Symbolic dynamics based method for rigorous study of the existence of short cycles for chaotic systems

Zbigniew Galias
Department of Electrical Engineering
AGH — University of Science and Technology
al. Mickiewicza 30, 30–059 Kraków, Poland
Email: galias@agh.edu.pl

Warwick Tucker
Department of Mathematics
University of Bergen
Johannes Brunsgate 12, 5008 Bergen, Norway
Email: warwick.tucker@math.uib.no

***Abstract*— It is shown that the problem of existence of periodic orbits can be studied rigorously by means of a symbolic dynamics approach combined with interval methods. Symbolic dynamics is used to find approximate initial positions of periodic points and interval operators are used to prove the existence of periodic orbits in a neighborhood of the computer generated solution. As an example the Lorenz system is studied. All 2536 periodic orbits of the Poincaré map with the period $n \leq 14$ are found.**

I. INTRODUCTION

Interval arithmetic provides tools to find all short periodic orbits for discrete and continuous dynamical systems. An interval operator is used to test the existence of periodic points in small boxes (interval vectors) and the generalized bisection allows us to make the full search in the region of interest. This method has been successfully applied to the Hénon map, Ikeda map and the Rössler system [**?**] for which all short periodic orbits up to a relatively large period have been found.

In [**?**] this method was used to study the existence of short cycles for the Lorenz system. All periodic orbits with the period $n \leq 4$ of the Poincaré map associated with the Lorenz system have been found. For longer periods the method failed due to very long computation time caused by the necessity of searching the whole region covering the chaotic attractor.

For a certain class of systems it is possible to use dynamical information to restrict the search space. In this work, we present a symbolic dynamics based approach to find initial positions of periodic points. When combined with interval tools for testing the existence of periodic orbits this method allows us to find all short cycles with longer periods. The method is applicable to systems for which there exists symbolic dynamics which uniquely characterizes periodic solutions.

II. A SYMBOLIC DYNAMICS BASED METHOD TO FIND ALL SHORT PERIODIC ORBITS

In this section, we briefly describe a general method which can be used to find all short periodic orbits for continuous dynamical systems. The first step is a reduction of the continuous-time system to a discrete system using the concept of the Poincaré map. The *Poincaré map* $P : \Sigma \mapsto \Sigma$ is defined as $P(x) = \varphi(\tau(x), x)$, where $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_m$ is the union of hyperplanes and $\tau(x)$ is the return time after which the trajectory $\varphi(t, x)$ returns to Σ. Periodic points of P correspond to periodic orbits of the continuous system.

In order to study the existence of period–n orbits of P we construct the map F defined by

$$[F(z)]_k = x_{(k+1) \bmod n} - P(x_k), \quad k = 1, \ldots, n, \qquad (1)$$

where $z = (x_0, \ldots, x_{n-1})^{\mathrm{T}}$. Zeros of F correspond to period–n points of P, i.e. $F(z)=0$ if and only if $P^n(x_0) = x_0$.

A. Interval methods

Interval methods provide simple computational tests for uniqueness, existence, and nonexistence of zeros of a map within a given interval vector. In order to investigate the existence of zeros of F in the interval vector $\mathbf{z}$ one evaluates an interval operator over $\mathbf{z}$. In this work we use the Krawczyk operator [**?**]:

$$\mathrm{K}(\mathbf{z}) = \hat{z} - CF(\hat{z}) - (CF'(\mathbf{z}) - I)(\mathbf{z} - \hat{z}), \qquad (2)$$

where $\hat{z} \in \mathbf{z}$ and C is an invertible matrix.

If $\mathrm{K}(\mathbf{z}) \subset \operatorname{int} \mathbf{z}$, where $\operatorname{int} \mathbf{z}$ denotes the interior of $\mathbf{z}$, then F has exactly one zero in $\mathbf{z}$. This property allows us to prove the existence and uniqueness of zeros. If $\mathrm{K}(\mathbf{z}) \cap \mathbf{z} = \emptyset$, then there are no zeros of F in $\mathbf{z}$.

In order to evaluate the interval operator for the map F defined by Eq. (**??**), we need a method to find an enclosure of $P(\mathbf{x})$ and an enclosure of the Jacobian $P'(\mathbf{x})$. These enclosures are found in interval arithmetic by rigorous integration of the differential equation and its variational equation. For details see [?].

B. Finding all short cycles

As mentioned before it is possible to combine interval operators and generalized bisection to find all short periodic orbits enclosed in a certain region. First, the region of interest is covered by boxes (interval vectors) and the dynamics of P is represented in a form of the directed graph. Next, for each cycle in the graph an interval operator is used to study the existence of periodic orbits in the interval vector corresponding to this cycle. For details see [?].

Since usually many cycles correspond to a single periodic orbit this approach may fail due to very long computation time needed to check all cycles.

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C. Symbolic dynamics approach

For a certain class of systems it is possible to construct symbolic dynamics which uniquely characterizes periodic orbits. Let us assume that the state space is divided into disjoint sets $N_1, N_2 \ldots, N_p$. We associate with a trajectory (x_k) a sequence of symbols (s_k), in such a way that $x_k \in N_{s_k}$. In the following we assume that each periodic orbit corresponds to a unique periodic symbol sequence.

Given a symbol sequence $s = (s_0, s_1, \ldots, s_{n-1})$ and a long computer generated trajectory $(x_i)_{i=1,\ldots N}$ with the associated symbol sequence $(t_i)_{i=1,\ldots N}$ it is easy to guess an approximate position of the periodic orbit with the symbol sequence s. For each s_k we find j such that the number $\{l \colon t_{j+i} = s_{i \bmod n}$ for all $i = 0, 1, \ldots, l\}$ is maximum and we define $y_k = x_j$. The vector $y = (y_1, y_2, \ldots, y_n)$ is used as an initial guess. Next, we use a Newton iteration to find a periodic orbit in a neighborhood of y and finally we prove the existence of a nearby true periodic orbit using the Krawczyk operator.

To verify that a given symbol sequence is not admissible (there is no periodic orbit with this sequence) we may use generalized bisection to exclude the possibility of existence of periodic orbits with this sequence.

Since according to the assumption each periodic orbit has a unique symbol sequence, it is sufficient to check all symbol sequences to locate all periodic orbits.

III. Short periodic orbits for the Lorenz system

The Lorenz system is described by the following set of equations

$$\begin{aligned} \dot{x}_1 &= s x_2 - s x_1, \\ \dot{x}_2 &= r x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 &= x_1 x_2 - q x_3. \end{aligned} \tag{3}$$

We consider the Lorenz system with the classical parameter values: $s = 10$, $r = 28$, $q = 8/3$. An example trajectory of the Lorenz system is shown in Fig. **??**.

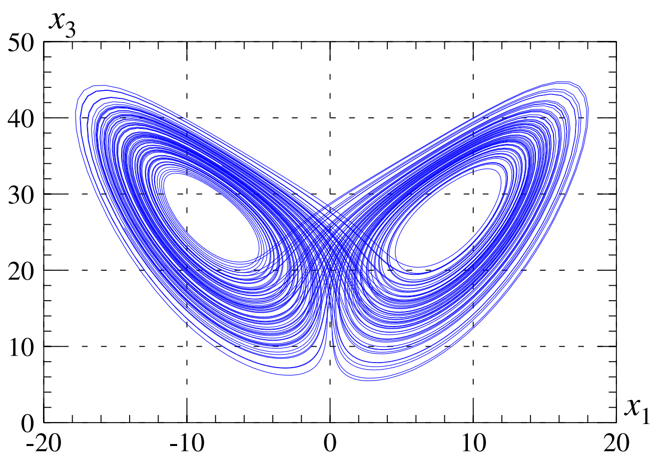

Fig. 1. A trajectory of the Lorenz system

Let us choose the Poincaré map defined by the hyperplane $\Sigma = \{x = (x_1, x_2, x_3) \colon x_3 = 27, \dot{x}_3 < 0\}$. A trajectory of the Poincaré map composed of 1000000 points is shown in Fig. **??**.

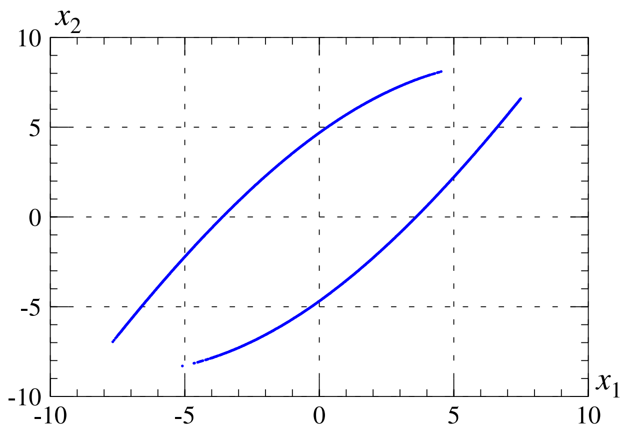

Fig. 2. A 1000000 points computer generated trajectory of P

Let γ denotes the first intersection of the (two-dimensional) stable manifold of the origin with the return plane Σ. We will label each trajectory in the following way: if the trajectory intersects Σ to the left of γ, then the intersection point is labelled with L, otherwise it is labelled with R. In order to study periodic orbits we consider periodic symbol sequences $s = (s_0, s_1, \ldots, s_{n-1})$, where $s_k \in \{\mathrm{L}, \mathrm{R}\}$, for $k = 0, 1, \ldots n-1$. In [**?**], it was established that the Poincaré map induces a stable foliation of the forward invariant part of Σ. From this, it follows that a periodic symbol sequence corresponds to at most one periodic orbit. Non-rigorous computations indicate that each periodic symbol sequence with no more than 24 repeating symbols corresponds to a periodic orbit (compare [**?**]).

We have applied the method presented in Section **??** to find all short periodic orbits for the map P. In the first step, a trajectory of P composed of 10^6 points was generated (compare Fig. **??**). The trajectory should be long and should cover the attractor as densely as possible. If this is the case then all short admissible symbol sequences can be found in this trajectory. In Table **??** the numbers of symbol sequences of various length realized by a generated trajectory are reported. The total number l_p of admissible symbol sequences of length p is given for comparison. Two cases are considered. In the first case the length of the trajectory is 10^5. One can see that all symbol sequences up to length 12 are present. Note that only 16% (9%) of symbol sequences of length 19 (20) are present. For a longer data set composed of 10^6 points all symbol sequences up to length 15 are present and 76% (54%) of symbol sequences of length 19 (20) are present. It is clear that the longer data set provides much better approximations of positions of points with a given sequence of symbols.

In the second step we have considered all periodic symbol sequences with the principal period $n \leq 14$. We have shown that both symbol sequences with the principal period $n = 1$, i.e. $s = (\mathrm{L})$ and $s = (\mathrm{R})$ are not admissible and that every other sequence corresponds to exactly one periodic orbit of P.

Let us note that there are exactly 2536 periodic symbol

TABLE I

THE TOTAL NUMBER l_n OF SYMBOL SEQUENCES OF LENGTH n, THE NUMBERS l'_n AND l''_n OF SYMBOL SEQUENCES OF LENGTH n PRESENT IN THE DATA FILE CONTAINING 10^5 AND 10^6 POINTS

n	l_n	l'_n	l''_n
1	2	2	2
2	4	4	4
3	8	8	8
4	16	16	16
5	32	32	32
6	64	64	64
7	128	128	128
8	256	256	256
9	512	512	512
10	1024	1024	1024
11	2048	2048	2048
12	4096	4096	4096
13	8192	8185	8192
14	16384	16051	16384
15	32768	29061	32768
16	65536	45880	65521
17	131072	62713	129931
18	262144	76315	242853
19	524288	85920	400018
20	1048576	92002	569806

sequences with period $n \in \{2, 3, \ldots, 14\}$ and recall that each periodic symbol sequence corresponds to at most one periodic orbit. Hence, we have confirmed that there are exactly 2536 periodic orbits of P with the period $n \leq 14$ (compare [?]).

TABLE II

SHORT PERIODIC ORBITS OF P, n — THE PERIOD OF THE ORBIT, $\mathbf{T}$ — THE LENGTH OF THE CORRESPONDING PERIODIC ORBIT OF THE FLOW

n	$\mathbf{T}$	s	n	$\mathbf{T}$	s
2	1.55865^3_2	LR	7	5.39421^8_6	LLRLLRR
3	2.30590^8_6	LLR	7	5.42912^7_4	LLRLRLR
4	3.02358^5_3	LLLR	8	5.7834^{13}_{09}	LLLLLLLR
4	3.08427^8_6	LLRR	8	5.92499^6_2	LLLLLLRR
5	3.72564^3_1	LLLLR	8	5.99044^5_2	LLLLLRRR
5	3.82025^5_3	LLLRR	8	5.9973^{21}_{18}	LLLLLRLR
5	3.8695^{40}_{38}	LLRLR	8	6.01002^8_5	LLLLRRRR
6	4.41776^8_5	LLLLLR	8	6.03523^4_1	LLLLRLLR
6	4.53410^9_7	LLLLRR	8	6.08235^9_6	LLLLRLRR
6	4.56631^2_0	LLLRRR	8	6.08382^8_5	LLLLRRLR
6	4.59381^6_3	LLLRLR	8	6.10805^6_3	LLLRLRRR
6	4.63714^2_0	LLRLRR	8	6.12145^6_3	LLLRLLRR
7	5.1030^{40}_{37}	LLLLLLR	8	6.12233^8_5	LLLRRLLR
7	5.234^{200}_{197}	LLLLLRR	8	6.13512^9_6	LLLRRLRR
7	5.28634^3_0	LLLLRRR	8	6.1547^{21}_{18}	LLLRLRLR
7	5.301^{202}_{199}	LLLLRLR	8	6.1758^{80}_{77}	LLRLLRLR
7	5.33091^4_1	LLLRLLR	8	6.1875^{21}_{18}	LLRLRRLR
7	5.3698^{82}_{79}	LLLRLRR	8	6.19460^3_1	LLRLRLRR
7	5.37052^7_4	LLLRRLR			

The results concerning periodic orbits with the period $n \leq 8$ are collected in Table **??**. For each orbit its period n, the interval $\mathbf{T}$ containing the flow-time and the corresponding symbol sequence are reported. Periodic orbits with the period $n \leq 6$ are plotted in Fig. **??**. In case of a pair of symmetric orbits only one of them is plotted.

The number of periodic orbits of P with a given minimum period n and bounds for flow times are collected in Table **??**. It is interesting to note that some of period–11 orbits are longer

TABLE III

SHORT PERIODIC ORBITS OF P, THE NUMBER p_n OF PERIOD–n CYCLES, RIGOROUS BOUNDS $\mathbf{T}_n$ FOR THE LENGTH OF PERIOD–n CYCLES

n	p_n	$\mathbf{T}_n$
2	1	$[1.5586, 1.5587]$
3	2	$[2.3059, 2.3060]$
4	3	$[3.0235, 3.0843]$
5	6	$[3.7256, 3.8696]$
6	9	$[4.4177, 4.6372]$
7	18	$[5.1030, 5.4292]$
8	30	$[5.7834, 6.1947]$
9	56	$[6.4602, 6.9880]$
10	99	$[7.1346, 7.7531]$
11	186	$[7.8073, 8.5467]$
12	335	$[8.4792, 9.3117]$
13	630	$[9.1509, 10.1054]$
14	1161	$[9.8231, 10.8703]$
1–14	2536	

than some period–12 orbits. Similar phenomena is observed for larger n. Positions of periodic orbits of P with the period $n \leq 13$ are plotted in Fig. **??**. One can see that they fill densly the middle part of the attractor, and that the area occupied by period–n orbits grows with n.

The method has also been applied for longer orbits. Three examples are presented in Fig. **??**. In the last example the symbol sequence contains all possible subsequences of length $p \leq 8$. The examples show that the method proposed makes it possible to find long periodic orbits with a prescribed symbol sequence.

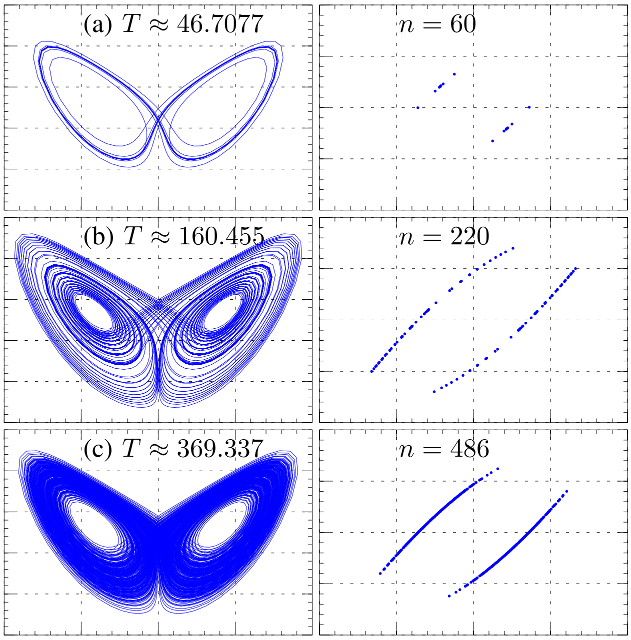

Fig. 5. Examples of long periodic orbits

IV. Conclusion

We have described a symbolic dynamics based method for finding all short periodic orbits for chaotic systems. The method has been applied to the Poincaré map associated with

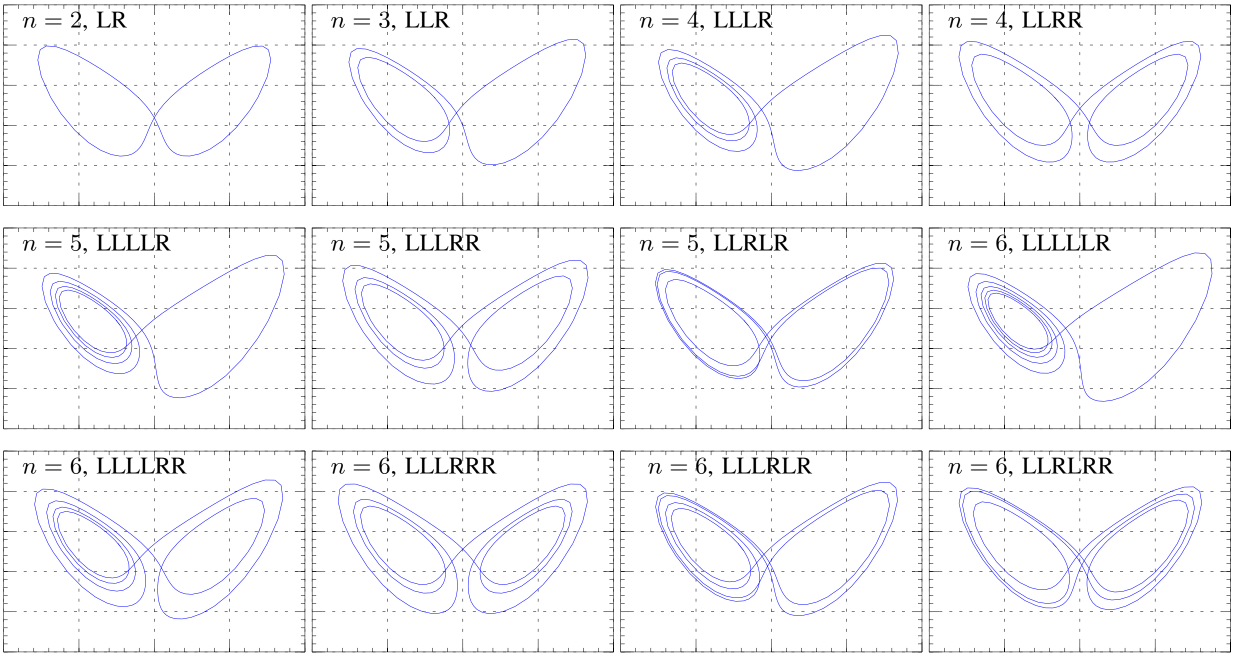

Fig. 3. The shortest periodic orbits for the Lorenz system

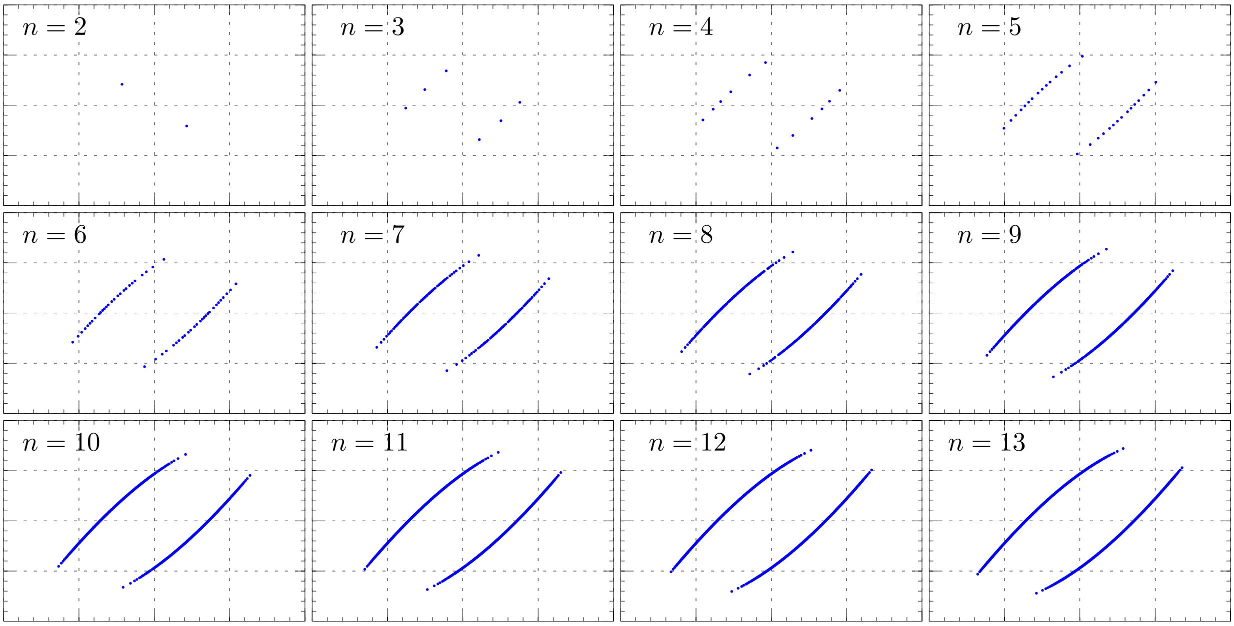

Fig. 4. All periodic orbits of P with the period $n \leq 13$

the Lorenz system. All periodic orbits with the period $n \leq 14$ have been found. Several long periodic orbits with specific symbol sequences have been located and their existence have been proved.

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