On the global stability of a peer-to-peer network model

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Abstract

In this paper, we analyze the stability properties of a system of ordinary differential equations describing the thermodynamic limit of a microscopic and stochastic model for file sharing in a peer-to-peer network introduced in [1]. We show, under certain assumptions, that this BitTorrent-like system has a unique locally attracting equilibrium point which is also computed explicitly. Local and global asymptotic stability is also shown.

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1. Introduction

A significant fraction of Internet traffic is due to peer-to-peer networking activities. According to some accounts, as much as 50 % of overall traffic is due to the exchange (down- and uploading) of huge files. To-date, controlling this kind of activities is not possible. However, issues touching upon net neutrality may, in the near future, impose certain controls via, say, pricing schemes or rate control throttles, Political issues aside, it is clear that understanding, even roughly, the behavior of peer-to-peer networking traffic can be invaluable in providing a better picture of what is going on and what possible solutions may exist.

Popular file sharing protocols include Kazaa [2] and BitTorrent [3]. Applications using these systems include exchange of huge files, such as MP3 music and video-on-demand. BitTorrent, in particular, operates under the following principle: a huge file is almost never available in its entirety. Rather, pieces of it are spread around the network. The protocol aims at collecting the pieces. However, while this is happening, the peer also uploads pieces it already possesses, making them available for downloading by other peers. Most of the time, peers are selfish and will switch off from the network once they possess the whole file or the parts they need. By splitting the file into many pieces (hundreds or thousands), there is an implicit incentive for peers to stay on for longer time and thus exchange their pieces with others.

A highly idealistic, yet still mathematically complex, model of a BitTorrent-like system was introduced in [1]. In this model, there are, initially, N peers (where N is a very large

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integer), each possessing certain pieces of a file $F = \{1, \dots, n\}$. The number of pieces n is of the order of 10^2 or 10^3. However, even the case $n = 2$ or 3 is mathematically interesting. Each peer is labeled by a certain set $A \subset F$ containing all pieces currently possessed by the peer. A peer labeled $\varnothing$ possesses nothing and a peer labeled F possesses everything (and, if selfish, will switch off immediately or very quickly). New peers arrive in the system according to a Poisson process. Peers may leave the system after an exponential amount of time with parameter depending on their label. Peers come into contact and either exchange pieces or one downloads a piece from the other. Assuming spatial "homogeneity" and "uniformity", the state of the system is a vector x with entries x^A, indexed by the subsets A of F, where x^A denotes the number of peers with label A currently in the system. If no peers arrive or depart, then $\sum_A x^A = N$ and there is no stability issue (we call this a conservative system). If peers only depart then, again, there is no stability issue and it is also clear that after some (perhaps long) time the system will empty out (we call this a dissipative system). The case we are interested in here is the case where there are arrivals and departures (we call this an open system).

It is reasonable to assume that the rate of interaction of two peers labeled A and B coming into contact at a rate proportional to x^A times x^B, mimics the stochastic modeling in chemical reactions or epidemiological models. The stochastic model is thus a continuous-time Markov chain in a huge state space. By letting $N \to \infty$ and speeding time up proportionally with N, we arrive at an ordinary differential equation (ODE), one which was described and proved in [1]. The goal of this paper is to provide some glimpses into the behavior of this ODE under some simplifying assumptions, thus, hopefully, shedding some light into the behavior of this highly complex, yet important, system in the networking practice of nowadays. The assumptions are that (i) there are arrivals of peers only missing one piece and (ii) there are departures of peers only when they possess all pieces. Assumption (i) is highly restrictive but can be justified in two ways: first, it provides a mathematically tractable ODE; second, it may describe a model of a file-sharing network where arrivals of peers possessing small number of pieces is rare compared to the arrivals of peers possessing almost the entire file. We are looking, essentially, at the behavior of a peer-to-peer network, under the missing piece assumption (a piece which could be rare, yet important for the assembly of the file).

The paper is organized as follows: In Section 2 we discuss some preliminary concepts regarding the rates of individual peer interactions and introduce some notation. In Section 3 we describe the ODE. We do so in general, without any assumptions on the arrival/departure rates or system parameters. This will be useful in future research. In Section 4 we derive an expression for the unique equilibrium of the ODE, under simplified assumptions for the arrival and departure rates. Local and global stability are then examined in detail. The latter makes use of Dulac's criterion. An open problem and conclusions are discussed in Section 5.

2. Definition of interaction rates and notation

We first define the rates of interactions between two peers, according to the principle that they are proportional to the two populations. We adopt the following notations for set operations: $A \subset B$ means that A is contained in B, possibly equal to it. $A \subsetneq B$ (or $B \supsetneq A$) means that A is strictly contained in B. We write $B \setminus A$ for the set $B \cap A^c$. If $A \subset B$ we write $B - A$. If $i \notin A$, we write $A + i$ for $A \cup \{i\}$; and if $j \in A$, we write $A - j$ for $A - \{j\}$.

The notation "$A \sim B$" stands for "$A \subset B$ or $B \subset A$" and "$A \not\sim B$" stands for its negation or, equivalently, "$A \setminus B \neq \varnothing$ and $B \setminus A \neq \varnothing$".

Suppose there are, currently, X^A peers in the system labeled possessing the set $A \subset F$. We refer to such a peer as a "peer labeled A" or, simply, an "A-peer". If an A-peer comes into contact with a B-peer, exactly one of the following may happen:

Case 1. *Nothing.* This happens when $A = B$.

Case 2. A *download* of a chunk from one peer to the other. This can happen if $A \subsetneq B$ or $B \subsetneq A$. The rate of this is defined to be $\beta X^A X^B$ for some constant $\beta \geq 0$ (the download parameter). Suppose $A \subsetneq B$. Then A picks a chunk $i \in B - A$ and changes label to $A + i$, while B retains its label. Since there are $|B - A|$ choices for i, the rate of the particular interaction is $\beta \frac{X^A X^B}{|B-A|}$.

Case 3. A *swap.* This happens when $A \not\sim B$. The rate of swaps between A-peers and B-peers is defined to be $\gamma X^A X^B$ for some constant $\gamma \geq 0$ (the swap parameter). In such a case, A picks a chunk $j \in B \setminus A$ and B picks a chunk $i \in A \setminus B$. The labels of the two peers change from (A, B) to $(A + j, B + i)$. Since there are $|B \setminus A|$ ways to choose j and $|A \setminus B|$ ways to choose i, the rate of the swap changing (A, B) to $(A + j, B + i)$ is equal to $\gamma \frac{X^A X^B}{|A \setminus B| \cdot |B \setminus A|}$.

Note that Case 2 occurs when $A \sim B$ and $A \neq B$. So, clearly, the three cases are mutually exclusive. The discussion above will help us understand the form the differential equation for the evolution of the system takes.

Note also that besides the parameters β, γ, we also have arrival and departure rates as parameters. We let α^A be the rate of arrivals of A-peers from the outside world. We let $\delta^A X^A$ be the rate of departures of A-peers to the outside world. The parameter δ^A can be interpreted as the impatience rate of an A-peer: the higher the δ^A, the sooner it quits. The case we have in mind is that $\delta^A = \delta$ when $A = F$, with all the other δ^A:s negligible or equal to zero.

3. A differential equation description

The state of the system is a vector $x = (x^A, A \subset F)$, with components indexed by subsets of F. Since $|F| = n$, we have that $x \in \mathbb{R}_+^{2^n}$. We let $\|x\|$ be the ℓ^1-norm of x. Initially, the state is $x(0)$ and we assume $\|x(0)\| = 1$. If N is the actual number of peers present at time 0, then $x^A(0)N$ is the number of peers labeled A. At time $t \geq 0$, the state is $x^A(t)$. This number represents the fraction of peers labeled A, present in the system at time t, in relation to the total number of initial peers. The (system of) ODE is of the form

$$\frac{dx^A}{dt} = v^A(x), \quad A \subset F, \tag{1}$$

where $v^A(x)$ is a polynomial of degree at most 2 in the variables $x^A, A \subset F$. The component x^A may increase either due to an arrival, a download or a swap; it may decrease either due to a departure, a download or a swap. There are therefore 6 terms in the right-hand side of the ODE:

$$v^A(x) = \alpha^A + \Psi_d^A(x) + \Psi_s^A(x) - \delta^A x^A - \Phi_d^A(x) - \Phi_s^A(x). \tag{2}$$

The first and fourth terms are clear from the discussion in the previous section. (Note in passing that if $\beta = \gamma = 0$, then the system is trivial, and can be solved explicitly, as all the other terms vanish.) The function $\Phi_d^A(x)$ (respectively $\Psi_d^A(x)$) is the rate of decrease of x^A (respectively increase) due to a download. Similarly, $\Phi_s^A(x)$, $\Psi_s^A(x)$ for a swap. Let us explain these four functions, based on the cases analyzed earlier.

To have a decrease due to a download, we must have a peer labeled A come into contact with a peer labeled B for some B which strictly contains A. Let $i \in B - A$. The rate of downloads of the specific chunk i between is $\beta x^A x^B / |B - A|$. So the total rate of decrease of x^A due to a download is

$$\Phi_d^A(x) = \beta \sum_{B \supsetneq A} \sum_{i \in B-A} \frac{x^A x^B}{|B-A|} = \beta x^A \underbrace{\sum_{B \supsetneq A} x^B}_{\substack{\#\text{peers an } A\text{-peer} \\ \text{can download from}}} . \tag{3}$$

Similarly, the total rate of decrease of x^A due to a swap is

$$\Phi_s^A(x) = \gamma x^A \underbrace{\sum_{B \not\sim A} x^B}_{\substack{\#\text{peers an } A\text{-peer} \\ \text{can swap with}}} . \tag{4}$$

To have an increase in x^A due a download, it must be the case that a peer labeled $A - i$ downloads some chunk $i \in B$, from some peer labeled B. For this to happen we must have $i \in A$ and $A \subset B$. Such an interaction has rate $\beta x^{A-i} x^B / |B - (A - i)|$. Summing up over all $i \in A$ and all B containing A we obtain $\Psi_d^A(x)$:

$$\Psi_d^A(x) = \beta \sum_{B \supset A} \sum_{i \in A} \frac{x^{A-i} x^B}{|B - (A-i)|} = \beta \sum_{B \supset A} \frac{x^B}{1 + |B| - |A|} \underbrace{\sum_{i \in A} x^{A-i}}_{\substack{\#\text{peers which can} \\ \text{assume label } A \\ \text{after a download}}} , \tag{5}$$

where we made use of the observation that $|B - (A - i)| = 1 + |B| - |A|$, if $i \in A$.

Finally, to have an increase of the A-population due to a swap, we must have a peer labeled $A - j$ interact with a peer labeled B, for some $B \not\sim A - j$. This interaction transfers a chunk j from the B-peer to $(A - j)$-peer and a chunk i from in the reverse direction, for some $i \neq j$. The swap of chunks results into a peer labeled A and a peer labeled $B + i$. We need to have $j \in A$ with $j \in B \setminus (A - j)$ and $i \in (A - j) \setminus B$, or, equivalently,

$$j \in A \cap B, \quad i \in A \setminus B. \tag{6}$$

The set B can be any set such that $A - j \not\sim B$. Equivalently, $(A - j) \setminus B \neq \varnothing$ and $B \setminus (A - j) \neq \varnothing$. The latter holds true, while the former means that $A \setminus B \neq \varnothing$ or that

$$A \not\subset B. \tag{7}$$

The rate of the swap of an $(A - j)$-peer with a B-peer making them peers labeled A and $B + i$, respectively, equals (see Case 3 above) $\gamma x^{A-j} x^B$ and so the rate of swap of the specific pair of chunks, i, j, equals $\gamma x^{A-j} x^B / |B \setminus (A-j)| \cdot |(A-j) \setminus B|$. So the total rate is obtained by summing over the possible i, j and B, ranging over the sets specified by (6) and (7):

$$\begin{aligned}
\Psi_s^A(x) &= \gamma \sum_{B \not\supset A} \sum_{i \in A \setminus B} \sum_{j \in A \cap B} \frac{x^{A-j}}{|B \setminus (A-j)|} \cdot \frac{x^B}{|(A-j) \setminus B|} \\
&= \gamma \sum_{B \not\supset A} \sum_{j \in A \cap B} \frac{x^{A-j} x^B}{|B \setminus (A-j)|} \sum_{i \in A \setminus B} \frac{1}{|(A-j) \setminus B|} \\
&= \gamma \sum_{B \not\supset A} \sum_{j \in A \cap B} \frac{x^{A-j} x^B}{1 + |B \setminus A|} \\
&= \gamma \sum_{B \not\supset A} \frac{x^B}{1 + |B \setminus A|} \underbrace{\sum_{j \in A \cap B} x^{A-j}}_{\substack{\#\text{peers which can assume} \\ \text{label } A \text{ after a swap} \\ \text{with a } B\text{-peer}}} ,
\end{aligned} \tag{8}$$

where, for the third equality, we used the fact that $(A-j) \setminus B = A \setminus B$ if $j \in A \cap B$, so the last sum in the second line equals 1; we also used the fact that $|B \setminus (A-j)| = 1 + |B \setminus A|$.

To summarize, we have explained the derivation of ODE (1) with right-hand side defined by (2), (3), (4), (5) and (8). Note that each of the last four equations has a physical meaning. We will use the following notation below:

$$\begin{aligned}
\varphi_d^A(x) &:= \sum_{B \supsetneq A} x^B = \#\text{peers an } A\text{-peer can download from} \\
\varphi_s^A(d) &:= \sum_{B \not\subset A} x^B = \#\text{peers an } A\text{-peer can swap with} \\
\psi_d^A(x) &:= \sum_{i \in A} x^{A-i} = \#\text{peers which can assume label } A \text{ after a download} \\
\psi_s^{A,B}(x) &:= \sum_{j \in A \cap B} x^{A-j} = \#\text{peers which can assume label } A \text{ after a swap with a } B\text{-peer.}
\end{aligned}$$

3.1. Conservation equation

The ODE exhibits an important conservation principle. Denote by

$$|x| := \sum_{A \subset F} x^A$$

the total volume of peers in the system. Physically, this quantity ought to remain nonnegative at all times. (We will see this below.) The total rate of increase of $|x|$ due to a download is $\sum_{A \subset F} \Phi_d^A(x)$. This should be balanced by the total rate of decrease of $|x|$ due to a download. Similarly, for swaps. We thus expect:

Lemma 1. *For all x,*

$$\sum_{A \subset F} \Phi_d^A(x) = \sum_{A \subset F} \Psi_d^A(x),$$
$$\sum_{A \subset F} \Phi_s^A(x) = \sum_{A \subset F} \Psi_s^A(x).$$

Proof. Consider (5) for $\Psi_d^A(x)$. We have

$$\sum_{A\subset F}\Psi_d^A(x) = \beta\sum_{A\subset F}\sum_{B\supset A}\sum_{i\in A}\frac{x^{A-i}x^B}{1+|B|-|A|}.$$

We are summing over the set

$$\{(i,A,B)\in F\times\mathscr{P}(F)\times\mathscr{P}(F):\ i\in A\subset B\}.$$

We may change variables from (i,A,B) to $(i,\widetilde{A},B)$, where

$$A := \widetilde{A}+i,$$

Then the triple sum above equals

$$\sum_{(i,\widetilde{A},B)\in F\times\mathscr{P}(F)\times\mathscr{P}(F)}\beta\frac{x^{\widetilde{A}}x^B}{|B|-|\widetilde{A}|}\mathbf{1}(\widetilde{A}\subsetneq B)\mathbf{1}(i\notin\widetilde{A})\mathbf{1}(i\in B)$$

$$=\beta\sum_{\widetilde{A}\subset F}\sum_{B\supsetneq\widetilde{A}}x^{\widetilde{A}}x^B\sum_{i\in F}\frac{\mathbf{1}(i\notin\widetilde{A})\mathbf{1}(i\in B)}{|B|-|\widetilde{A}|},$$

and the last sum is clearly equal to 1. Hence the right-hand side of the last display equals $\sum_{\widetilde{A}\subset F}\Phi_d^{\widetilde{A}}(x)$, as claimed. The second claim is proved similarly, by another change of variables. ☐

As a corollary, we obtain:

Corollary 1. *We have*

$$\frac{d}{dt}|x| = \sum_{A\subset F}\alpha^A - \sum_{A\subset F}\delta^A x^A,$$

and $|x(t)|\geq 0$ *for all* t.

Proof. To see the last claim, just notice that $\frac{d}{dt}|x|\geq(\sum_A\alpha^A)-(\max_A\delta^A)|x|$, and use an integrating factor. ☐

3.2. Classification.

Depending on the values of the arrival and departure rates, we may classify the ODE as follows:

- Open system: $\sum_{A\subset F}\alpha^A>0$ and $\sum_{A\subset F}\delta^A>0$.
- Closed conservative system: $\sum_{A\subset F}\alpha^A=0$ and $\sum_{A\subset F}\delta^A=0$.
- Closed dissipative system: $\sum_{A\subset F}\alpha^A=0$ and $\sum_{A\subset F}\delta^A>0$.
- Unstable system: $\sum_{A\subset F}\alpha^A>0$ and $\sum_{A\subset F}\delta^A=0$.

An unstable system necessarily satisfies $\lim_{t\to\infty} |x(t)| = \infty$, and will not be considered further. By Lemma 1, a closed conservative system satisfies

$$\frac{d}{dt}|x| = 0$$

and so $|x(t)| = |x(0)|$ for all $t \geq 0$. On the other hand, a closed dissipative system satisfies

$$\frac{d}{dt}|x| = -\sum_A \delta^A x^A$$

and so $|x(t)|$ is a decreasing function of t. If $\min_A \delta^A > 0$ then $\lim_{t\to\infty} |x(t)| = 0$; otherwise, the ODE may have many equilibria (limit cycles cannot be excluded either). An open system has a unique equilibrium point.

3.3. Examples

The equation for $\dot{x}^\varnothing$. A peer labeled $\varnothing$ can download from any other peer with label $B \neq \varnothing$. Hence

$$\Phi_d^\varnothing(x) = \beta x^\varnothing \sum_{B\neq\varnothing} x^B.$$

But a peer label $\varnothing$ cannot swap with any other peer. So

$$\Phi_s^\varnothing(x) = 0.$$

We also see that

$$\Psi_d^\varnothing(x) = \Psi_s^\varnothing(x) = 0.$$

The reason is that no peer can assume label $\varnothing$ after a download or a swap. Hence

$$\dot{x}^\varnothing = \alpha^\varnothing - \delta^\varnothing x^\varnothing - \beta x^\varnothing \sum_{B\neq\varnothing} x^B.$$

The equation for $\dot{x}^F$. A peer labeled F cannot download from or swap with any other peer. So

$$\Phi_d^F(x) = \Phi_s^F(x) = 0.$$

On the other hand, (5) and (8) give:

$$\Psi_d^F(x) = \beta x^F \sum_{i\in F} x^{F-i}$$

$$\Psi_s^F(x) = \gamma \sum_{B\subsetneq F} x^B \sum_{j\in B} x^{F-j} = \gamma \sum_{j\in F} x^{F-j} \sum_{\substack{B\subsetneq F\\ j\in B}} x^B.$$

Hence

$$\dot{x}^F = \alpha^F - \delta^F x^F + \beta x^F \sum_{i\in F} x^{F-i} + \gamma \sum_{j\in F} x^{F-j} \sum_{\substack{B\subsetneq F\\ j\in B}} x^B.$$

The ODE for $n = 2$. The equation for $\dot{x}^{\varnothing}$ and $\dot{x}^{12}$ have been explained earlier. Using (3) and (4), we have

$$\Phi^1_d(x) = \beta x^1 x^{12}$$
$$\Phi^1_s(x) = \gamma x^1 x^2.$$

Using (5) we have

$$\Psi^1_d(x) = \beta\big(x^1 x^{\varnothing} + \frac{1}{2} x^{12} x^{\varnothing}\big).$$

On the other hand, it is clear that no peer assumes label 1 after a swap, i.e. $\Psi^1_s(x) = 0$. The quantities Φ^2_d, Φ^2_s, Ψ^2_d, and Ψ^2_s are obtained by replacing 1 with 2. Hence

$$\begin{aligned}
\dot{x}^{\varnothing} &= \alpha^{\varnothing} - \delta^{\varnothing} x^{\varnothing} - \beta x^{\varnothing}(x^1 + x^2 + x^{12}) \\
\dot{x}^1 &= \alpha^1 - \delta^1 x^1 - x^1(\beta x^{12} + \gamma x^2) + \beta x^{\varnothing}(x^1 + \tfrac{1}{2} x^{12}) \\
\dot{x}^2 &= \alpha^2 - \delta^2 x^2 - x^2(\beta x^{12} + \gamma x^1) + \beta x^{\varnothing}(x^2 + \tfrac{1}{2} x^{12}) \\
\dot{x}^{12} &= \alpha^{12} - \delta x^{12} + \beta(x^1 + x^2) x^{12} + 2\gamma x^1 x^2.
\end{aligned}$$

The ODE for $n = 1$: no BitTorrent incentives. When $n = 1$, the file is not split into chunks. We refer to this situation as the no BitTorrent incentives case. Clearly,

$$\begin{aligned}
\dot{x}^{\varnothing} &= \alpha^{\varnothing} - \delta^{\varnothing} x^{\varnothing} - \beta x^{\varnothing} x^1 \\
\dot{x}^1 &= \alpha^1 - \delta^1 x^1 + \beta x^{\varnothing} x^1.
\end{aligned}$$

A closed dissipative system is, in this case, described by

$$\begin{aligned}
\dot{x}^{\varnothing} &= -\beta x^{\varnothing} x^1 \\
\dot{x}^1 &= -\delta^1 x^1 + \beta x^{\varnothing} x^1,
\end{aligned}$$

if $\delta^{\varnothing} = 0$. This is the classical equation of the so-called SIR epidemic model.

3.4. The ODE as a limit of a stochastic system

In a previous paper [1], we proved that this differential equation arises as a limit of a continuous-time Markov chain with state space $\mathbb{Z}_+^{\mathscr{P}(F)}$ under appropriate scaling of the parameters (arrival, departure, download and swap rates) and the state space itself. We refer the reader to that paper for the passage from the microscopic to the current, macroscopic, description.

4. The symmetric case

We notice the following symmetry of the vector field $v(x) = (v^A(x), A \subset F)$. Let σ be a permutation of F (a bijection from F into F). For $x = (x^A, A \subset F)$ define σx as a vector in $\mathbb{R}^{\mathscr{P}(F)}$ with components

$$(\sigma x)^A := x^{\sigma A},$$

with $\sigma A := \{\sigma(j) : \; j \in A\}$. From the form of the vector field (2), (3), (4), (5) and (8) we notice that

$$v^{\sigma A}(x) = v^A(\sigma x).$$

Definition 1. *We say that the ODE corresponds to a symmetric network if the rates α^A and δ^A depend on A through its cardinality $|A|$.*

We henceforth consider a symmetric open network such that the ODE has a unique fixed point $x^\star$, namely, $v(x^\star) = 0$. Define $g : \mathbb{R}^{n+1} \to \mathbb{R}^{\mathscr{P}(F)}$ by

$$z = (z^0, z^1, \ldots, z^n) \mapsto gz; \quad (gz)^A := \binom{n}{|A|}^{-1} z^{|A|}.$$

Define also

$$V^k(z^0, \ldots, z^n) = \sum_{|A|=k} v^A(gz). \tag{9}$$

It is then easy to see that if the vector $z = (z^0, \ldots, z^n)$ solves $V^k(z) = 0$, for all $k = 0, \ldots, n$, then the vector $x = gz$ solves $v(x^\star) = 0$. Hence computing the fixed point of a symmetric open network is reduced from having to solve 2^n equations to having to solve $n+1$ equations.

Lemma 2. *The equilibrium point $x^\star$ of an open symmetric network is given by*

$$x^{\star A} = \binom{n}{|A|}^{-1} z^{|A|},$$

where $z = (z^0, \ldots, z^n)$ satisfies

$$V^k(z) = 0, \quad k = 0, \ldots, n.$$

4.1. The fixed point in a special case

We use this simple observation in order to explicitly compute the fixed point $x^\star$ in a special case. We first assume that the network is symmetric and set

$$\lambda_k := \alpha^A, \text{ for all } A \text{ with } |A| = k.$$

Second, we assume that

$$\delta^A = 0, \text{ if } A \neq F$$

and that

$$\delta := \delta^F > 0, \quad \beta > 0.$$

Consider the system of equations

$$V^k(z^0, \ldots, z^n) = 0, \ 0 \le k \le n. \tag{10}$$

We now use (9) and (2) to obtain an expression for $V^k(z)$. This is of the form

$$V^k(z) = \sum_{|A|=k} [\alpha^A - \delta^A (gz)^A] + V_+^k(z) - V_-^k(z), \tag{11}$$

where $V_+^k(z) = \sum_{|A|=k} [\Psi_d^A(gz) + \Psi_s^A(gz)]$, and $V_-^k(z) = \sum_{|A|=k} [\Phi_d^A(gz) + \Phi_s^A(gz)]$. We observe that $V_+^k(z) = V_-^{k-1}(z)$, $k = 1, \ldots, n$ and so, after adding the first k equations together, we obtain

$$0 = \lambda_0 - \beta z^0 \sum_{m=1}^{n} z^m, \tag{12}$$

$$0 = \sum_{m=0}^{k} \binom{n}{m} \lambda_m - z^k \Bigg(\gamma \sum_{m=1}^{n} z^m - \frac{\gamma}{\binom{n}{k}} \sum_{m=1}^{k} \binom{n-m}{k-m} z^m + \frac{\beta - \gamma}{\binom{n}{k}} \sum_{m=k+1}^{n} \binom{m}{k} z^m \Bigg), \quad 1 \le k \le n-1, \tag{13}$$

$$0 = \sum_{m=0}^{n} \binom{n}{m} \lambda_m - \delta z^n. \tag{14}$$

The case $n = 1$ has already been examined in [1] so suppose now that $n \geq 2$. We further assume

$$\lambda_{n-1} + \lambda_n > 0,$$

and, to simplify the model substantially, we also set

$$\lambda_0 = \cdots = \lambda_{n-2} = 0.$$

Then there is a unique solution to (12)–(14) with $z^k \geq 0$ for all $k \in \{0, \ldots, n\}$. To see this, solve (14) first. This gives a positive solution z^n. Since $\binom{n-m}{k-m} < \binom{n}{k}$ and $\binom{m}{k} \leq \binom{n}{k}$ the solution of (13) results in a nonnegative z^k. The complete solution is given by:

$$z^n = \frac{n\lambda_{n-1} + \lambda_n}{\delta}, \tag{15}$$

$$z^{n-1} = \begin{cases} -\frac{n\beta z^n}{2\gamma(n-1)} + \sqrt{\left(\frac{n\beta z^n}{2\gamma(n-1)}\right)^2 + \frac{n^2 \lambda_{n-1}}{\gamma(n-1)}}, & \text{if } \gamma > 0, \\ \frac{n\lambda_{n-1}}{\beta z^n}, & \text{if } \gamma = 0, \end{cases} \tag{16}$$

$$z^{n-2} = \cdots = z^0 = 0.$$

Define

$$w^k := x^{\star F \setminus \{k\}}.$$

We will now show that there are no fixed points with $w^j \neq w^k$ for some $j, k \in \{1, \ldots, n\}$, if $\alpha^A = 0$ for all A such that $|A| < n-1$. If $\alpha^A = 0$ for all A such that $|A| < n-1$, we have $x^{\star A} = 0$ if $|A| < n-1$ for all fixed points $x^\star$. The reason is that the amount of peers with fewer than $n-1$ chunks will vanish, since there is no inflow from such peers other than from peers with fewer chunks. We now have

$$0 = \lambda_{n-1} - w^1 \Bigg(\beta x^{\star F} - \gamma w^1 + \gamma \sum_{m=1}^{n} w^m \Bigg) \tag{17}$$

$$0 = \lambda_{n-1} - w^2 \Bigg(\beta x^{\star F} - \gamma w^2 + \gamma \sum_{m=1}^{n} w^m \Bigg) \tag{18}$$

$$\vdots$$

$$0 = \lambda_{n-1} - w^n \Bigg(\beta x^{\star F} - \gamma w^n + \gamma \sum_{m=1}^{n} w^m \Bigg)$$

$$0 = \lambda_n + \beta x^{\star F} \sum_{k=1}^{n} w^k + \gamma \sum_{k=1}^{n} w^k \Bigg(-w^k + \sum_{m=1}^{n} w^m \Bigg) - \delta x^{\star F}. \tag{19}$$

Adding up these equations results in

$$x^{\star F} = (\lambda_n + n\lambda_{n-1})/\delta$$

which is strictly positive, due to our assumptions. Suppose now that $w^1 \neq w^2$. Subtracting (18) from (17) gives

$$0 = (w^2 - w^1)(\beta x^{\star F} + \sum_{k=1}^{n} w^k - (w^1 + w^2)),$$

and so

$$\sum_{k=3}^{n} w^k + \beta x^{\star F} = 0,$$

But the left-hand side is positive, and so we obtain a contradiction. Therefore, $w^1 = w^2$.

Hence, there is a unique fixed point, given by (15)–(16), with $z^k = 0$ for all $k < n - 1$, if $\alpha^A = 0$ for all A such that $|A| < n - 1$.

4.2. Local stability

We now examine the local stability of the system around the equilibrium point computed in the previous section. It will be shown that the fixed point is locally attracting by examining the eigenvalues of the Jacobian matrix of $\mathbf{V} := (V^0, \dots, V^n)$ at the fixed point $z := (z^0, \dots, z^n)$. We have

$$\begin{aligned}
V^0(z) &= \lambda_0 - \beta z^0 \sum_{m=1}^{n} z^m, \\
V^1(z) &= n\lambda_1 - z^1\left(\gamma \sum_{m=1}^{n} z^m - \frac{\gamma}{n} z^1 + \frac{\beta - \gamma}{n} \sum_{m=2}^{n} m z^m\right) + \beta z^0 \sum_{m=1}^{n} z^m, \\
V^k(z) &= \binom{n}{k}\lambda_k - z^k\left(\gamma \sum_{m=1}^{n} z^m - \frac{\gamma}{\binom{n}{k}} \sum_{m=1}^{k} \binom{n-m}{k-m} z^m + \frac{\beta - \gamma}{\binom{n}{k}} \sum_{m=k+1}^{n} \binom{m}{k} z^m\right) \\
&\quad + z^{k-1}\left(\gamma \sum_{m=1}^{n} z^m - \frac{\gamma}{\binom{n}{k-1}} \sum_{m=1}^{k-1} \binom{n-m}{k-1-m} z^m + \frac{\beta - \gamma}{\binom{n}{k-1}} \sum_{m=k}^{n} \binom{m}{k-1} z^m\right), \\
&\qquad\qquad 2 \leq k \leq n-1, \\
V^n(z) &= \lambda_n + z^{n-1}\left(\beta z^n + \frac{\gamma}{n} \sum_{m=1}^{n-1} m z^m\right) - \delta z^n.
\end{aligned}$$

The Jacobian matrix J of $\mathbf{V}$ at the fixed point is almost lower triangular. The only nonzero entry in the upper triangular part of J is the entry at row $n - 1$ and column n, which is equal to $-\beta z^{n-1}$. The matrix J itself equals

$$\begin{pmatrix}
-\frac{0\times\gamma+(n-0)\beta}{n} z^{n-1} - \beta z^n & 0 & \cdots & \cdots & \cdots & 0 \\
j_{1,0} & -\frac{1\times\gamma+(n-1)\beta}{n} z^{n-1} - \beta z^n & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
j_{n-2,0} & \cdots & j_{n-2,n-3} & -\frac{(n-2)\times\gamma+2\beta}{n} z^{n-1} - \beta z^n & 0 & 0 \\
j_{n-1,0} & \cdots & \cdots & j_{n-1,n-2} & -2\gamma\frac{n-1}{n} z^{n-1} - \beta z^n & -\beta z^{n-1} \\
j_{n,0} & \cdots & \cdots & j_{n,n-2} & 2\gamma\frac{n-1}{n} z^{n-1} + \beta z^n & \beta z^{n-1} - \delta
\end{pmatrix}.$$

The characteristic polynomial is given by

$$\det(\kappa I - J) = \left(\kappa^2 + \kappa\left(\beta z^n + \delta + 2\gamma\frac{n-1}{n}z^{n-1} - \beta z^{n-1}\right) + \delta\left(\beta z^n + 2\gamma\frac{n-1}{n}z^{n-1}\right)\right)$$
$$\times \prod_{k=0}^{n-2}\left(\kappa + \frac{\gamma k + \beta(n-k)}{n}z^{n-1} + \beta z^n\right).$$

All eigenvalues have negative real parts if

$$\beta z^n + \delta + 2\gamma\frac{n-1}{n}z^{n-1} - \beta z^{n-1} > 0. \tag{20}$$

Assume first $\gamma > 0$. If we use (16), condition (20) is equivalent to

$$\frac{n\beta z^n}{2\gamma(n-1)} + \frac{\delta}{\beta} > \sqrt{\left(\frac{n\beta z^n}{2\gamma(n-1)}\right)^2 + \frac{n^2\lambda_{n-1}}{\gamma(n-1)}\left(1 - \frac{2\gamma(n-1)}{\beta n}\right)}. \tag{21}$$

If $\beta \le \frac{2\gamma(n-1)}{n}$, the right-hand side of (21) is nonpositive and so (21) holds. Suppose then that

$$\beta > \frac{2\gamma(n-1)}{n}. \tag{22}$$

After squaring (21) and using (15) we obtain the condition

$$\frac{\beta^2\lambda_n n}{\gamma(n-1)} + \delta^2 + 4n\beta\lambda_{n-1} - 4\gamma(n-1)\lambda_{n-1} > (\beta z^n)^2\left(1 - \frac{n\beta}{\gamma(n-1)}\right). \tag{23}$$

Due to (22) the left hand side of (23) is positive and the right hand side is negative so (23) holds. The case $\gamma = 0$ is simpler. Let $\gamma = 0$ in (20) and substitute (15) and (16). The new condition is

$$\beta z^n + \frac{\delta\lambda_n}{n\lambda_{n-1} + \lambda_n} > 0,$$

which is clearly true. We conclude that all eigenvalues have strictly negative real part, and, therefore, the fixed point is locally attracting. It is easy to see that local stability at the fixed point also holds for the case $n = 1$.

4.3. Global stability

We next consider the problem of global stability, for the simplified open network. Let $m \in \{0, \dots, n-1\}$ and let $\lambda_0 = \cdots = \lambda_m = 0$. Define

$$\mathbb{S}_m := \{0\}^{m+1} \times [0, \infty)^{n-m}.$$

Then $\mathbb{S}_m$ is a trapping region. This follows since $V^k(z) = 0$ for every $k \in \{0, \dots, m\}$ when $z \in \mathbb{S}_m$ and

$$V^k(z) = \binom{n}{k}\lambda_k + z^{k-1}\left(\gamma\sum_{m=1}^{n} z^m - \frac{\gamma}{\binom{n}{k-1}}\sum_{m=1}^{k-1}\binom{n-m}{k-1-m}z^m + \frac{\beta-\gamma}{\binom{n}{k-1}}\sum_{m=k}^{n}\binom{m}{k-1}z^m\right) \ge 0$$

for every $k \in \{m+1, \ldots, n\}$ when $z \in \mathbb{S}_m$ and $z^k = 0$. Similarly, if $\lambda_0 = \cdots = \lambda_n = 0$ we have the trivial trapping region $\{0\}^{n+1}$. Consider now the set $\mathbb{S}_{n-2} = \{0\}^{n-1} \times [0, \infty)^2$. The dynamical system reduces to

$$\frac{dy_{n-1}}{dt} = V^{n-1}(0, \ldots, 0, y_{n-1}, y_n)$$
$$\frac{dy_n}{dt} = V^n(0, \ldots, 0, y_{n-1}, y_n)$$

or, equivalently,

$$\frac{dy_{n-1}}{dt} = n\lambda_{n-1} - \gamma(n-1)y_{n-1}^2/n - \beta y_{n-1}y_n$$
$$\frac{dy_n}{dt} = \lambda_n + \gamma(n-1)y_{n-1}^2/n + \beta y_{n-1}y_n - \delta y_n.$$

Our goal is to show that z (solving $V(z) = 0$) is globally attracting. We will make use of Dulac's criterion [4] which states that there are no closed orbits lying entirely in $(0, \infty)^2$, if there exists a scalar function $u(y_n, y_{n-1})$ such that the divergence of the vector field $u(y_n, y_{n-1})\big(\frac{dy_{n-1}}{dt}, \frac{dy_n}{dt}\big)$ is strictly negative everywhere or strictly positive everywhere. Choose

$$u(x, y) := \frac{1}{xy}, \quad x, y > 0.$$

For this choice, we have

$$\begin{aligned}
\nabla \bullet \left(u(y_{n-1}, y_n)\Big(\frac{dy_{n-1}}{dt}, \frac{dy_n}{dt}\Big)\right) &= \frac{\partial}{\partial y_{n-1}}\left(\frac{n\lambda_{n-1}}{y_{n-1}y_n} - \frac{\gamma(n-1)y_{n-1}}{ny_n} - \beta\right) \\
&+ \frac{\partial}{\partial y_n}\left(\frac{\lambda_n}{y_{n-1}y_n} + \frac{\gamma(n-1)y_{n-1}}{ny_n} + \beta - \frac{\delta}{y_{n-1}}\right) \\
&= -\frac{n\lambda_{n-1}}{y_{n-1}^2 y_n} - \frac{\gamma(n-1)}{ny_n} - \frac{\lambda_n}{y_{n-1}y_n^2} - \frac{\gamma(n-1)y_{n-1}}{ny_n^2},
\end{aligned}$$

which is strictly negative for all $y_{n-1} > 0$ and $y_n > 0$. We conclude that there exist no closed orbits lying entirely in $(0, \infty)^2$.

In addition, however, we need to ensure that there is no orbit touching the boundary of the positive orthant. To see this, we examine the direction of the vector field (V^{n-1}, V^n) at points $(0, y_n)$ or $(y_{n-1}, 0)$.

Suppose that $\lambda_{n-1} > 0$ and $\lambda_n + \gamma > 0$. Then, for all $(y_{n-1}, y_n) \in [0, \infty)^2 \setminus \{(0, 0)\}$,

$$\begin{aligned}
V^{n-1}(0, \ldots, 0, 0, y_n) &= n\lambda_{n-1} > 0, \\
V^n(0, \ldots, 0, y_{n-1}, 0) &= \lambda_n + \gamma(n-1)y_{n-1}^2/n > 0. \qquad (24)
\end{aligned}$$

From this, it is clear that there are no closed orbits in $[0, \infty)^2$ at all. This, together with the result of local stability at z, implies that z is globally attracting.

4.4. Conclusions

We summarize with the theorem

Theorem 1. *Let $\delta\beta(\lambda_{n-1}+\lambda_n) > 0$ and $\lambda_0 = \ldots = \lambda_{n-2} = 0$ if $n \geq 2$. Then there is a unique fixed point*

$$\begin{aligned} z^n &= \frac{n\lambda_{n-1}+\lambda_n}{\delta}, \\ z^{n-1} &= \begin{cases} -\frac{n\beta z^n}{2\gamma(n-1)} + \sqrt{\left(\frac{n\beta z^n}{2\gamma(n-1)}\right)^2 + \frac{n^2\lambda_{n-1}}{\gamma(n-1)}}, & \text{if } \gamma > 0, \\ \frac{n\lambda_{n-1}}{\beta z^n}, & \text{if } \gamma = 0 \text{ or } n = 1, \end{cases} \\ z^{n-2} &= \ldots = z^0 = 0 \text{ if } n \geq 2 \end{aligned}$$

to the system of equations

$$\frac{d(y_0,\ldots,y_n)}{dt} = (V^0,\ldots,V^n),\ y_k \geq 0\ \forall k \in \{0,\ldots,n\}.$$

Moreover, the fixed point is locally attracting. If $n \geq 2$, $\lambda_{n-1} > 0$ and $\lambda_n + \gamma > 0$ the fixed point is even globally attracting in the trapping region $\{0\}^{n-1} \times [0,\infty)^2$. If $n = 1$, $\lambda_{n-1} > 0$ and $\lambda_n > 0$, the fixed point is even globally attracting in the trapping region $[0,\infty)^2$.

5. Open problems and future research

If $\gamma = 0$, $\beta > 0$, $\delta > 0$ and $\lambda_0 + \lambda_1 + \ldots + \lambda_n > 0$, the unique fixed point is given by the recursion formula

$$\begin{aligned} z^n &= \frac{\sum_{m=0}^n \binom{n}{m}\lambda_m}{\delta}, \\ z^k &= \frac{\binom{n}{k}\sum_{m=0}^k \binom{n}{m}\lambda_m}{\beta\sum_{m=k+1}^n \binom{m}{k} z^m}, \text{ if } k \in \{n-1,\ldots,0\}. \end{aligned}$$

The entries of the $(n+1)\times(n+1)$ Jacobian matrix are now

$$j_{k,m} := \frac{\partial V^k}{\partial z^m} = \begin{cases} -\beta\sum_{l=1}^n z^l & \text{if } k = m = 0, \\ -\beta z^0 & \text{if } m > k = 0, \\ \frac{\beta}{\binom{n}{k-1}}\sum_{l=k}^n \binom{l}{k-1} z^l & \text{if } m+1 = k \in \{1,\ldots,n\}, \\ \frac{\beta}{\binom{n}{k}}\left((n-k+1)z^{k-1} - \sum_{l=k+1}^n \binom{l}{k} z^l\right) & \text{if } m = k \in \{1,\ldots,n-1\}, \\ \frac{\beta\binom{m}{k}}{\binom{n}{k}}\left(z^{k-1}\frac{n-k+1}{m-k+1} - z^k\right) & \text{if } m > k \in \{1,\ldots,n-1\}, \\ \beta z^{n-1} - \delta & \text{if } m = k = n, \\ 0 & \text{otherwise.} \end{cases}$$

It is not known, however, if the fixed point is locally attracting. We have tested many randomized parameter values for $n = 6$ and found numerical evidence that an eigenvalue with positive real part exists. For example if

$$\begin{aligned}
\lambda_0 &= 0.227982627176583 \\
\lambda_1 &= 0.890228681868976 \\
\lambda_2 &= 0.967185953221958 \\
\lambda_3 &= 0.104979958526759 \\
\lambda_4 &= 0.024896230450952 \\
\lambda_5 &= 0.085678511554328 \\
\lambda_6 &= 0.134252480012138 \\
\beta &= 0.008450605167450 \\
\delta &= 0.962784200412016,
\end{aligned}$$

then the maximal real part of the eigenvalues is 0.011036574008719. It should be pointed out, however, that the real part of this eigenvalue is small, and this could be a numerical artifact. Consequently, it seems that the fixed point is not locally attracting for every parameter value if $n = 6$. For $n = 5$ however, we have not found an eigenvalue with nonnegative real part.

We note that the work on this kind of models was initially motivated by Massoulié and Vojnovic [5], who introduced a model based on coupon collection. More recent work by Norros *et al.* [6] and Zhu and Hajek [7] considers the possibility of totally selfish peers. Namely, a peer leaves the network immediately, once he acquires all the chunks of the file. To accommodate this, a seeder, present at all times, is required. The corresponding stochastic system is then seen to be, in some cases, unstable. A variety of differential equations associated with this phenomenon, but only in the case of $n = 2$ chunks, is proposed by Norros *et al.* We believe that generalizations of these equations in higher dimensions, as well as their study of local global asymptotic stability will provide better understanding of peer-to-peer networks.

Finally, we point out that, in this paper, we assumed that the majority of the arrival rates are zero. This may not be a realistic assumption, but it yields explicit and tractable results.

References

[1] George Kesidis, Takis Konstantopoulos and Perla Sousi, A stochastic epidemiological model and a deterministic limit for BitTorrent-like peer-to-peer file-sharing networks, Springer Lecture Notes in Computer Science, 5425, 2009, pp. 26-36. Full version at: arXiv:0811.1003

[2] www.kazaa.com.

[3] B. Cohen, BitTorrent specification: www.bittorrent.org.

[4] S.H. Strogatz, Nonlinear dynamics and chaos: with application to physics, biology, chemistry, and engineering, Addison–Wesley, Reading, 1994.

[5] L. Massoulié, M. Vojnovic, Coupon replication systems, Proc. ACM SIGMETRICS, Banf, Canada, 2005.

[6] I. Norros, H. Reittu, T. Eirola, On the stability of two-chunk file-sharing systems, Queueing Systems 67 (2009) 183-206.

[7] J. Zhu, B. Hajek, The missing piece syndrome in peer-to-peer communication, Proc. ISIT 2010, pp. 1748-1752.