for $x \in (1-v,1)$.

$$\frac{1-c_1^+}{v}e^{-\pi} \le \psi'(x) \le \frac{1-c_1^+}{v}e^{\pi}. \tag{3.21}$$

Therefore, we get for $x > y$ in $[0,c)$

$$f_0(x)-f_0(y) \le \frac{c_1^-}{u}e^{\pi}\rho\frac{u}{c}(x-y)\left(\frac{c-y}{c}\right)^{\rho-1} = \frac{e^{\pi}\rho c_1^-}{c}(x-y)\left(\frac{c-y}{c}\right)^{\rho-1}.$$

The lower bound is obtained as follows:

$$f_0(x)-f_0(y) \ge \frac{c_1^-}{u}e^{-\pi}\rho\frac{u}{c}(x-y)\left(\frac{c-x}{c}\right)^{\rho-1} = \frac{e^{-\pi}\rho c_1^-}{c}(x-y)\left(\frac{c-x}{c}\right)^{\rho-1}.$$

Bounds on the difference of f_1 can be obtained in a similar way. □

Let us introduce the following notation for the sake of brevity:

$$\alpha \equiv \frac{e^{-\pi}}{\rho}, \quad \eta \equiv \frac{e^{-\pi}\mu}{(1-c_1^+)\rho}, \quad \kappa \equiv \frac{e^{-\pi}c}{c_1^-\rho}, \quad \gamma \equiv \frac{e^{2\pi}}{\rho}, \quad \nu \equiv \frac{\mu}{(1-c_1^+)^{\frac{1}{\rho}}}, \quad \xi \equiv \frac{c}{(c_1^-)^{\frac{1}{\rho}}}. \tag{3.22}$$

Since $R \subset f^{m+1}(R)$, we have that $f_0^{-1}(c) \in f^m(R)$, and for monotone combinatorics $f_0^{-1}(c) > c_1^+$. Similarly, $f_1^{-1}(c) < c_1^-$. The next lemma uses this fact, and provides a lower bound on the length of the intervals $[f_0^{-1}(c), p]$ and $[q, f_1^{-1}(c)]$, which is also a lower bound on the length of the intervals $[c_1^+, p]$ and $[q, c_1^-]$.

Lemma 14. *Let $f \in \mathcal{K}^\pi \cap \mathcal{L}_\omega$ for some $\pi > 0$ and $\omega = (\omega_-, \omega_+) \in \mathcal{M}$ with $|\omega_-| = n+1$, $|\omega_+| = m+1$. Then*

$$|p - f_0^{-1}(c)| \ \ge \ \left(\kappa\left(\frac{c}{c-c_1^+}\right)^{\rho-1}\left(\nu^{\frac{\rho}{\rho-1}}e^{\frac{-\pi}{\rho-1}}\right)\right)^{\frac{\rho^n}{\rho^n-1}} \equiv \Delta, \tag{3.23}$$

$$|q - f_1^{-1}(c)| \ \ge \ \left(\eta\left(\frac{\mu}{c_1^- - c}\right)^{\rho-1}\left(\xi^{\frac{\rho}{\rho-1}}e^{\frac{-\pi}{\rho-1}}\right)\right)^{\frac{\rho^m}{\rho^m-1}} \equiv \Theta. \tag{3.24}$$

Proof. We will first demonstrate that

$$f_1^{-n}(x) \ge c + \nu^{\frac{\rho}{\rho-1}}e^{\frac{-\pi}{\rho-1}}\left(x-c_1^+\right)^{\frac{1}{\rho^n}} \tag{3.25}$$

for all $x > c_1^+$. To prove (3.25) we use the following expressions for the inverse branches of a Lorenz map:

$$f_0^{-1}(x) \ = \ c - c\left(\frac{|\phi^{-1}([x,c_1^-])|}{|\phi^{-1}([0,c_1^-])|}\right)^{\frac{1}{\rho}} = c - c\left(\frac{u-\phi^{-1}(x)}{u}\right)^{\frac{1}{\rho}}, \tag{3.26}$$

$$f_1^{-1}(x) \ = \ c + \mu\left(1 - \frac{|\psi^{-1}([x,1])|}{|\psi^{-1}([c_1^+,1])|}\right)^{\frac{1}{\rho}} = c + \mu\left(1 - \frac{1-\psi^{-1}(x)}{v}\right)^{\frac{1}{\rho}}, \tag{3.27}$$

and start with

$$f_1^{-1}(x) \geq c + \mu \left(e^{-\pi} \frac{x - c_1^+}{1 - c_1^+} \right)^{\frac{1}{\rho}} = c + \mu \left(\frac{e^{-\pi}}{1 - c_1^+} \right)^{\frac{1}{\rho}} \left(x - c_1^+ \right)^{\frac{1}{\rho}},$$

for $x > c_1^+$, and use induction on this inequality to obtain

$$f_1^{-n}(x) \geq c + \left(\mu \left(\frac{e^{-\pi}}{1 - c_1^+} \right)^{\frac{1}{\rho}} \right)^{1 + \ldots \rho^{-(n-1)}} \left(x - c_1^+ \right)^{\frac{1}{\rho^n}} \geq c + \nu^{\frac{\rho}{\rho-1}} e^{\frac{-\pi}{\rho-1}} \left(x - c_1^+ \right)^{\frac{1}{\rho^n}}.$$

According to Lemma 13:

$$|f_0(p) - c| \leq \kappa^{-1} |p - f_0^{-1}(c)| \left(\frac{c - f_0^{-1}(c)}{c} \right)^{\rho - 1}.$$

On the other hand, $f_0(p) = f_1^{-n}(p)$, and according to (3.25),

$$f_1^{-n}(p) \geq c + \nu^{\frac{\rho}{\rho-1}} e^{\frac{-\pi}{\rho-1}} \left(p - c_1^+ \right)^{\frac{1}{\rho^n}}. \tag{3.28}$$

Therefore,

$$f_0(p) - c = f_1^{-n}(p) - c \geq \nu^{\frac{\rho}{\rho-1}} e^{\frac{-\pi}{\rho-1}} |p - c_1^+|^{\frac{1}{\rho^n}},$$

and

$$\kappa^{-1} |p - f_0^{-1}(c)| \left(\frac{c - f_0^{-1}(c)}{c} \right)^{\rho - 1} \geq \nu^{\frac{\rho}{\rho-1}} e^{\frac{-\pi}{\rho-1}} |p - c_1^+|^{\frac{1}{\rho^n}} \geq \nu^{\frac{\rho}{\rho-1}} e^{\frac{-\pi}{\rho-1}} |p - f_0^{-1}(c)|^{\frac{1}{\rho^n}}, \tag{3.29}$$

which results in the required bound (3.23).

The bound on $|q - f_1^{-1}(c)|$ is obtained in a similar way. □

Lower bounds on the differences $|p - f_0^{-1}(c)|$ and $|f_1^{-1}(c) - q|$ can be used to bound c_1^- and $1 - c_1^+$ from below.

Lemma 15. *Let $f \in \mathcal{K}^\pi \cap \mathcal{L}_\omega$ for some $\pi > 0$ and $\omega = (\omega_-, \omega_+) \in \mathcal{M}$ with $|\omega_-| = n + 1$, $|\omega_+| = m + 1$. Then,*

$$c_1^+ \geq \frac{\kappa^m \Delta}{1 - \kappa^m}, \quad 1 - c_1^- \geq \frac{\eta^n \Theta}{1 - \eta^n}. \tag{3.30}$$

Proof. To get the lower bound on c_1^+ we notice that the derivatives of the inverse branches of $Q(x)$ (formulae (3.26) and (3.27) with $\phi = \psi = \mathrm{id}$) are increasing functions, while the derivatives of ϕ and ψ are bounded as in 3.20 and (3.21). This can be used to get a straightforward bound

$$D f_0^{-1}(x) \geq \frac{e^{-\pi} c}{c_1^- \rho} = \kappa,$$

for all $0 < x < c_1^-$. Therefore,

$$f^{-m}(p) \geq \left(D f^{-1}(0) \right)^m p \geq \kappa^m p,$$

so

$$p \geq c_1^+ + \Delta \geq \kappa^m p + \Delta \implies p \geq \frac{\Delta}{1-\kappa^m},$$

and

$$c_1^+ \geq \frac{\kappa^m \Delta}{1-\kappa^m}.$$

The lower bound on $1-c_1^-$ is obtained in a similar way. □

We will now turn our attention to the bounds on L and R.

LEMMA 16. *Let $f \in \mathcal{K}^\pi \cap \mathcal{L}_\omega$ where $0 < 2\pi < \ln \rho$ and $\omega = (\omega_-, \omega_+) \in \mathcal{M}$ with $|\omega_-| = n+1$, $|\omega_+| = m+1$. Then there exist a constant K, such that*

$$|L| \leq \left((c_1^- - c)\frac{c^\rho e^\pi}{c_1^-}\right)^{\frac{1}{\rho+1}} \left(\frac{\gamma^{-1}-1}{\gamma^{-n}-1}\right)^{\frac{1}{\rho+1}}, \tag{3.31}$$

$$|L| \leq \left(\mu^2 \left|\frac{\mu}{\Theta+|R|}\right|^{\rho-1} \frac{\gamma c^\rho}{(1-c_1^+)c_1^-}\right)^{\frac{1}{\rho+\frac{1}{\rho^{n-1}}}}, \tag{3.32}$$

$$|L| \geq \left(\frac{e^{-\pi}c^\rho}{c_1^-}\eta^n\right)^{\frac{1}{\rho-1}} \exp\left(K\frac{\eta^n \Theta}{\mu(1-\eta^n)}\sum_{k=1}^{n}\left(\frac{e^{-2\pi}}{\eta}\frac{(\Theta+|R|)^{\rho-1}}{\mu^{\rho-1}}\right)^{k-1}\right), \tag{3.33}$$

and

$$|R| \leq \left((c-c_1^+)\frac{\mu^\rho e^\pi}{(1-c_1^+)}\right)^{\frac{1}{\rho+1}} \left(\frac{\gamma^{-1}-1}{\gamma^{-m}-1}\right)^{\frac{1}{\rho+1}}, \tag{3.34}$$

$$|R| \leq \left(c^2 \left|\frac{c}{\Delta+|L|}\right|^{\rho-1} \frac{\gamma \mu^\rho}{(1-c_1^+)c_1^-}\right)^{\frac{1}{\rho+\frac{1}{\rho^{m-1}}}}, \tag{3.35}$$

$$|R| \geq \left(\frac{e^{-\pi}\mu^\rho}{1-c_1^+}\kappa^n\right)^{\frac{1}{\rho-1}} \exp\left(K\frac{\kappa^n \Delta}{c(1-\kappa^m)}\sum_{k=1}^{m}\left(\frac{e^{-2\pi}}{\kappa}\frac{(\Delta+|L|)^{\rho-1}}{c^{\rho-1}}\right)^{k-1}\right). \tag{3.36}$$

Proof. 1) Upper bounds. Denote $p_i = f^i(p)$ and $q_i = f^i(q)$ (notice, $p_{n+1} = p$ and $q_{m+1} = q$), and, as before, $c_i^\pm = f^{i-1}(c_1^\pm)$. Suppose, point x_1 is in the interval I_1, and denote points in the orbit of x_1 as x_k: $x_k = f_1^{k-1}(x_1)$. Then, according to (3.25),

$$p_k \equiv f_1^{-(n-k)}(x_n) \geq c + \nu^{\frac{\rho}{\rho-1}} e^{\frac{-\pi}{\rho-1}} \left(x_n - c_1^+\right)^{\frac{1}{\rho^{n-k}}} \equiv \tilde{p}_k,$$

and one gets for all $n+1>k>0$

$$
\begin{aligned}
Df_1^{-1}(x_{k+1}) &\le \left(\frac{\mu e^{\pi}}{\rho(1-c_1^+)}\right)\left(e^{-\pi}\frac{x_{k+1}-c_1^+}{1-c_1^+}\right)^{\frac{1-\rho}{\rho}} \le \frac{\mu e^{\pi}}{\rho(1-c_1^+)}(1-c_1^+)^{\frac{\rho-1}{\rho}}\left(e^{-\pi}(x_{k+1}-c)\right)^{\frac{1-\rho}{\rho}} \\
&\le \quad \frac{\mu e^{\pi}}{\rho}(1-c_1^+)^{-\frac{1}{\rho}}\left(e^{-\pi}\left(\frac{\mu}{(1-c_1^+)^{\frac{1}{\rho}}}\right)^{\frac{\rho}{\rho-1}} e^{\frac{-\pi}{\rho-1}}(x_n-c_1^+)^{\frac{1}{\rho^{n-k-1}}}\right)^{\frac{1-\rho}{\rho}} \\
&\le \frac{e^{2\pi}}{\rho}\left(x_n-c_1^+\right)^{\frac{1-\rho}{\rho^{n-k}}} = \gamma\left(x_n-c_1^+\right)^{\frac{1-\rho}{\rho^{n-k}}},
\end{aligned}
$$

and

$$Df_1(x_k) = \left(Df_1^{-1}(x_{k+1})\right)^{-1} \ge \gamma^{-1}\left(x_n-c_1^+\right)^{\frac{\rho-1}{\rho^{n-k}}}.$$

We can now see that

$$Df_1(x_k) \ge \gamma^{-1}\left(p_n-c_1^+\right)^{\frac{\rho-1}{\rho^{n-k}}}$$

for all $x_k \in I_k$. Therefore,

$$
\begin{aligned}
|c_k^- - p_k| \;&\ge\; |p_1-c_1^-|\prod_{i=1}^{k-1}\min_{x\in I_i} Df_1(x) \ge |p_1-c_1^-|\prod_{i=1}^{k-1}\gamma^{-1}\left(p_n-c_1^+\right)^{\frac{\rho-1}{\rho^{n-i}}} \\
&\ge\; |p_1-c_1|\gamma^{1-k}\left(p_n-c_1^+\right)^{\frac{1}{\rho^{n-k}}}.
\end{aligned}
$$

Notice, that for monotone combinatorics all images $f^k(L)$, $1\le k\le n$, are contained in the interval (c,c_1^-), while the images $f^k(R)$, $1\le k\le m$, are all contained in (c_1^+,c). Therefore,

$$c_1^- - c > \sum_{k=1}^{n}|c_k^- - p_k| \ge |p_1-c_1|\sum_{k=1}^{n}\gamma^{1-k}\left(p_n-c_1^+\right)^{\frac{1}{\rho^{n-k}}} \ge \frac{c_1^-}{e^{\pi}}\left(\frac{|L|}{c}\right)^{\rho}\sum_{k=1}^{n}\gamma^{1-k}|L|^{\frac{1}{\rho^{n-k}}}. \tag{3.37}$$

We can now use the fact that $\gamma^{-1}=\rho/e^{2\pi}>1$ for all π as in the hypothesis of the Lemma, to simplify the above expression.

$$c_1^- - c \;\ge\; \frac{c_1^-}{e^{\pi}}\left(\frac{|L|}{c}\right)^{\rho}\gamma^{1-n}|L|\sum_{k=0}^{n-1}\gamma^k = \frac{c_1^-}{e^{\pi}c^{\rho}}|L|^{\rho+1}\frac{\gamma^{1-n}-\gamma}{1-\gamma}.$$

and the upper bound (3.31) from the claim follows. The bound (3.34) on R is obtained in a similar way.

To derive the upper bound (3.32), we return to (3.37), and notice that for monotone combinatorics all images $f^k(L)$, $1\le k\le n-1$, are contained in the interval $(f_1^{-1}(c),c_1^-)$, while the images $f^k(R)$, $1\le k\le m-1$, are all contained in $(c_1^+,f_0^{-1}(c))$. Therefore,

$$c_1^- - f_1^{-1}(c) \;\ge\; \frac{c_1^-}{e^{\pi}}\left(\frac{|L|}{c}\right)^{\rho}\sum_{k=0}^{n-2}\gamma^{1-k}|L|^{\frac{1}{\rho^{n-k}}} = \frac{c_1^-}{e^{\pi}c^{\rho}}|L|^{\rho+\frac{1}{\rho^{n-1}}}, \tag{3.38}$$

while on the other hand, according to Lemma 13,

$$
\begin{aligned}
|c_2^- - c| &= |f_1(c_1^-) - c| \ge \frac{e^{-\pi}\rho(1-c_1^+)}{\mu} |c_1^- - f_1^{-1}(c)| \left| \frac{f_1^{-1}(c) - c}{\mu} \right| \\
&\ge \frac{e^{-\pi}\rho(1-c_1^+)}{\mu} |c_1^- - f_1^{-1}(c)| \left| \frac{\Theta + |R|}{\mu} \right|^{\rho-1} \implies \\
\mu &\ge \frac{e^{-\pi}\rho(1-c_1^+)}{\mu} |c_1^- - f_1^{-1}(c)| \left| \frac{\Theta + |R|}{\mu} \right|^{\rho-1} \implies \\
|c_1^- - f_1^{-1}(c)| &\le \mu^2 \left| \frac{\mu}{\Theta + |R|} \right|^{\rho-1} \frac{e^{\pi}}{\rho(1-c_1^+)},
\end{aligned}
$$

which together with (3.38) results in

$$
\mu^2 \left| \frac{\mu}{\Theta} \right|^{\rho-1} \frac{e^{\pi}}{\rho(1-c_1^+)} \ge \frac{c_1^-}{e^{\pi} c^{\rho}} |L|^{\rho + \frac{1}{\rho^{n-1}}},
$$

and the second upper bound (3.32) follows. The bound (3.35) is obtained in a similar way.

2) Lower bounds. We will use the fact that $L \subset f^{n+1}(L)$, or

$$
|L| \le |f_1^n(p_1) - f_1^n(c_1^-)|.
$$

Then, according to the previous Lemma,

$$
\begin{aligned}
|L| &\le |f_1(p_n) - f_1(c_n^-)| \le \frac{e^{\pi}(1-c_1^+)\rho}{\mu} \left|p_n - c_n^-\right| \left| \frac{c_n^- - c}{\mu} \right|^{\rho-1} \\
&\le \eta^{-2} \left|p_{n-1} - c_{n-1}^-\right| \left| \frac{c_n^- - c}{\mu} \right|^{\rho-1} \left| \frac{c_{n-1}^- - c}{\mu} \right|^{\rho-1} \\
&\le \eta^{-n} \left|p_1 - c_1^-\right| \prod_{k=1}^{n} \left| \frac{c_k^- - c}{\mu} \right|^{\rho-1} \le \eta^{-n} e^{\pi} c_1^- \left| \frac{L}{c^{\rho}} \right|^{\rho} \prod_{k=1}^{n} \left| \frac{c_k^- - c}{\mu} \right|^{\rho-1}. \qquad (3.39)
\end{aligned}
$$

We will now obtain an estimate on $(c_k^- - c)/\mu$. To that end, first notice, that

$$
Df_1(x) \ge \frac{e^{-\pi}(1-c_1^+)\rho}{\mu} \left(\frac{x-c}{\mu} \right)^{\rho-1} \ge \frac{e^{-\pi}(1-c_1^+)\rho}{\mu} \left(\frac{\Theta + |R|}{\mu} \right)^{\rho-1},
$$

for all $x \geq f_1^{-1}(c)$, therefore,

$$\begin{aligned}
c_k^- - c &\leq 1 - \min_{x \geq f_1^{-1}(c)} \{Df_1(x)\}(1 - c_1^-) \\
&\leq 1 - c - \left(\frac{e^{-\pi}(1 - c_1^+)\rho}{\mu} \left(\frac{\Theta + |R|}{\mu} \right)^{\rho-1} \right)^{k-1} (1 - c_1^-) \\
&\leq \mu - \left(\frac{e^{-2\pi}}{\eta} \left(\frac{\Theta + |R|}{\mu} \right)^{\rho-1} \right)^{k-1} \frac{\eta^n \Theta}{1 - \eta^n} \implies \\
\frac{c_k^- - c}{\mu} &\leq 1 - \left(\frac{e^{-2\pi}}{\eta} \left(\frac{\Theta + |R|}{\mu} \right)^{\rho-1} \right)^{k-1} \frac{\eta^n \Theta}{\mu(1 - \eta^n)} \implies \\
\prod_{k=1}^{n} \left| \frac{c_k^- - c}{\mu} \right|^{\rho-1} &\leq \prod_{k=1}^{n} \left(1 - \left(\frac{e^{-2\pi}}{\eta} \left(\frac{\Theta + |R|}{\mu} \right)^{\rho-1} \right)^{k-1} \frac{\eta^n \Theta}{\mu(1 - \eta^n)} \right)^{\rho-1} \\
&\leq \exp\left(-K(\rho - 1) \frac{\eta^n \Theta}{\mu(1 - \eta^n)} \sum_{k=1}^{n} \left(\frac{e^{-2\pi}}{\eta} \frac{(\Theta + |R|)^{\rho-1}}{\mu^{\rho-1}} \right)^{k-1} \right),
\end{aligned}$$

where K is some immaterial constant of order 1. Finally, (3.39) becomes

$$L \leq \eta^{-n} e^{\pi} c_1^- \left| \frac{L}{c^\rho} \right|^\rho \exp\left(-K(\rho - 1) \frac{\eta^n \Theta}{\mu(1 - \eta^n)} \sum_{k=1}^{n} \left(\frac{e^{-2\pi}}{\eta} \frac{(\Theta + |R|)^{\rho-1}}{\mu^{\rho-1}} \right)^{k-1} \right).$$

which results in the required lower bound for L.

The lower bound for R is obtained in a similar way. □

4. *A priori bounds*

Recall that by Lemma 11, the diffeomorphic coefficients of the renormalized map are

$$\tilde{\phi} = \xi_C^{-1} \circ f_1^n \circ \phi \circ \xi_{\phi^{-1} \circ f_1^{-n}(C)}, \quad \tilde{\psi} = \xi_C^{-1} \circ f_0^m \circ \psi \circ \xi_{\psi^{-1} \circ f_0^{-m}(C)}.$$

The next Proposition establishes the conditions for the invariance of the distortion of the coefficients under renormalization.

Proposition 17. *(Invariance of distortion). For every $\rho > 1$ and every $0 < \pi < 1/2 \ln \rho$, there exist $N = N(\rho, \pi) > 1$, such that if $f \in \mathcal{K}_\varepsilon^\pi \cap \mathcal{L}_\omega$ where $\omega = (\omega^-, \omega^+)$ with $|\omega^-| \geq N$ and $|\omega^+| \geq N$, then*

$$\operatorname{dist}[\tilde{\phi}] \leq \pi, \quad \text{and} \quad \operatorname{dist}[\tilde{\psi}] \leq \pi.$$

Proof. We consider the exponential of the distortion of $\tilde{\phi}$ on $[0,1]$. For any $x, y \in [0,1]$,

$$\frac{D\tilde{\phi}(x)}{D\tilde{\phi}(y)} = \frac{D(f_1^n \circ \phi)(\xi_{\phi^{-1} \circ f_1^{-n}(C)}(x)))}{D(f_1^n \circ \phi)(\xi_{\phi^{-1} \circ f_1^{-n}(C)}(y)))}. \tag{4.40}$$

Recall, that $C=[p,q]$, and that, by Lemma 12, f_1^{-n} and f_0^{-m} are defined at least on $(c_1^+,1)$ and $(0,c_1^-)$, respectively. By Koebe Principle 1

$$\frac{D\left(f_1^n\circ\phi\right)(z)}{D\left(f_1^n\circ\phi\right)(w)}\le\left(\frac{1+\tau}{\tau}\right)^2,$$

where $z,w\in\phi^{-1}(f_1^{-n}(C))$, and

$$\tau=\max\{\tau_1,\tau_2\},\quad \tau_1=\frac{1-q}{q-p},\quad \tau_2=\frac{p-c_1^+}{q-p}.$$

Similarly, for $z,w\in\psi^{-1}(f_0^{-m}(C))$,

$$\frac{D\left(f_0^m\circ\psi\right)(z)}{D\left(f_0^m\circ\psi\right)(w)}\le\left(\frac{1+\zeta}{\zeta}\right)^2,$$

where

$$\zeta=\max\{\zeta_1,\zeta_2\},\quad \zeta_1=\frac{p}{q-p},\quad \zeta_2=\frac{c_1^- -q}{q-p}.$$

Therefore,

$$\begin{aligned}\max\{\mathrm{dist}[\tilde\phi],\mathrm{dist}[\tilde\psi]\} &\le \max\left\{\left(\frac{1+\tau_2}{\tau_2}\right)^2,\left(\frac{1+\zeta_2}{\zeta_2}\right)^2\right\}\\ &= \max\left\{\left(\frac{q-c_1^+}{p-c_1^+}\right)^2,\left(\frac{c_1^- -p}{c_1^- -q}\right)^2\right\}. \qquad (4.41)\end{aligned}$$

Below we will demonstrate that (4.41) is less than e^π for sufficiently large n and m.

Recall, that Δ from Lemma 14 serves as a lower bound on $p-f_0^{-1}(c)$, while Θ is a lower bound on $q-f_1^{-1}(c)$. Then, using that $p-f_0^{-1}(c)<p-c_1^+$, together with the upper bounds on L and R from Lemma 16, we get

$$\begin{aligned}\frac{q-c_1^+}{p-c_1^+} &\le 1+\frac{q-p}{p-c_1^+}\le 1+\frac{|C|}{\Delta}\le 1+\frac{|L|+|R|}{\Delta}\\ &\le 1+\left(\left((c_1^- -c)\frac{c^\rho e^\pi}{c_1^-}\frac{\gamma^{-1}-1}{\gamma^{-n}-1}\right)^{\frac{1}{\rho+1}}+\left((c-c_1^+)\frac{\mu^\rho e^\pi}{(1-c_1^+)}\frac{\gamma^{-1}-1}{\gamma^{-m}-1}\right)^{\frac{1}{\rho+1}}\right)\times\\ &\quad\times\left(\frac{c_1^-\rho}{c}\frac{\left(1-\frac{c_1^+}{c}\right)^{\rho-1}}{\nu^{\frac{\rho}{\rho-1}}e^{-\pi\frac{\rho}{\rho-1}}}\right)^{\frac{\rho^n}{\rho^n-1}}\\ &\le 1+\left(\left(\mu c^{\rho-1}e^\pi\frac{\gamma^{-1}-1}{\gamma^{-n}-1}\right)^{\frac{1}{\rho+1}}+\left(c\mu^{\rho-1}e^\pi\frac{\gamma^{-1}-1}{\gamma^{-m}-1}\right)^{\frac{1}{\rho+1}}\right)\left(\frac{c_1^-\rho}{c}\frac{\left(1-\frac{c_1^+}{c}\right)^{\rho-1}}{\nu^{\frac{\rho}{\rho-1}}e^{-\pi\frac{\rho}{\rho-1}}}\right)^{\frac{\rho^n}{\rho^n-1}}.\end{aligned}$$

Notice, that the function $(1-x)x^{\rho-1}$ assumes its maximum at $x=(\rho-1)/\rho$, therefore,

$$\mu c^{\rho-1} \le \left(1-\frac{\rho-1}{\rho}\right)\left(\frac{\rho-1}{\rho}\right)^{\rho-1} \le \frac{e^{-1}}{\rho-1},$$

and similarly for $c\mu^{\rho-1}$. Now, let $s=\min\{n,m\}$, then

$$\begin{aligned}
\frac{q-c_1^+}{p-c_1^+} &\le 1+2\left(\frac{e^{\pi-1}}{\rho-1}\frac{\gamma^{-1}-1}{\gamma^{-s}-1}\right)^{\frac{1}{\rho+1}} \left(\frac{c_1^-\rho}{c}\frac{\left(1-\frac{c_1^+}{c}\right)^{\rho-1}}{\nu^{\frac{\rho}{\rho-1}}e^{-\pi\frac{\rho}{\rho-1}}}\right)^{\frac{\rho^n}{\rho^n-1}}\\
&\le 1+2\left(\frac{e^{\pi-1}}{\rho-1}\frac{\gamma^{-1}-1}{\gamma^{-s}-1}\right)^{\frac{1}{\rho+1}} \left(\frac{\rho}{(1-\varepsilon)\varepsilon^{\frac{\rho}{\rho-1}}e^{-\pi\frac{\rho}{\rho-1}}}\right)^{\frac{\rho^n}{\rho^n-1}}.
\end{aligned}$$

where we have also used that the minimum if $c\mu^{\frac{\rho}{\rho-1}}$ for $c\in[\varepsilon,1-\varepsilon]$, is $(1-\varepsilon)\varepsilon^{\frac{\rho}{\rho-1}}$.

One can now see that if s sufficiently large, then the small factor $1/(\gamma^{-s}-1)$ dominates other terms, and for every π as in the hypothesis, there exists a sufficiently large s, such that $(q-c_1^+)/(p-c_1^+)$ is smaller than $e^{\pi/2}$ whenever the distortion of the coefficients of f is smaller than π.

In a similar way $(c_1^- - p)/(c_1^+ - q)$ is less than $e^{\pi/2}$ for sufficiently large n and m. □

Recall the definition of the subset $\mathcal{M}_N$ from the Introduction: this is the subset of $\mathcal{M}$ (monotone types) of all ω's such that the length of words in ω satisfies $|\omega^-|\ge N$ and $|\omega^+|\ge N$. Also, recall that according to Lemma 11, the critical point of a renormalized Lorenz map is given by

$$\tilde{c}=\frac{|L|}{|C|}. \tag{4.42}$$

Proposition 18. *(Invariance of the bounds on the critical point). Let π satisfy $0<\pi<1/2\ln\rho$, and let $\rho>2$. Then there exist $\varepsilon>0$ such that if $f\in\mathcal{K}_\varepsilon^\pi\cap\mathcal{L}_\mathcal{M}$ then the critical point $\tilde{c}$ of the renormalization satisfies*

$$\tilde{c}\in[\varepsilon,1-\varepsilon].$$

Proof. Our immediate goal is to show, that for f as in the hypothesis of the Proposition, there exists a positive ε, such that $\tilde{c}$ lies in $[\varepsilon,1-\varepsilon]$ whenever c does. We will start with the lower bound on $\tilde{c}$.

According to (4.42), for $\tilde{c}$ to be larger or equal to some $\epsilon>0$ it is sufficient that

$$\frac{1}{1+\max\frac{|R|}{|L|}}\ge\epsilon \Leftrightarrow 1\ge\epsilon\left(1+\max\frac{|R|}{|L|}\right). \tag{4.43}$$

The maximum of the ratio of the lengths of R and L can be estimated using bounds from Lemma 16:

$$\frac{|R|}{|L|}\le\frac{\left((c-c_1^+)\frac{\mu^\rho e^\pi}{(1-c_1^+)}\right)^{\frac{1}{\rho+1}}\left(\frac{\gamma^{-1}-1}{\gamma^{-m}-1}\right)^{\frac{1}{\rho+1}}}{\left(\frac{e^{-\pi}c^\rho}{c_1^-}\eta^n\right)^{\frac{1}{\rho-1}}\exp\left(K\frac{\eta^n\Theta}{\mu(1-\eta^n)}\sum_{k=1}^n\left(\frac{e^{-2\pi}}{\eta}\frac{(\Theta+|R|)^{\rho-1}}{\mu^{\rho-1}}\right)^{k-1}\right)}. \tag{4.44}$$

We will identify the behaviour of the right hand side of the above inequality as $\mu \to 0$.

$$\begin{aligned}\frac{|R|}{|L|} &\leq A \frac{\mu^{\frac{\rho}{\rho+1}}}{\mu^{\frac{n}{\rho-1}} \exp\left(B\eta^{1+\frac{\rho^m}{\rho^m-1}} \mu^{-1-(\rho-1)(n-1)} \left(D\eta^{\frac{\rho^m}{\rho^m-1}} + P\mu^{\frac{\rho}{\rho+1}} \right)^{(\rho-1)(n-1)} \right)} \\ &\leq A\mu^{\frac{\rho}{\rho+1}-\frac{n}{\rho-1}} \exp\left(-K\mu^{\frac{\rho^m}{\rho^m-1}-\frac{\rho-1}{\rho+1}(n-1)} \right),\end{aligned}$$

where A, B, D, P and K are some bounds that depend on n, m, ρ and π, but do not depend on μ. Therefore, for (4.43) to hold whenever μ is small, its is sufficient that

$$\epsilon + A\epsilon\mu^{\frac{\rho}{\rho+1}-\frac{n}{\rho-1}} \exp\left(-K\mu^{\frac{\rho^m}{\rho^m-1}-\frac{\rho-1}{\rho+1}(n-1)} \right) \leq 1.$$

For sufficiently large n and m, the above inequality is of the form

$$\epsilon + A\epsilon\mu^{-an} \exp\left(-T\mu^{-bn} \right) \leq 1,$$

where a, b, A and T are some constants.

Since μ strictly less than 1, for sufficiently large n the small exponential dominates the large factor in front of it, and there exists $\epsilon > 0$ and a natural number $N > 1$, such that the inequality holds for $\mu \leq \epsilon$ and all $n, m > N$.

We now turn to the case $c \to 0$. Here we will use the upper bound (3.35) on $|R|$:

$$\frac{|R|}{|L|} \leq \frac{\left(c^2 \left| \frac{c}{\Delta} \right|^{\rho-1} \frac{\gamma\mu^\rho}{(1-c_1^+)c_1^-} \right)^{\frac{1}{\rho+\frac{1}{\rho^{n-1}}}}}{\left(\frac{e^{-\pi}c^\rho}{c_1^-}\eta^n \right)^{\frac{1}{\rho-1}} \exp\left(K\frac{\eta^n\Theta}{\mu(1-\eta^n)} \sum_{k=1}^n \left(\frac{e^{-2\pi}}{\eta} \frac{(\Theta+|R|)^{\rho-1}}{\mu^{\rho-1}} \right)^{k-1} \right)}, \tag{4.45}$$

and we isolate all powers of c and c_1^-:

$$\begin{aligned}\frac{|R|}{|L|} &\leq \frac{\left(c^2 \left(\frac{c}{\Delta} \right)^{\rho-1} \frac{\gamma\mu^\rho}{(1-c_1^+)c_1^-} \right)^{\frac{1}{\rho+\frac{1}{\rho^{n-1}}}}}{\left(\frac{e^{-\pi}c^\rho}{c_1^-}\eta^n \right)^{\frac{1}{\rho-1}}} \leq A \frac{\left(\frac{c^2}{c_1^-} \left(\frac{c}{\kappa^{\frac{\rho^n}{\rho^n-1}}} \right)^{\rho-1} \right)^{\frac{1}{\rho+\frac{1}{\rho^{n-1}}}}}{\left(\frac{c^\rho}{c_1^-} \right)^{\frac{1}{\rho-1}}} \\ &\leq A \frac{\left(\frac{c^2}{c_1^-} \left(c^{1-\frac{\rho^n}{\rho^n-1}} (c_1^-)^{\frac{\rho^n}{\rho^n-1}} \right)^{\rho-1} \right)^{\frac{1}{\rho+\frac{1}{\rho^{n-1}}}}}{\left(\frac{c^\rho}{c_1^-} \right)^{\frac{1}{\rho-1}}} \\ &\leq Ac^{2\frac{\rho^{n-1}}{\rho^n+1}-\frac{\rho-1}{\rho^n-1}\frac{\rho^{n-1}}{\rho^n+1}-\frac{\rho}{\rho-1}} (c_1^-)^{\left(\frac{\rho^n(\rho-1)}{\rho^n-1}-1 \right)\frac{\rho^{n-1}}{\rho^n+1}+\frac{1}{\rho-1}} \\ &\leq Ac^{\frac{(\rho-2)\rho^{2n}+\rho^n(1-\rho^2)+\rho}{(\rho^{2n}-1)(\rho^2-\rho)}-1} (c_1^-)^{\frac{(\rho-1)\rho^n+\rho^{2n}(\rho^2-2\rho+2)-\rho}{(\rho^{2n}-1)(\rho^2-\rho)}},\end{aligned}$$

for all $\rho > 1$ the power of c_1^- is positive, and we can bound it from above by 1, then the condition

$$\epsilon + \epsilon \max\left\{ \frac{|R|}{|L|} \right\} \leq 1$$

is implied by

$$\epsilon + A\epsilon c^{\frac{(\rho-2)\rho^{2n}+\rho^n(1-\rho^2)+\rho}{(\rho^{2n}-1)(\rho^2-\rho)}-1} \leq 1,$$

and it is sufficient for all $c \leq \epsilon$ that

$$\epsilon + A\epsilon^{\frac{(\rho-2)\rho^{2n}+\rho^n(1-\rho^2)+\rho}{(\rho^{2n}-1)(\rho^2-\rho)}} \leq 1. \tag{4.46}$$

For all $\rho > 2$ and sufficiently large n the power of ϵ in the second term is positive, and we get that the invariance condition is satisfied by a sufficiently small ϵ.

To summarise, denote the maximum of the upper bounds (4.44) and (4.45) by $M(c)$ (we suppress the dependence of M on ρ, π, $c_1^{\pm}$, n and m in our notation), then we have shown that there exists $\epsilon > 0$ such that

$$\frac{1}{1+M(x)} \geq \epsilon,$$

for all $x \leq \epsilon$, and that

$$\frac{1}{1+M(x)} \geq \epsilon,$$

for all $x \geq 1-\epsilon$. Since $1/(1+M(x))$ is clearly a continuous function of x, it achieves a minimum ϵ_1 on any interval $[\epsilon, 1-\epsilon]$. We can now choose $\varepsilon_1 = \min\{\epsilon_1, \epsilon\}$ to be the lower bound on c.

Existence of $\varepsilon_2 > 0$ such that $\tilde{c} < 1 - \epsilon_2$ is proved in a similar way by considering the maximum of the ratio of the lengths of L and R. Finally, take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. □

Remark 19. *One can now see that the reason for a somewhat restrictive condition $\rho > 2$ is the positivity of the exponent in* (4.46). *One might hope that with more work, for example, a better upper bound on c_1^-, which would add to smallness in* (4.46)*, one can relax this constraint.*

The following results is an immediate corollary of Propositions 17 and 20.

Proposition 20. *(A priori bounds). Let π satisfy $0 < \pi < 1/2 \ln \rho$. Then, for every $\rho > 2$ there exists a natural $N > 1$ and $\varepsilon > 0$, such that $\mathcal{R}[\mathcal{K}_\varepsilon^\pi \cap \mathcal{L}_{\mathcal{M}_N}] \subset \mathcal{K}_\varepsilon^\pi$.*

5. *Periodic points of renormalization*

We consider a restriction $\mathcal{R}_\omega$ of the renormalization operator to some

$$\omega = (0\overbrace{1\ldots1}^{n}, 1\overbrace{0\ldots0}^{m}) \in \mathcal{M}, \quad n \geq N, \; m \geq N,$$

where N is as in Propositions 17 and 20.

In this Section we will demonstrate that $\mathcal{R}_\omega$ has a fixed point. We will generally follow the approach of Section 3.3 from [**11**] (and we will make a conscientious attempt to keep the notation in line with that work). One important difference with the case considered in [**11**], however, is that we are looking at a different class of return times. This will introduce some extra difficulties, especially evident in the proof of Lemma (26), somewhat more involved than its analogue from [**11**].

We will start by quoting several previously established results.

Definition 21. *A branch I of f^n is full if f^n maps I onto the domain of f. I is trivial if f^n fixes both endpoints of I.*

We will now quote several facts about Lorenz maps, established in [**7**].

Definition 22. *A slice in the parameter plane is any set of the form*

$$\mathcal{S} = [0,1]^2 \times \{c\} \times \{\phi\} \times \{\psi\},$$

where c, ϕ and ψ are fixed. We will use the simplified notation $(u,v) \in \mathcal{S}$.

A slice $\mathcal{S}$ induces a family of Lorenz maps

$$\mathcal{S} \ni (u,v) \mapsto (u,v,c,\phi,\psi) \subset \mathcal{L}^0.$$

any family induced by a slice is *full*, that is it contains maps of all possible combinatorics. Specifically (see [**7**] for details),

Proposition 23. *(Theorem A from [**7**]). Let $(u,v) \mapsto (u,v,c,\phi,\psi)$ be a family induced by a slice. Then this family intersects $\mathcal{L}^0_{\bar\omega}$ for every $\bar\omega$ (finite or infinite) such that $\mathcal{L}^0_{\bar\omega} \neq \emptyset$.*

Lemma 24. *(Lemma 4.1 from [**7**]). Assume that f is renormalizable. Let $(l,c) \supset L$ be the branch of f^{n+1} and $(c,r) \supset R$ be that of f^{m+1}. Then*

$$f^{n+1}(l) \leq l, \quad f^{m+1}(r) \geq r.$$

Let π, ε and $\mathcal{K}^\pi_\varepsilon$ be as in the previous Section. Consider the set

$$\mathcal{Y} = \mathcal{L}^S_\omega \cap \mathcal{K}^\pi_\varepsilon. \tag{5.47}$$

Proposition 25. *The boundary of $\mathcal{Y}$ consists of three parts: $f \in \partial\mathcal{Y}$ iff at least one of the following holds:*

C1. the left and the right branches of $\mathcal{R}[f]$ are full or trivial;

C2. $\mathrm{dist}[\phi] = \pi$ *or* $\mathrm{dist}[\psi] = \pi$;

C3. $c(f) = \varepsilon$, or $c(f) = 1 - \varepsilon$.

Proof. Consider the boundary of $\mathcal{L}^0_\omega$. If either branch of $\mathcal{R}_\omega[f]$ is full or trivial, then there exists an perturbation of f, however small, such that f is no longer renormalizable. Hence C1 holds on $\partial\mathcal{L}^0_\omega$. If $f \in \mathcal{L}^0_\omega$ does not satisfy C1 then, according to Lemma 24, all small perturbations of it will be still renormalizable.

Conditions C2 and C3 are part of the boundary of $\mathcal{K}^\pi_\varepsilon$. By Proposition 23 these boundaries intersect $\mathcal{L}^S_\omega$, and hence C2 and C3 are also the boundary conditions for $\mathcal{Y}$. □

Fix $c_0 \in (\varepsilon, 1-\varepsilon)$, and let $\mathcal{S} = [0,1]^2 \times \{c_0\} \times \{\mathrm{id}\} \times \{\mathrm{id}\}$. Recall, that the linear structure on the space $\mathcal{D}^2$ is defined via the nonlinearity operator:

$$\alpha\phi + \beta\psi = N^{-1}_{\left(\alpha N_\phi + \beta N_\psi\right)}.$$

Introduce the *deformation retract* onto $\mathcal{S}$ as

$$\begin{aligned} \pi_t(u,v,c,\phi,\psi) &\equiv (u,v,c+t(c_0-c),\phi_t,\psi_t) \\ &= (u,v,c+t(c_0-c),(1-t)\phi + t\ \mathrm{id},(1-t)\psi + t\ \mathrm{id}). \end{aligned} \tag{5.48}$$

Let

$$\mathcal{R}_t = \pi_t \circ \mathcal{R}.$$

We will strengthen the conditions on the set $\mathcal{Y}$ and consider a smaller set

$$\mathcal{Y}_\delta = \mathcal{Y} \cap \{f \in \mathcal{L}^S_\omega : c(\mathcal{R}[f]) \geq \delta\}. \tag{5.49}$$

The boundary of $\mathcal{Y}_\delta$ is given by conditions $C1$-$C3$ together with

C4. $\{f \in \mathcal{Y} : c(\mathcal{R}[f]) = \delta\}$.

LEMMA 26. *The exists a choice of* c_0 *in* (5.48) *and* $\delta \in (1-\varepsilon, \varepsilon)$*, such that* $\mathcal{R}$ *has a fixed point in* $\partial\mathcal{Y}_\delta$ *iff* $\mathcal{R}_t$ *has a fixed point in* $\partial\mathcal{Y}_\delta$ *for some* $t \in [0,1]$.

Proof. The direct statement is obvious since $\mathcal{R} \equiv \mathcal{R}_0$.

Assume that $f \in \partial\mathcal{Y}_\delta$ with the coefficients (ϕ,ψ) is such that $\mathcal{R}_t f = f$ for some $t \in (0,1]$, and assume that $\mathcal{R}$ has no fixed point on $\partial\mathcal{Y}_\delta$. We will demonstrate that this is impossible.

Choose c_0 close to $1-\varepsilon$: $c_0 = 1-\varepsilon-\nu$ for some small ν. By Proposition 20, $c(\mathcal{R}[f]) \in [\varepsilon, 1-\varepsilon]$ whenever $c(f)$ is. Together with the condition $c(\mathcal{R}[f]) \geq \delta$ this implies that

$$c(\mathcal{R}[f]) \in [\delta, 1-\varepsilon].$$

Since $t > 0$, by formula (5.48) $c(\mathcal{R}_t[f])$ is strictly in the interior of $[\delta, 1-\varepsilon]$ for all $t \in (0,1]$. Therefore, neither C3 nor C4 can hold for $f = \mathcal{R}_t[f]$ for $t \in (0,1]$.

The distortion of the coefficients of $\mathcal{R}[f]$ is not greater than π by Proposition 17. For $t \in (0,1]$ distortion of the diffeomorphic parts $(\tilde\phi_t, \tilde\psi_t)$ of $\mathcal{R}_t[f]$ is strictly smaller than that of $(\tilde\phi, \tilde\psi)$ (diffeomorphic coefficients for $\mathcal{R}[f]$). This can be seen from the following computation:

$$\frac{D\tilde\phi_t(x)}{D\tilde\phi_t(y)} = \frac{\frac{\exp\left[\int_0^y (1-t)N_{\tilde\phi}(s) + tN_{\mathrm{id}}(s)ds\right]}{\int_0^1 \exp\left[\int_0^r (1-t)N_{\tilde\phi}(s) + tN_{\mathrm{id}}(s)ds\right]dr}}{\frac{\exp\left[\int_0^x (1-t)N_{\tilde\phi}(s) + tN_{\mathrm{id}}(s)ds\right]}{\int_0^1 \exp\left[\int_0^r (1-t)N_{\tilde\phi}(s) + tN_{\mathrm{id}}(s)ds\right]dr}} = \frac{\exp\left[\int_0^y (1-t)N_{\tilde\phi}(s)ds\right]}{\exp\left[\int_0^x (1-t)N_{\tilde\phi}(s)ds\right]} = \left(\frac{D\tilde\phi(y)}{D\tilde\phi(x)}\right)^{1-t} < e^\pi.$$

Similarly for $\tilde\psi_t$. Therefore, we have that C2 does not hold for $f = \mathcal{R}_t[f]$ for $t \in (0,1]$.

The only possibility is that, if $f = \mathcal{R}_t[f] \in \partial\mathcal{Y}_\delta$ then it belongs to the part of the boundary described by C1.

Suppose that either branch of $\mathcal{R}[f]$ is full; for definitiveness, suppose $c_1^-(\mathcal{R}[f]) = 1$. Since ϕ fixes both end points of the unit interval, this implies that $u(\mathcal{R}[f]) = 1$, and since the deformation retract does not change the value of u, $u(\mathcal{R}_t[f]) = 1$. Since ϕ_t fixes 1 as well, we get that $c_1^-(\mathcal{R}_t[f]) = 1$, and therefore, the corresponding branch of $\mathcal{R}_t[f]$ is full as well. This shows that f can not be fixed by $\mathcal{R}_t$ since a renormalizable map can not have a full branch. Therefore, one of the branches of $\mathcal{R}[f]$ must be trivial.

Before we proceed with the last case of trivial branches, we will derive an upper bound on $\phi_t(u)$ and a lower bound on $\psi_t(1-v)$. Recall, that $\phi_t = (1-t)\phi + t \ \mathrm{id}$ where the linear structure is given by (2.8). Then, on one hand,

$$\begin{aligned} \phi_t(x) &= \frac{\int_0^x (D\phi(r))^{1-t} dr}{\int_0^1 (D\phi(r))^{1-t} dr} = 1 - \frac{\int_x^1 (D\phi(r))^{1-t} dr}{\int_0^1 (D\phi(r))^{1-t} dr} \leq 1 - \frac{\int_x^1 (D\phi(r))^{1-t} dr}{\left(\int_0^1 D\phi(r) dr\right)^{1-t}} \\ &= 1 - \int_x^1 (D\phi(r))^{1-t} dr = \int_0^x (D\phi(r))^{1-t} dr \leq \int_0^x \left(\frac{c_1^-}{u} e^{\pi}\right)^{1-t} dr \leq \left(\frac{c_1^-}{u} e^{\pi}\right)^{1-t} x, \end{aligned}$$

and

$$\phi_t(u) \leq (c_1^-)^{1-t} u^t e^{\pi(1-t)}. \tag{5.50}$$

On the other hand,

$$\phi_t(x) \leq \int_0^x (D\phi(r))^{1-t} dr \leq \int_0^x D\phi(r) dr \sup_{r \in (0,x)} (D\phi(r))^{-t} \leq \phi(x) \left(e^{\pi} \frac{x}{\phi(x)}\right)^t,$$

and

$$\phi_t(u) \leq (c_1^-)^{1-t} u^t e^{\pi t}. \tag{5.51}$$

We can now take a linear combination of (5.50) and (5.51) as an upper bound on $\phi_t(u)$. A particularly convenient choice is

$$\phi_t(u) \leq (c_1^-)^{1-t} u^t \left(t e^{\pi(1-t)} + (1-t) e^{\pi t}\right), \tag{5.52}$$

which insures that $\phi_0(u) \leq c_1^-$ and $\phi_1(u) \leq u$. Notice, that the maximum of the function $\left(t e^{\pi(1-t)} + (1-t) e^{\pi t}\right)$ is achieved at $t = 1/2$.

In a similar way,

$$\psi_t(1-v) \geq (c_1^+)^{1-t} (1-v)^t \left(t e^{-\pi(1-t)} + (1-t) e^{-\pi t}\right). \tag{5.53}$$

Suppose, the left branch is trivial: $c(\mathcal{R}[f]) \geq c_1^-(\mathcal{R}[f])$. Recall, that for a map renormalizable with monotone combinatorics ω $c_1^- > f_1^{-1}(c)$, and according to Lemmas 16 and 14 the differences $c_1^- - c > |R| + \Theta \geq K$ and $c - c_1^+ > |L| + \Delta \geq J$, where K and J depend on ρ, π, ε, n and m, but do not depend on the particular form of the map. Suppose ν is small: $\nu << K$. Then, on one hand,

$$c_1^-(\mathcal{R}_t[f]) - c(\mathcal{R}_t[f]) \leq (c_1^-)^{1-t} u^t \left(t e^{\pi(1-t)} + (1-t) e^{\pi t}\right) - c(\mathcal{R}[f]) - t(c_0 - c(\mathcal{R}[f])).$$

Recall, that by Proposition 17, π can be chosen small if one considers large n and m. Since $c_1^-(\mathcal{R}[f]) - c(\mathcal{R}[f]) < 0$, the expression $(c_1^-)^{1-t} u^t \left(t e^{\pi(1-t)} + (1-t) e^{\pi t}\right) - c(\mathcal{R}[f]) < K - \nu$ if π is small. Then since $c_0 - c(\mathcal{R}[f])$ is larger than $-\nu$, we have that

$$(c_1^-)^{1-t} u^t \left(t e^{\pi(1-t)} + (1-t) e^{\pi t}\right) - c(\mathcal{R}[f]) - t(c_0 - c(\mathcal{R}[f])) < K,$$

and hence $f = \mathcal{R}_t[f]$ is not renormalizable with the monotone combinatorics ω.

Now, suppose that the right branch is trivial. Then

$$c(\mathcal{R}_t[f]) - c_1^+(\mathcal{R}_t[f]) \le c(\mathcal{R}[f]) + t(c_0 - c(\mathcal{R}[f])) - (c_1^+)^{1-t}(1-v)^t \left(te^{-\pi(1-t)} + (1-t)e^{-\pi t}\right).$$

Since $c(\mathcal{R}[f]) - c_1^+(\mathcal{R}[f]) < 0$, we have that for a sufficiently small π,

$$c(\mathcal{R}[f]) - (c_1^+)^{1-t}(1-v)^t \left(te^{-\pi(1-t)} + (1-t)e^{-\pi t}\right) < \frac{J}{2},$$

while

$$c(\mathcal{R}[f]) - (c_1^+)^{1-t}(1-v)^t \left(te^{-\pi(1-t)} + (1-t)e^{-\pi t}\right) + t(c_0 - c(\mathcal{R}[f])) \le \frac{J}{2} + (1-\varepsilon-\nu-\delta).$$

Therefore, the map $\mathcal{R}_t[f]$ is not renormalizable with the monotone combinatorics ω for $t \in [0,1]$, if we chose δ so that

$$1 - \varepsilon - \nu - \delta < J/2. \tag{5.54}$$

We now notice, that according to Lemmas 16 and 14

$$J = O\left(\left(\delta(1-\delta)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho^n}{\rho^n-1}}\right) + O\left(\delta^{\frac{\rho}{\rho-1}}(1-\delta)^{\frac{n}{\rho-1}}\right) \exp\left(O\left((1-\delta)^{\frac{\rho^m}{\rho^m-1} - \frac{\rho-1}{\rho+1}(n-1)}\right)\right). \tag{5.55}$$

If δ is small, then the above expression demonstrates that $J = O\left(\delta^{\frac{\rho^n}{\rho-1}}\right)$, and the inequality (5.54) is not satisfied. On the other hand, if δ is close to $1-\varepsilon-\nu$, then the exponential in (5.55) becomes large and dominates others terms, and (5.54) is easily satisfied. Therefore, there exists $\delta \in (0, 1-\varepsilon-\nu)$, not necessarily very close to $1-\varepsilon-\nu$, such that (5.54) holds for all $c > \delta$.

We conclude that $f = \mathcal{R}_t[f] \notin \partial\mathcal{Y}_\delta$ which is a contradiction with the assumption in the beginning of the proof.

□

According to the Theorem B in [**7**] the intersection of $\mathcal{S}$ contains a connected component I of the interior, called a *full island*, such that the family $I \ni (u,v) \mapsto f$ is full.

LEMMA 27. *Any extension of $\mathcal{R}_1|_{\partial\mathcal{Y}_\delta}$ to $\mathcal{Y}_\delta$ has a fixed point.*

Proof. Assume that $\mathcal{R}_1$ has no fixed point in $\partial\mathcal{Y}_\delta$ (otherwise the theorem is trivial).

Let $\mathcal{S} = [0,1]^2 \times \{c_0\} \times \{\mathrm{id}\} \times \{\mathrm{id}\}$, where c_0 is as in the previous Lemma. This set contains a full island I with $\partial I \subset \partial\mathcal{Y}_\delta$.

Pick any $R : I \mapsto \mathcal{S}$ such that $R|_{\partial I} = \mathcal{R}_1|_{\partial I}$. Define the displacement map $d : \partial I \mapsto \mathbb{T}^1$ by

$$d(x) = \frac{x - R(x)}{|x - R(x)|},$$

which is well-defined since R does not have fixed points on $\partial I \subset \partial\mathcal{Y}_\delta$. The degree of d is non-zero since I is full. Therefore, R has a fixed point in I (otherwise d would extend to all of I, and would have a degree zero). □

To finish the proof of the existence of the fixed points we will require the following theorem from [**2**]:

Theorem 5.1. *Let* $X \subset Y$ *where* X *is closed and* Y *is a normal topological space. If* $f : X \mapsto Y$ *is homotopic to a map* $g : X \mapsto Y$ *with the property that every extension of* $g|_{\partial X}$ *to* X *has a fixed point in* X*, and if the homotopy* h_t *has no fixed point on* ∂X *for every* $t \in [0, 1]$*, then* f *has a fixed point in* X*.*

Proposition 28. $\mathcal{R}_\omega$ *has a fixed point.*

Proof. $\mathcal{R}_1$ either has a fixed point in $\partial \mathcal{Y}_\delta$, or otherwise by Lemma 27 any of extensions of $\mathcal{R}_1|_{\partial \mathcal{Y}_\delta}$ to $\mathcal{Y}_\delta$ has a fixed point. In the second case we can apply Theorem 5.1 to immediately obtain the required result. □

Now we can finish the proof of the Main Theorem 1.

Proof of the Main Theorem 1. Suppose that N is as in Propositions 17 and 20. Pick a sequence $\bar{\omega} = (\omega_0, \omega_1, \ldots, \omega_{k-1})$, $\omega_j \in \mathcal{M}_N$. One can use $\mathcal{R}_{\omega_{k-1}} \circ \ldots \circ \mathcal{R}_{\omega_0}$ in place of $\mathcal{R}_\omega$ in the previous Proposition to demonstrate that $\mathcal{R}_{\omega_{k-1}} \circ \ldots \circ \mathcal{R}_{\omega_0}$ has a fixed point, which, hence, is a periodic point of $\mathcal{R}$ of combinatorial type $\bar{\omega}$. □

The Main Theorem 2, now, is a direct consequence of the fact that the set $\mathcal{K}^\pi_\varepsilon$ is renormalization invariant, in particular, that the family $\{cR^k[f]\}$ is relatively compact for all $f \in \mathcal{L}^S_{\bar{\omega}} \cap \mathcal{K}^\pi_\varepsilon$. Its proof follows word by word that in [**11**], and will not be included here.

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