FIXED POINTS OF A DESTABILIZED KURAMOTO-SIVASHINSKY EQUATION

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ABSTRACT. We consider the family of destabilized Kuramoto-Sivashinsky equations in one spatial dimension $u_t + \nu u_{xxxx} + \beta u_{xx} + \gamma u u_x = \alpha u$ for $\alpha, \nu \geq 0$ and $\beta, \gamma \in \mathbb{R}$. For certain parameter values, shock-like stationary solutions have been numerically observed. In this work we verify the existence of several such solutions using the framework of self-consistent bounds and validated numerics.

Keywords: Kuramoto-Sivashinsky equations; boundary value problem; Galerkin projection; self-consistent bounds; rigorous computations; interval arithmetic

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1. INTRODUCTION

The Kuramoto-Sivashinsky (KS) equation

$$u_t = \mu\nabla^2 u - \nu\nabla^2\nabla^2 u - \lambda(u\cdot\nabla)u, \tag{1.1}$$

was derived by Kuramoto *et al.* [14] as a reaction-diffusion system modeling the Belousov-Zhabotinsky reaction [6, 28], and by Sivashinsky [20] for studying hydrodynamic instabilities in laminar flames. In one dimension, the KS-equation is usually studied in the form

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0,$$

where the parameter ν is emphasized as the viscosity parameter; taking ν positive introduces dissipation in the system. The Kuramoto-Sivashinsky equation has been subject to extensive studies in various works; the reader is referred to [3, 11, 12, 18, 22], without attempting to be comprehensive. Due to its dissipative nature, the KS-equation diplays long-term behaviour (fixed points, periodic solutions) that appear to be rather low-dimensional. This indicates that the higher modes are not very influential, and explains why approximation techniques such as Galerkin projections are suitable. We will use this fact in what follows.

By introducing a destabilizing term αu, shock-like stationary solutions were observed by Wittenberg [23], and Rademacher and Wittenberg [19]. Motivated by this, we consider the

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general form

$$u_t + \nu u_{xxxx} + \beta u_{xx} + \gamma u u_x = \alpha u, \tag{1.2}$$

where $\alpha \geq 0$ and $\beta, \gamma \in \mathbb{R}$. Without the destabilizing term αu, this is the same form that has been subjected to rigorous, computer-assisted studies by Zgliczyński and Mischaikow [27] and Zgliczyński [24, 25, 26], validating stationary and periodic solutions, and providing means for time integration.

When looking for odd, stationary solutions, we may fix $\nu = 1$ and introduce a new parameter by considering solutions of period L. We are then interested in solving the boundary-value problem

$$u_{xxxx} + \beta u_{xx} + \gamma u u_x = \alpha u, \quad u(-L/2,t) = u(L/2,t), \quad u(x,t) = -u(-x,t). \tag{1.3}$$

Combining the methods and observations of Zgliczyński, Rademacher and Wittenberg, we have validated several odd, shock-like, stationary solutions of (1.2) using the framework of self-consistent bounds discussed in [27, 24, 25, 26]. Theorem 1.1 is an example from the obtained results; the reader may find all the solutions that we have validated in Section 2.7.

Theorem 1.1. *For the parameters $\beta = 2$, $\gamma = 1$, $\alpha = 0.5$, and $L = 30$, the KS-equation (1.3) has a shock-like stationary solution $u^*(t,x)$ such that*

$$\|u^*(t,x) - u(t,x)\|_{C^0} \leq 3.757 \times 10^{-4} \text{ and } \|u^*(t,x) - u(t,x)\|_{L^2} \leq 1.353 \times 10^{-4},$$

where $u(t,x) = \sum_{k=1}^{\infty} -2a_k \sin(\frac{2\pi x}{L})$ is given by its Fourier coefficients in Table 2.7.

We have already used the term *rigorous computations*; this refers to calculations that are done using a computer program, and which give validated results, every possible numerical error is controlled. We achieve this by using the framework of interval analysis, allowing us to prove *mathematical theorems* from the obtained outputs. For more information about interval analysis, computer-aided proofs, and rigorous numerics, the reader is referred to Moore [16], Alefeld [2], Tucker [21], and Nedialkov *et al.* [17].

Outline. In the first part of the paper we will give some necessary definitions and notations, in Section 1.1 we give an overview of the method of self-consistent bounds. In the second part we discuss how to apply the previously presented techniques in a rigorous manner adopted to the destabilized Kuramoto-Sivashinsky equation (1.2). Section 2.1 deals with transforming our problem into an infinite ladder of ordinary differential equations. We consider a Galerkin-projection, that is a finite projection of this infinite ladder of ODEs in Section 2.2 and thus, introducing proper error terms, obtain a finite dimensional differential inclusion. In Section 2.3 we obtain a non-rigorous candidate solution. This allows us to construct a block decomposition and corresponding local coordinates in Section 2.4. According to the method of self-consistent bounds, fulfilling certain isolation inequalities ensures that we have a valid enclosure of a stationary solution for (1.2). These inequalities are discussed in Section 2.5. We include the algorithm from [24] for obtaining such bounds in Section 2.6 and present our results in Section 2.7. In the final part of the paper we comment on future directions.

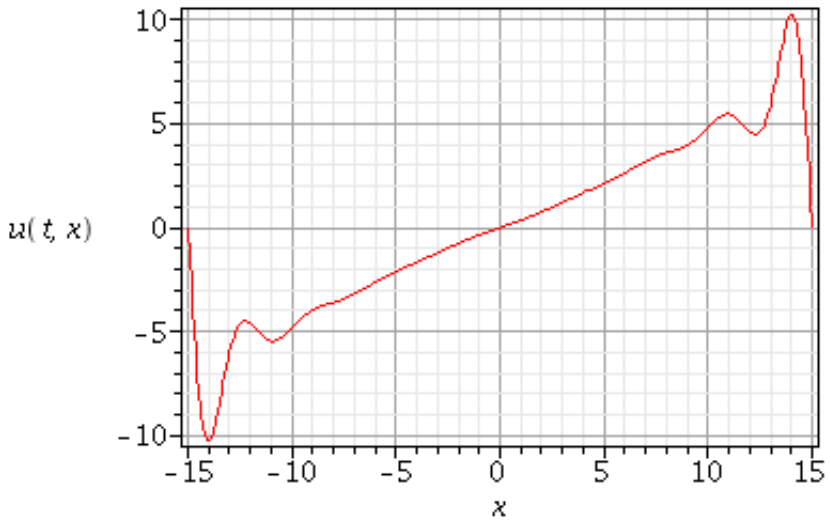


FIGURE 1. $u(t,x)$ for $\beta = 2$, $\gamma = 1$, $\alpha = 0.5$, and $L = 30$.

Notation. We denote by $\mathbb{N}, \mathbb{N}_0, \mathbb{R}$ and $\mathbb{C}$ the set of positive integers, nonnegative integers, reals and complex numbers, respectively. For a set $N \subset \mathbb{R}^n$, $\mathrm{bd}(N)$ denotes its boundary and $\mathrm{int}(N)$ its interior.

Let $\mathscr{L}^2(\mathbb{R}^2)$ and $l^2(\mathbb{N})$ be the Hilbert space of the real square-integrable functions from $\mathbb{R}^2$ to $\mathbb{R}$, and the Hilbert space of the complex square-summable series from $\mathbb{N}$ to $\mathbb{C}$, respectively. Given a differentiable function $u(t,x)\colon \mathbb{R}^2 \to \mathbb{R}$ by $D^s u$ we denote the partial derivative $\frac{\partial^s}{\partial x^s} u$.

1.1. **The method of self-consistent bounds.** In this section we provide a quick overview of the method of self-consistent bounds in a more abstract setting than we will use later on. For a detailed and thorough introduction the reader is referred to Zgliczyński and Mischaikow [27] and Zgliczyński [24, 25, 26].

Let $\mathscr{H}_0 = \mathscr{L}^2(\mathbb{R}^2)$ and consider the autonomous differential equation for $u \in \mathscr{H}_0$

$$\begin{cases} \dot{u} & = F(u), \\ u(0,x) & = u_0(x), \qquad u_0 \in \mathscr{H}_0, \end{cases} \tag{1.4}$$

where $F\colon \mathscr{H}_0 \to \mathscr{H}_0$ is a differential operator. By a solution of (1.4) we understand a differentiable function $u\colon [0,t_{max}) \times \mathscr{H}_0 \to \mathscr{H}_0$ such that (1.4) is satisfied for $t \in [0,t_{max})$. In this paper, we shall consider stationary solutions to (1.4) that are periodic in x.

For a real separable Hilbert space $\mathscr{H}$, given a subspace $X_m \subset \mathscr{H}$ with $\dim(X_m) < \infty$, we can investigate the evolution equation $\dot{u} = F(u)$ by decomposing $\mathscr{H}$ into the direct sum of the finite dimensional X_m and the infinite dimensional $Y_m = X_m^{\perp}$:

$$\mathscr{H} = X_m \oplus Y_m.$$

Let P_m, Q_m denote the orthogonal projections from $\mathscr{H}$ onto X_m and Y_m respectively. By introducing $u = \mathbf{x} \oplus \mathbf{y}$, we may split the evolution equation (1.4) into the two parts

$$\begin{aligned} \mathbf{x}' &= P_m F(\mathbf{x} \oplus \mathbf{y}), \\ \mathbf{y}' &= Q_m F(\mathbf{x} \oplus \mathbf{y}). \end{aligned} \tag{1.5}$$

Consider now a compact $W \subset X_m$ representing the *main modes* and a closed $T \subset Y_m$ representing the *high modes* or the so-called *tail*. For an initial condition $\mathbf{x}_0 \oplus \mathbf{y}_0 \in W \oplus T$, let us study the differential inclusion

$$\begin{cases} \mathbf{x}' & \in P_m F(\mathbf{x} \oplus T), \qquad \mathbf{x} \in W, \\ \mathbf{x} & = \mathbf{x}_0. \end{cases} \tag{1.6}$$

If T is "sufficiently small", then robust properties of the finite dimensional system

$$\begin{cases} \mathbf{x}' & = P_m F(\mathbf{x} \oplus \mathbf{0}), \qquad \mathbf{x} \in W, \\ \mathbf{x} & = \mathbf{x}_0. \end{cases} \tag{1.7}$$

should carry over to (1.6) and in turn to (1.5) and (1.4). The finite dimensional system (1.7) is called the *m-Galerkin* projection of (1.4). The vector $\mathbf{0}$ in (1.7) denotes the zero vector of Y_m. We define the finite dimensional vector field $F_m \colon X_m \to X_m$ as $F_m(\mathbf{x}) = P_m F(\mathbf{x} \oplus \mathbf{0})$.

The finite dimensional system (1.7) may produce good approximations to solutions of the full system (1.5) if F is such that the tail of the solutions are "small" in a particular sense. In light of this, we impose several conditions on F: we require the following decomposition into a linear and nonlinear part

$$F(u) = Lu + N(u, Du, \ldots, D^r u), \qquad u \in \mathscr{H}_0,$$

where the linear operator L is diagonal in the Fourier basis $\mathscr{B}_F = \{e^{ikx}\}_{k \in \mathbb{Z}}$, and the nonlinear part is given as a polynomial N of spatial partial derivatives of u up to order r. In addition, we require that (1.4) is *dissipative* in the sense that the eigenvalues ρ_k of L satisfy

$$\rho_k = -g(|k|)|k|^p, \tag{1.8}$$

for some $p > r$, where $g \colon \mathbb{R}_0^+ \to \mathbb{R}$, and there exists $k_0 \in \mathbb{R}^+$ such that $g(z)$ is positive, uniformly bounded away from zero and from above for $z > k_0$. Condition (1.8) suggests that for large $|k|$, that is for large scales, the linear part of F will be dominant. Note that expressing $u \in \mathscr{H}_0$ in $\mathscr{B}_F$ results in time-dependent Fourier coefficients

$$u(t,x) = \sum_{k=-\infty}^{\infty} c_k(t) e^{ikx}.$$

Since we are currently interested in stationary solutions, we omit the variable t when it is not absolutely necessary. Also, rather than working directly with (1.4), we consider the Hilbert space of the coefficients with respect to $\mathscr{B}_F$, that is $\mathscr{H}_1 = l^2(\mathbb{Z}) \cong \mathscr{H}_0$. This means that we

will identify u with its coefficient vector. As a consequence, the operators F, L and N then act on $\mathscr{H}_1$ as well. Equation (1.4) on $\mathscr{H}_0$ then transforms into the infinite ladder of ODEs

$$c_k' = \mathscr{P}_k F(u) = \rho_k c_k + \mathscr{P}_k N(u), \quad k \in \mathbb{Z} \tag{1.9}$$

where $\mathscr{P}_k \colon \mathscr{H}_1 \to \mathbb{C}$ is the projection to the k-th component.

Returning to the abstract setting, let H_k denote the span of the kth basis vector of $\mathscr{H}$. For $m \in \mathbb{N}$, we may decompose $\mathscr{H}$ into the direct sum of the finite dimensional $X_m = \oplus_{|k|\le m} H_k$ and the infinite dimensional $Y_m = X_m^\perp = \oplus_{|k|>m} H_k$ that is

$$\mathscr{H} = X_m \oplus Y_m.$$

The orthogonal projections from $\mathscr{H}$ onto H_k, X_m and Y_m are given by A_k, P_m and Q_m, respectively. Later on we shall obtain a candidate solution at which the Jacobian of the m-Galerkin projection attains only a finite number of complex and a finite number of positive eigenvalues. We shall then impose the additional requirement that m is large enough so that all the eigenvalues ρ_k are negative for $|k| \ge m$.

In order to be able to study (1.4) through the m-Galerkin projections (1.7), we need uniform convergence, and thus regularity conditions for F, W and T. This can be constuctively achieved by the use of *isolation blocks* and *self-consistent bounds*.

Definition 1.2. *Let $\varphi \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous flow generated by a differential equation $y' = f(y)$, where $f \colon \mathbb{R}^n \to \mathbb{R}^n$. The compact set $N \subset \mathbb{R}^n$ is said to be an* isolating neighbourhood *if*

$$\mathrm{Inv}(N, \varphi) := \{z \in N : \varphi(\mathbb{R}, z) \subset N\} \subset \mathrm{int}(N).$$

In addition, if for any $z \in \mathrm{bd}(N)$, there exists $t_z > 0$ such that

$$\varphi((0, t_z), z) \cap N = \emptyset \quad \text{or} \quad \varphi((-t_z, 0), z) \cap N = \emptyset,$$

then we call N an isolating block.

Remark 1.3. *The definition essentially says that trajectories trapped in an isolating block stay in its interior. A trajectory entering or leaving the block crosses the boundary, and is not tangential to it at any point. As a consequence we may divide the boundary of the block into* entry *and* exit *sides.*

Definition 1.4. *Assume that F is such that $F_n \colon X_n \to X_n$ is a C^1 function for all $n \in \mathbb{N}$, and let $0 < m \le M < \infty$. Consider the structure*

$$S = W \oplus \prod_{|k|>m} B_k,$$

where $W \subset X_m$ and $B_k \subset H_k$ for $|k| > m$ are compact sets. The conditions **C1**, **C2**, **C3**, **C4** *and* **C5** *are defined as follows.*

- **C1** *$0 \in B_k$ for $|k| > M$.*
- **C2** *$\sum_{|k|>M} a_k^2 < \infty$, where $a_k = \max_{a \in B_k} |a|$ for $|k| > M$. This implies that $S \subset \mathscr{H}$.*
- **C3** *$u \mapsto F(u)$ is continuous on S and $\sum_{|k|>M} f_k^2 < \infty$, where $f_k = \max_{u \in S} |A_k F(u)|$ for $|k| > M$.*

C4 *For each $|k| > m$ we have $B_k = [\underline{a_k}, \overline{a_k}]$ that is, an interval. For $u \in S$ and $|k| > m$ it holds that*

$$u_k = \underline{a_k} \Rightarrow A_k F(u) > 0,$$
$$u_k = \overline{a_k} \Rightarrow A_k F(u) < 0,$$

using the notation $u_k = A_k u$.

C5 *W is an isolating block for the m-Galerkin projection $\mathbf{x} = P_m(F(\mathbf{x} \oplus \mathbf{y}))$ for all $\mathbf{y} \in Y_m$.*

We say that the set S forms self-consistent bounds for F *if the conditions* **C1**, **C2** *and* **C3** *are satisfied. If* **C4** *holds in addition, we speak about* topologically self-consistent bounds for F. *We call W the* main part *and*

$$T = \prod_{|k|>m} B_k \subset Y_m$$

the tail. *In addition we refer to $\prod_{|k|>M} B_k$ as the* far-tail *and to $\prod_{m<|k|\leq M} B_k$ as the* near-tail.

As shown in the papers of Zgliczyński, conditions **C1**, **C2** and **C3** imply that given self-consistent bounds $W \oplus T$, the set $W \oplus T$ is a compact subset of $\mathscr{H}$. Moreover, the convergence $\lim_{n\to\infty} P_n(F(u)) = F(u)$ is uniform for $u \in W \oplus T$. If in addition **C4** is satisfied it is possible to establish the existence and uniqueness of solutions of (1.4) inside $W \oplus T$. Condition **C5** may be formulated in a similar manner to **C4**, that is inequalities for a scalar product with the outward normal on the boundary of W. We shall present the details later on when it comes to implementation. Having all conditions **C1**-**C5** satisfied implies that $W \oplus T$ contains a fixed point.

The presented results suggest the following steps for a computer-aided proof:

1. Find a non-rigorous candidate solution that we will try to validate.
2. Construct a good direct sum decomposition and use the Jacobian of the Galerkin projection at the candidate solution to establish local coordinates for the main modes.
3. Find topologically self-consistent bounds forming an isolating block with the main modes represented in local coordinates.

One crucial issue is how to represent self-consistent bounds in a computer-aided proof. The most immediate obstacle we have to overcome is its infinite structure. The dominant property of L for large k with negative eigenvalues suggests that the Fourier coefficients of a solution decay to zero at least at a polynomial speed, and thus we may uniformly enclose them. We do this for the far-tail $\prod_{|k|>M} B_k$ by the polynomial bounds $B_k = \frac{C}{|k|^s}[-1,1]$ for $|k| > M$, where $C \in \mathbb{R}^+$ and $s \in \mathbb{N}$ are independent of k. For a more detailed desciption of this procedure, the reader is referred to [27, 24].

2. Implementation

In this part of the paper we discuss in detail how to apply the previously presented techniques to the destabilized Kuramoto-Sivashinsky equation (1.2). As certain algebraic estimates are essentially identical for (1.2) and the KS equation discussed in [27, 24], we omit them and refer to these papers. The software for producing our results is available from [1].

2.1. **Infinite ladder of ODEs.** In this section we convert (1.2) into an infinite ladder of ODEs. Since we are looking for period-L solutions of

$$u_t + u_{xxxx} + \beta u_{xx} + \gamma u u_x = \alpha u,$$

it is natural to consider the Fourier expansion $u(t,x) = \sum_{k\in\mathbb{Z}} c_k(t) e^{i\eta kx}$ using the notation $\eta = \frac{2\pi}{L}$. It is easy to see that the nonlinear term may be expanded in two different ways $uu_x = \sum i\eta n c_n c_{k-n}$ and $uu_x = \sum i\eta (k-n) c_n c_{k-n}$, therefore $uu_x = i\eta \frac{k}{2} \sum c_n c_{k-n}$ holds. This leads to the following infinite ladder of complex ODEs

$$c_k' = (-\eta^4 k^4 + \beta\eta^2 k^2 + \alpha) c_k - i\gamma\eta \tfrac{k}{2} \sum_{n=-\infty}^{\infty} c_n c_{k-n}, \text{ for } k \in \mathbb{Z}. \tag{2.1}$$

According to our assumption on $u(t,x)$, we are interested in odd solutions. This results in $c_k = -c_{-k}$, in particular $c_0 = 0$. Since $u(t,x)$ is real, the imaginary part of the Fourier sum must vanish. One may easily check that this implies that c_k is pure imaginary. Introducing $a_k = -ic_k$, (2.1) transforms into

$$a_k' = (-\eta^4 k^4 + \beta\eta^2 k^2 + \alpha) a_k + \gamma\eta \tfrac{k}{2} \sum_{n=-\infty}^{\infty} a_n a_{k-n}, \text{ for } k \in \mathbb{Z}. \tag{2.2}$$

Using that $a_k = -a_{-k}$, we may decompose the infinite sum as

$$\begin{aligned} \sum_{n=-\infty}^{\infty} a_n a_{k-n} &= \sum_{n=1}^{k-1} a_n a_{k-n} + \sum_{n=-\infty}^{-1} a_n a_{k-n} + \sum_{n=k+1}^{\infty} a_n a_{k-n} \\ &= \sum_{n=1}^{k-1} a_n a_{k-n} - \sum_{n=1}^{\infty} a_n a_{k+n} - \sum_{n=1}^{\infty} a_n a_{k+n}. \end{aligned} \tag{2.3}$$

We obtain from (2.2), (2.3), and by introducing $\rho_k = -\eta^4 k^4 + \beta\eta^2 k^2 + \alpha$ that

$$a_k' = \rho_k a_k + \gamma\eta \tfrac{k}{2} \sum_{n=1}^{k-1} a_n a_{k-n} - \gamma\eta k \sum_{n=1}^{\infty} a_n a_{k+n} = \rho_k a_k + A_k N(\mathbf{a}), \text{ for } k \in \mathbb{N}, \tag{2.4}$$

where $\mathbf{a} \in l^2(\mathbb{N})$ in accordance with Section 1.1 (where we considered the equivalent $l^2(\mathbb{Z})$) and

$$u(t,x) = -2 \sum_{k=1}^{\infty} a_k(t) \sin(\eta k x).$$

We point out that the destabilizing effect of α becomes apparent in the expression for ρ_k.

2.2. **m-Galerkin projection.** We shall transform equation (2.4) rigorously into a computationally tractable finite dimensional differential inclusion. We obtain this form by considering an m-Galerkin projection, that is we discard the terms containing a_k with $k \geq m$ from (2.4). Therefore, the projection is given by

$$A_k F_m(a_1, \ldots, a_m) = \rho_k a_k + \gamma\eta \tfrac{k}{2} \sum_{n=1}^{k-1} a_n a_{k-n} - \gamma\eta k \sum_{n=1}^{m-k} a_n a_{k+n}, \text{ for } k = 1, \ldots, m, \tag{2.5}$$

while the error induced by the projection satisfies

$$A_k \mathscr{E}_G(m,\mathbf{a}) = -\gamma\eta k \sum_{n=m-k+1}^{\infty} a_n a_{k+n}, \text{ for } k = 1,\dots,m.$$

We shall refer to $\mathscr{E}_G(m,\mathbf{a}) \in \mathbb{R}^m$ as the Galerkin error. We may enclose this using the formulae given in Sections 3.1 and 3.2 of [27].

Assuming that $[\mathscr{E}_G(m,\mathbf{a})]$ is an enclosure of $\mathscr{E}_G(m,\mathbf{a})$, we obtain the following differential inclusion for the main modes

$$P_m(\mathbf{a})' \subseteq F_m(a_1,\dots,a_m) + [\mathscr{E}_G(m,\mathbf{a})]. \tag{2.6}$$

2.3. **Obtaining a non-rigorous candidate solution.** Consider now the M-Galerkin projection of (2.4), that is

$$\mathbf{b}' = F_M(\mathbf{b}), \qquad \mathbf{b} \in \mathbb{R}^M,$$

where $M \in \mathbb{N}$. We introduced $\mathbf{b}$ in order to emphasize the distinction from the infinite dimensional $\mathbf{a}$ in the previous sections. Finding a candidate solution means also finding an apparent zero of F_M. We may use either a zero finding method, like Newton's method, or if this zero is stable – a non-rigorous integrator to obtain the starting guess.

For this guess to be a reasonable approximate solution of (2.4) as well, we need to be able to neglect high-modes. When k is large enough, then $|\rho_k|$ is large and $\rho_k < 0$, thus the term $-|\rho_k|\mathbf{b}_k$ dominates in the k-th component of (2.4). As the high modes of the solutions to this projected ODE tend to zero, and since we are looking for a stationary solution for the infinite ladder of ODEs, we may indeed neglect the higher order terms and still obtain a reasonably good candidate solution if we choose M sufficiently large.

As an initial guess for a shock-like stationary solution, we construct a piecewise linear shock and use its approximate Fourier expansion; this is shown in Figure 2. Using these Fourier coefficients as an initial condition, we obtain a candidate solution for $F_M(\mathbf{b}) = \mathbf{0}$.

2.4. **Block decomposition and local coordinates.** Now we need to construct $W \oplus T$ enclosing our non-rigorous candidate that satisfies conditions **C1** - **C5**. This is done by solving the so-called isolation equations for W given in Section 2.5. These equations require a suitable local coordinate system on X_m which is obtained from the eigenvalue decomposition of $\mathrm{Jac}F_m$ at the candidate. Keeping the computations real, we eliminate the complex eigenvalues by switching to a real block decomposition.

Block decomposition. Assume that $\mathbf{b}^* = (b_1^*,\dots,b_m^*)$ is an approximate zero of F_m. This leads to the *approximation error*

$$\mathscr{E}_A(m,\mathbf{b}^*) = F_m(\mathbf{b}^*).$$

Note that having a good approximation, we expect $\|\mathscr{E}_A(m,\mathbf{b}^*)\|$ to be small. We calculate $\mathrm{Jac}_{F_m}(\mathbf{b}^*)$ and then obtain its eigenvalue decomposition. There are several algorithms producing such a decomposition, but the result is in general non-rigorous. We obtain two complex matrices Λ and E, where Λ is a diagonal matrix with the approximate complex eigenvalues λ_k on the diagonal, while E holds the respective approximate eigenvectors $\mathbf{c}_k$ in its rows. Thus $\mathrm{Jac}_{F_m}(\mathbf{b}^*)E \sim E\Lambda$, and consequently $\mathrm{Jac}_{F_m}(\mathbf{b}^*) \sim E\Lambda E^{-1}$ (given that E is invertible).

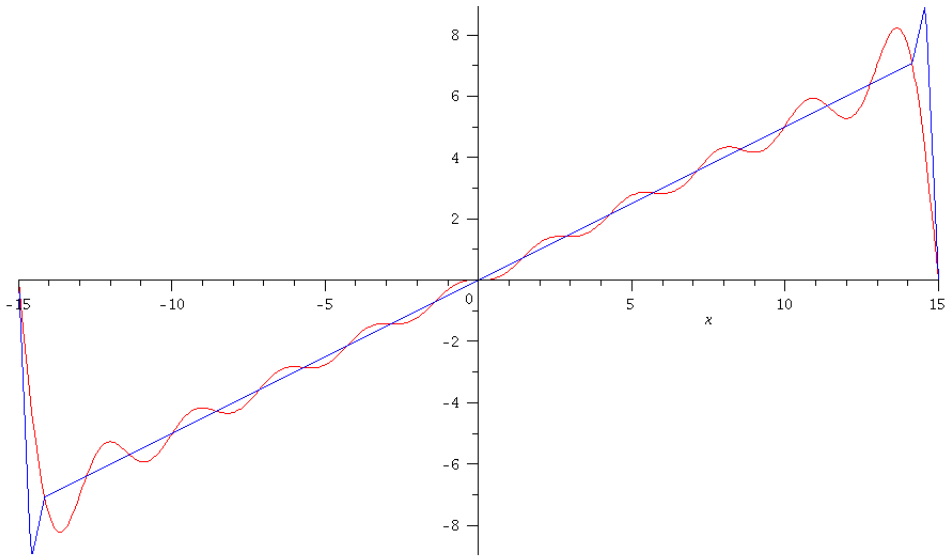


FIGURE 2. A piecewise linear shock and its Fourier approximation with 10 modes.

Now let us convert the complex eigenvalue decomposition into a real block decomposition. This is natural since we are interested in real solutions. This block decomposition is given by the direct sum of the one-dimensional eigenspaces spanned by the real eigenvectors and the two-dimensional eigenspaces spanned by the real and imaginary parts of a complex conjugate pair of eigenvectors. The new decomposition is given by two real matrices, the block diagonal matrix $\tilde{\Lambda}$ and the matrix $\tilde{E}$.

Assume that Λ is ordered such that $|\lambda_{k_1}| > |\lambda_{k_2}|$ given that $k_1 > k_2$, and $\lambda_k = \alpha_k + i\beta_k$ and $\lambda_{k+1} = \alpha_k - i\beta_k$ are conjugate pairs in Λ. In order to obtain the appropriate block in $\tilde{\Lambda}$, we replace the submatrix

$$\begin{pmatrix} \Lambda_{k,k} & \Lambda_{k,k+1} \\ \Lambda_{k+1,k} & \Lambda_{k+1,k+1} \end{pmatrix} = \begin{pmatrix} \alpha_k + i\beta_k & 0 \\ 0 & \alpha_k - i\beta_k \end{pmatrix}$$

by

$$\begin{pmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{pmatrix}.$$

Accordingly, the kth and $(k+1)$th columns of E holding the conjugate pairs $\mathbf{u}_k + i\mathbf{v}_k$ and $\mathbf{u}_k - i\mathbf{v}_k$ are replaced by $\mathbf{u}_k$ and $\mathbf{v}_k$, respectively. The resulting matrices $\tilde{\Lambda}$ and $\tilde{E}$ satisfy the approximation $\mathrm{Jac}_{F_m}(\mathbf{b}^*) \sim \tilde{E}\tilde{\Lambda}\tilde{E}^{-1}$ as before, but now all entries are real and $\tilde{\Lambda}$ is block diagonal. This leads to the error term for the block decomposition

$$\mathscr{E}_J(m,\mathbf{b}^*) = \mathrm{Jac}_{F_m}(\mathbf{b}^*) - \tilde{E}\tilde{\Lambda}\tilde{E}^{-1}.$$

We expect $\|\mathscr{E}_J(m,\mathbf{b}^*)\|$ to be small; this is usually the case, since the non-rigorous algorithms produce high-quality approximate decompositions.

The component-wise quadratic approximation for $F_m(\mathbf{b})$ centered at $\mathbf{b}^*$ is given by

$$A_k F_m(\mathbf{b}) = A_k\Big(F_m(\mathbf{b}^*) + \mathrm{Jac}_{F_m}(\mathbf{b}^*)(\mathbf{b}-\mathbf{b}^*)\Big) + (\mathbf{b}-\mathbf{b}^*)^T \mathrm{H}_{F_m}^{(k)}(\xi)(\mathbf{b}-\mathbf{b}^*), \quad k = 1,\ldots,m,$$

where $\mathrm{H}_{F_m}^{(k)}(\xi)$ is the corresponding Hessian matrix evaluated at an intermediate point. Notice that since F_m is of degree 2, see (2.5); $\mathrm{H}_{F_m}^{(k)} \in \mathbb{R}^{m\times m}$ is a constant matrix. Taking enclosures, substituting $\mathbb{R}^m \ni \mathbf{b} = P_m(\mathbf{a})$ and introducing the the component-wise defined error term

$$A_k \mathscr{E}_H(m,\mathbf{b}^*,\mathbf{a}) = (P_m(\mathbf{a})-\mathbf{b}^*)^T \mathrm{H}_{F_m}^{(k)}(P_m(\mathbf{a})-\mathbf{b}^*), \quad k = 1,\ldots,m,$$

we transform our differential inclusion (2.6) for the main modes into

$$\begin{aligned} P_m(\mathbf{a})' \in{}& \tilde{E}\tilde{\Lambda}\tilde{E}^{-1}(P_m(\mathbf{a})-\mathbf{b}^*) + [\mathscr{E}_A(m,\mathbf{b}^*)] + [\mathscr{E}_J(m,\mathbf{b}^*)](P_m(\mathbf{a})-\mathbf{b}^*) \\ &+ [\mathscr{E}_H(m,\mathbf{b}^*,\mathbf{a})] + [\mathscr{E}_G(m,\mathbf{a})], \end{aligned}$$

where $\mathbf{a} \in l^2(\mathbb{R})$.

Local coordinates. The transformation into local coordinates at $\mathbf{b}^*$, that is from $P_m(\mathbf{a})$ to $\mathbf{z}$, is given by

$$\mathbf{z} = \mathscr{P}_{\mathbf{b}^*}^{-1}(P_m(\mathbf{a})) = \tilde{E}^{-1}(P_m(\mathbf{a})-\mathbf{b}^*), \tag{2.7}$$

while the inverse transformation is

$$P_m(\mathbf{a}) = \mathscr{P}_{\mathbf{b}^*}(\mathbf{z}) = \tilde{E}\mathbf{z} + \mathbf{b}^*. \tag{2.8}$$

Using formulae (2.7) and (2.8) we introducing the error term

$$\begin{aligned} \mathscr{E}(m,\mathbf{b}^*,\mathbf{a},\mathbf{z}) = {}& \tilde{E}^{-1}\mathscr{E}_A(m,\mathbf{b}^*) + \tilde{E}^{-1}\mathscr{E}_J(m,\mathbf{b}^*)\tilde{E}\mathbf{z} \\ &+ \tilde{E}^{-1}\mathscr{E}_H(m,\mathbf{b}^*,\mathbf{a}) + \tilde{E}^{-1}\mathscr{E}_G(m,\mathbf{a}). \end{aligned}$$

Assuming that $[\mathscr{E}(m,\mathbf{b}^*,\mathbf{a},\mathbf{z})]$ is an enclosure of $\mathscr{E}(m,\mathbf{b}^*,\mathbf{a},\mathbf{z})$, we obtain the following formula as the inclusion in local coordinates for the main modes

$$\mathbf{z}' \in \tilde{\Lambda}\mathbf{z} + [\mathscr{E}(m,\mathbf{b}^*,\mathbf{a},\mathbf{z})], \text{ for } \mathbf{z} \in \mathbb{R}^m, \tag{2.9}$$

which is of the desired form.

2.5. **Isolation inequalities.** Given the m-Galerkin projection, we have transformed our equation for the first m coefficients (main modes) into a differential inclusion. For the tail modes we will work with (2.4). In order for the set $S = W \oplus T$ to be an isolating block, conditions **C4** and **C5** have to be satisfied. We shall use local coordinates on the space X_m, the set W is obtained as $W = \mathscr{P}_{\mathbf{b}^*} Z$, where Z is given in the local coordinates as follows. If the component k corresponds to a one-dimensional block then $[z]_k = [\underline{z_k}, \overline{z_k}]$. If the components k and $k+1$ correspond to a two-dimensional block, then using the notation $(k) := (k, k+1)$, $\mathbf{z}_{(k)}$ is given as the closed ball around zero with radius r_k, that is $\mathbf{z}_{(k)} = B(\mathbf{0}, r_k)$. In addition, let us define the notation

$$\lambda_{(k)} = \begin{pmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{pmatrix}$$

for the corresponding block in $\tilde{\Lambda}$. Now, we may formulate the isolation inequalities (see Lemma 4.3 in [24]) for the main modes:

- for a one-dimensional main mode block corresponding to component k:

$$\begin{aligned} \lambda_k \underline{z_k} + [\mathscr{E}(m, \mathbf{b}^*, \mathbf{a}, \mathbf{z})]_k > 0 \\ \lambda_k \overline{z_k} + [\mathscr{E}(m, \mathbf{b}^*, \mathbf{a}, \mathbf{z})]_k < 0, \end{aligned} \tag{2.10}$$

- for a two-dimensional main mode block corresponding to components $(k, k+1)$:

$$\lambda_{(k)} \mathbf{z}_{(k)} + \sqrt{[\mathscr{E}(m, \mathbf{b}^*, \mathbf{a}, \mathbf{z})]_k^2 + [\mathscr{E}(m, \mathbf{b}^*, \mathbf{a}, \mathbf{z})]_{k+1}^2} < 0, \tag{2.11}$$

together with the similar inequalities coming from **C4** for the tail:

- for the tail mode k:

$$\begin{aligned} \rho_k \underline{\mathbf{a}_k} + A_k N(\mathbf{a}) > 0 \\ \rho_k \overline{\mathbf{a}_k} + A_k N(\mathbf{a}) < 0. \end{aligned} \tag{2.12}$$

If these equations are satisfied, then the vector field points inwards on the boundary of the set $W \oplus T$ implying that W is an isolating block and $W \oplus T$ forms self-consistent bounds.

2.6. **The isolation algorithm.** We obtain the bounds through the following algorithm as seen in [24]. The function `Validate_FixedPoint` takes in as parameter our non-rigorous candidate solution. Note that this was obtained by using a "large" M for the Galerkin projection. We have the freedom to choose a smaller m and give initial values for some of the modes in the initial guess for the tail T^* as well.

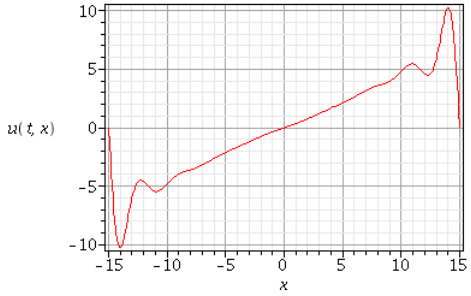
Algorithm 1 Validating a fixed point

1: **function** VALIDATE_FIXEDPOINT($\mathbf{b}^*, T^*$)
2: $Z \leftarrow [-\Delta, \Delta]^m$ ▷ Z is a small set around zero in local coordinates.
3: $\tilde{E}, \tilde{\Lambda} \leftarrow$ `Block Decomposition`$(\mathbf{b}^*)$ ▷ Obtain the block decomposition.
4: **repeat**
5: $W \leftarrow \mathscr{P}_{\mathbf{b}^*} Z$ ▷ W in the original coordinates.
6: $T \leftarrow T^*$ ▷ We always start from the candidate tail.
7: `Generate Tail`(W, T) ▷ Generation of the self-consistent tail T.
8: **if** `the tail is consistent` **then**
9: $\mathscr{E} \leftarrow [\mathscr{E}(m, \mathbf{b}^*, W \oplus T, \mathbf{z})]$ ▷ Enclose the error and obtain (2.9).
10: **for all** $k \in \mathbb{N}_m$ **do**
11: `Check Isolation`(Z, T, k) ▷ Check (2.10) and (2.11) for k.
12: **end for**
13: **if** `the isolation equations are not satisfied` **then**
14: `Increase` Z `where necessary`
15: **end if**
16: **end if**
17: **until** `we obtain self-consistent bounds`
18: **end function**

The function `Generate Tail` called in line 7 generates a topologically self-consistent tail, thus inequalities (2.12) are satisfied. Line 11 checks if (2.10) and (2.11) are satisfied for the k-th component. If they are, the function `Check Isolation` tightens the bounds maintaining the isolation. In line 14 we increase Z in the non-isolated directions. For a description of these procedures, the reader is referred to [27] and [24], respectively. In line 17 of Algorithm 1 we exit the cycle if the isolation is obtained for all modes, thus the existence of a fixed point is validated.

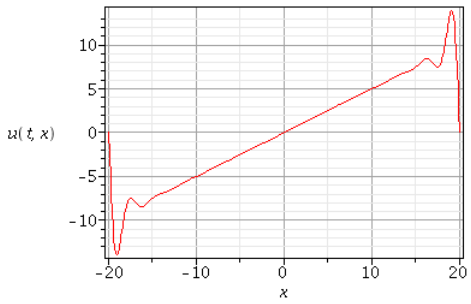
2.7. **Results of the computations.** We implemented our program in C++, using the CAPD Library [8] for rigorous computations. We have validated stationary solutions given various parameter sets. A table is provided for each of these situations to describe the stationary solution as follows.

For a given parameter set, we list the first 15 Fourier coefficients of $u(t,x) = \sum_{k=1}^{\infty} -2a_k \sin(\frac{2\pi x}{L})$ truncated to 6 digits; the full description of $u(t,x)$ may be found in the outputs [1]. Norm estimates are given to provide upper bounds on $\|u^*(t,x) - u(t,x)\|_{C^0}$ and $\|u^*(t,x) - u(t,x)\|_{L^2}$, where $u^*(t,x)$ is the found stationary solution of the destabilized Kuramoto-Sivashinsky equation. Note that selecting an optimal Δ – see line 2 in Algorithm 2.6 – is a delicate matter. Taking Δ too large will ruin all estimates needed for consistency and isolation. In addition to this, we cannot hope to obtain good (small) norm estimates unless Δ is small. On the other hand, our method will fail for small Δ unless we start with a good approximate solution. This situation calls for a small amount of trial and error; we start with a very small Δ, and gradually (and locally) inflate the neighbourhood – see line 14 in Algorithm 2.6.



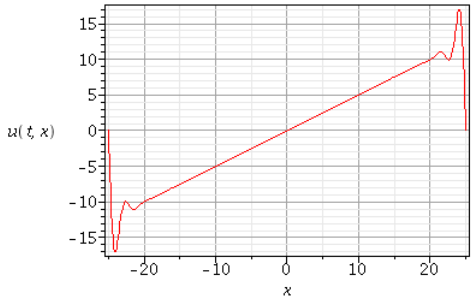
k	a_k	k	a_k	k	a_k
1	-2.17937	6	0.56107	11	-0.25329
2	1.20909	7	-0.55752	12	0.16891
3	-0.80087	8	0.55773	13	-0.11215
4	0.66633	9	-0.48520	14	0.07532
5	-0.58366	10	0.36620	15	-0.05136

(1) Viscous shock solution for the parameters $\alpha = 0.5$, $\beta = 2$, $\gamma = 1$, $\nu = 1$ and $L = 30$. In the algorithm, Δ is set to 10^{-5}. C^0 estimate: 3.7567×10^{-4}, L^2 estimate: 1.3523×10^{-4}.



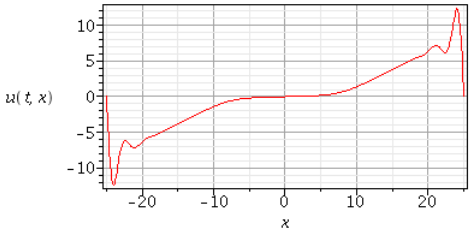
k	a_k	k	a_k	k	a_k
1	-3.19964	6	0.62280	11	-0.48604
2	1.61679	7	-0.56912	12	0.45776
3	-1.10128	8	0.53591	13	-0.41057
4	0.85067	9	-0.51590	14	0.34852
5	-0.70942	10	0.50192	15	-0.28211

(2) Viscous shock solution for the parameters $\alpha = 0.5$, $\beta = 2$, $\gamma = 1$, $\nu = 1$ and $L = 40$. In the algorithm, Δ is set to 10^{-8}. C^0 estimate: 2.2855×10^{-8}, L^2 estimate: 6.0610×10^{-9}.



k	a_k	k	a_k	k	a_k
1	-3.98764	6	0.72192	11	-0.48801
2	2.00717	7	-0.63934	12	0.47137
3	-1.35335	8	0.58130	13	-0.45724
4	1.03164	9	-0.53981	14	0.44270
5	-0.84323	10	0.50988	15	-0.42466

(3) Viscous shock solution for the parameters $\alpha = 0.5$, $\beta = 2$, $\gamma = 1$, $\nu = 1$ and $L = 50$. In the algorithm, Δ is set to 10^{-8}. C^0 estimate: 1.0506×10^{-6}, L^2 estimate: 2.2773×10^{-7}.



k	a_k	k	a_k	k	a_k
1	−1.84639	6	−1.84639	11	−0.38150
2	1.65079	7	1.65079	12	0.37451
3	−0.93640	8	−0.93640	13	−0.36967
4	0.70452	9	0.70452	14	0.35854
5	−0.61668	10	−0.61668	15	−0.33656

(4) Flat shock solution for the parameters $\alpha = 0.5$, $\beta = 2$, $\gamma = 1$, $\nu = 1$ and $L = 50$. In the algorithm, Δ is set to 10^{-8}. C^0 estimate: 7.0353×10^{-9}, L^2 estimate: 1.4926×10^{-9}.

Remark 2.1. *Note that the parameters of example (3) and (4) are identical. The initial guesses, however, were selected differently. This explains why different solutions were found.*

3. Future directions

The theory for time integration of differential inclusions was developed by Zgliczyński and Kapela [13]. The consequent works of Zgliczyński [25, 26] are aimed at validating and studying time-periodic solutions of dissipative PDEs and time integration. These techniques can be applied to the destabilized Kuramoto-Sivashinsky equation (1.2) and other equations as well. We have already implemented the rigorous time integration algorithm and we plan to study (1.2) in more detail.

After obtaining a fixed point, the next question is its stability. This usually requires a suitable norm; for block-decompositions like the ones we employ, we might use the so-called block-infinity norm, as seen in Zgliczyński [24]. Having obtained an enclosure for an asymptotically stable stationary solution, it is a natural task to study its basin of attraction. Assume that we obtain a larger trapping region in some way around the stationary solution on which we cannot prove its stability, thus cannot guarantee uniqueness either. Using rigorous time integration it may turn out that every solution starting from this larger set eventually enters the attracted isolating block obtained by Algorithm 1. This would imply that the new trapping region is inside the basin of attraction as well. Cyranka [9] provided such a computer-aided proof for the global stability of certain stationary solutions for the viscous Burgers equation with constant forcing using the techniques of Zgliczyński.

When time integration does not provide such results directly, possibly because the new candidate region is too large, we may still consider the time-h map given by the rigorous integrator. Studying this map on the finite dimensional space, where the self-consistent

bounds are represented, opens up the possibility for using graph representation techniques [10, 15, 5, 4] to estimate the basin of attraction.

As one can see, there are plenty of possibilities lying ahead. We plan to address these in the future.

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