Supplement worksheet for the paper: COMPUTING OF B-SERIES BY AUTOMATIC DIFFERENTIATION

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Setting up the enviroment

restart :
with(*VectorCalculus*) :

A function with well behaving derivatives

The derivatives of f behave well due to the simple combination of trigonometric functions

$f := (x, y, z) \rightarrow \langle \sin(x) \cdot \cos(2 \cdot y), \sin(x) \cdot \tan(y) + \cos(x^2), \tan(z) \cdot y^2 \cdot \cos(x) \rangle;$

$(x, y, z) \rightarrow VectorCalculus:\text{-}<,>(\sin(x)\cos(2y), \sin(x)\tan(y) + \cos(x^2), \tan(z)\, y^2 \cos(x))$ **(1.1)**

Some examples:

Diff(*f*(*x*, *y*, *z*), *x*\$5, *y*\$5) = *diff*(*f*(*x*, *y*, *z*), *x*\$5, *y*\$5);

$$\frac{\partial^{10}}{\partial y^5\, \partial x^5} (\sin(x)\cos(2y))e_x + (\sin(x)\tan(y) + \cos(x^2))e_y + (\tan(z)\, y^2\cos(x))e_z = -32\cos(x)\sin(2y)e_x + \left(88\cos(x)\left(1+\tan(y)^2\right)^2\tan(y)^2 + 16\cos(x)\left(1+\tan(y)^2\right)^3 + 16\cos(x)\tan(y)^4\left(1+\tan(y)^2\right)\right)e_y \quad \textbf{(1.2)}$$

Diff(*f*(*x*, *y*, *z*), *x*\$10, *y*\$5) = *diff*(*f*(*x*, *y*, *z*), *x*\$10, *y*\$5);

$$\frac{\partial^{15}}{\partial y^5\, \partial x^{10}} (\sin(x)\cos(2y))e_x + (\sin(x)\tan(y) + \cos(x^2))e_y + (\tan(z)\, y^2\cos(x))e_z = 32\sin(x)\sin(2y)e_x + \left(-88\sin(x)\left(1+\tan(y)^2\right)^2\tan(y)^2 - 16\sin(x)\left(1+\tan(y)^2\right)^3 - 16\sin(x)\tan(y)^4\left(1+\tan(y)^2\right)\right)e_y \quad \textbf{(1.3)}$$

Diff(*f*(*x*, *y*, *z*), *x*\$15, *y*\$15) = *diff*(*f*(*x*, *y*, *z*), *x*\$15, *y*\$15);

$$\frac{\partial^{30}}{\partial y^{15}\, \partial x^{15}} (\sin(x)\cos(2y))e_x + (\sin(x)\tan(y) + \cos(x^2))e_y + (\tan(z)\, y^2\cos(x))e_z = -32768\cos(x)\sin(2y)e_x + \left(-134094848\cos(x)\left(1+\tan(y)^2\right)^2\tan(y)^{12} - 13754155008\cos(x)\left(1+\tan(y)^2\right)^3\tan(y)^{10} - 182172651520\cos(x)\left(1+\tan(y)^2\right)^4\tan(y)^8 - 559148810240\cos(x)\left(1+\tan(y)^2\right)^5\tan(y)^6 - 460858269696\cos(x)\left(1+\tan(y)^2\right)^6\tan(y)^4 - 89702612992\cos(x)\left(1+\tan(y)^2\right)^7\tan(y)^2 - 1903757312\cos(x)\left(1+\tan(y)^2\right)^8 - 16384\cos(x)\tan(y)^{14}\left(1+\tan(y)^2\right)\right)e_y \quad \textbf{(1.4)}$$

Thus, obtaining the value at a given point (x,y,z) is relatively easy.

evalf(*subs*(*x* = 2, *y* = 3, *z* = 4, *diff*(*f*(*x*, *y*, *z*), *x*\$1, *y*\$1)));

$(-0.2325557512)e_x + (-0.4246027392)e_y + (-6.316823472)e_z$ **(1.5)**

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evalf(subs(x = 2, y = 3, z = 4, diff(f(x, y, z), x$10, y$10)));
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$$(894.0342999)e_x + (56039.54501)e_y + (0.)e_z \tag{1.6}$$

A function with derivatives that are given by more and more complicated formulas

The derivatives of g are getting more and more complicated as the order increases

$$g := (x, y, z) \to \left\langle \frac{\exp\left(\frac{\sin(x)}{\cos(2\cdot y)}\right)}{\ln(x^2\cdot y^2+5)\cdot z}, \frac{\exp\left(\frac{\sin(x)\cdot\tan(y)}{\ln(x^2+1)\cdot y\cdot z}\right)}{x^2+z}, \tan(z)\cdot y^2\cdot\cos(x) \right\rangle;$$

$$(x, y, z) \to VectorCalculus{:-}{<,>}\left(\mathrm{e}^{\frac{\sin(x)}{\cos(2y)}}\frac{1}{\ln(x^2y^2+5)\,z}, \mathrm{e}^{\frac{\sin(x)\tan(y)}{\ln(x^2+1)\,yz}}\frac{1}{x^2+z}, \tan(z)\,y^2\cos(x)\right) \tag{2.1}$$

Some examples:

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Diff(g(x, y, z), x$1, y$1) = diff(g(x, y, z), x$1, y$1);
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$$\frac{\partial^2}{\partial y\,\partial x}\left(\left(\frac{\mathrm{e}^{\frac{\sin(x)}{\cos(2y)}}}{\ln(x^2y^2+5)\,z}\right)e_x + \left(\frac{\mathrm{e}^{\frac{\sin(x)\tan(y)}{\ln(x^2+1)\,yz}}}{x^2+z}\right)e_y + \left(\tan(z)\,y^2\cos(x)\right)e_z\right) \tag{2.2}$$

$$= \left(\frac{2\cos(x)\,\mathrm{e}^{\frac{\sin(x)}{\cos(2y)}}\sin(2y)}{\cos(2y)^2\ln(x^2y^2+5)\,z} + \frac{2\cos(x)\sin(x)\sin(2y)\,\mathrm{e}^{\frac{\sin(x)}{\cos(2y)}}}{\cos(2y)^3\ln(x^2y^2+5)\,z}\right.$$

$$- \frac{2\cos(x)\,\mathrm{e}^{\frac{\sin(x)}{\cos(2y)}}x^2y}{\cos(2y)\ln(x^2y^2+5)^2z\,(x^2y^2+5)} - \frac{4\sin(x)\sin(2y)\,\mathrm{e}^{\frac{\sin(x)}{\cos(2y)}}x\,y^2}{\cos(2y)^2\ln(x^2y^2+5)^2z\,(x^2y^2+5)}$$

$$+ \frac{8\,\mathrm{e}^{\frac{\sin(x)}{\cos(2y)}}x^3y^3}{\ln(x^2y^2+5)^3z\,(x^2y^2+5)^2} - \frac{4\,\mathrm{e}^{\frac{\sin(x)}{\cos(2y)}}x\,y}{\ln(x^2y^2+5)^2z\,(x^2y^2+5)}$$

$$\left.+ \frac{4\,\mathrm{e}^{\frac{\sin(x)}{\cos(2y)}}x^3y^3}{\ln(x^2y^2+5)^2z\,(x^2y^2+5)^2}\right)e_x + \left(\frac{1}{x^2+z}\left(\left(\frac{\cos(x)\,(1+\tan(y)^2)}{\ln(x^2+1)\,y\,z}\right.\right.\right.$$

$$\left.- \frac{\cos(x)\tan(y)}{\ln(x^2+1)\,y^2z} - \frac{2\sin(x)\,(1+\tan(y)^2)\,x}{\ln(x^2+1)^2\,y\,z\,(x^2+1)} + \frac{2\sin(x)\tan(y)\,x}{\ln(x^2+1)^2\,y^2z\,(x^2+1)}\right)$$

$$e^{\frac{\sin(x)\tan(y)}{\ln(x^2+1)yz}}\Bigg) + \frac{1}{x^2+z}\Bigg(\Bigg(\frac{\cos(x)\tan(y)}{\ln(x^2+1)\,y\,z}$$

$$-\frac{2\sin(x)\tan(y)\,x}{\ln(x^2+1)^2\,y\,z\,(x^2+1)}\Bigg)\Bigg(\frac{\sin(x)\,(1+\tan(y)^2)}{\ln(x^2+1)\,y\,z}$$

$$-\frac{\sin(x)\tan(y)}{\ln(x^2+1)\,y^2\,z}\Bigg)e^{\frac{\sin(x)\tan(y)}{\ln(x^2+1)yz}}\Bigg)$$

$$-\frac{2\left(\frac{\sin(x)\,(1+\tan(y)^2)}{\ln(x^2+1)\,y\,z}-\frac{\sin(x)\tan(y)}{\ln(x^2+1)\,y^2\,z}\right)e^{\frac{\sin(x)\tan(y)}{\ln(x^2+1)yz}}\,x}{(x^2+z)^2}\Bigg)e_y$$

$$-2\tan(z)\,y\sin(x)e_z$$

Diff$(g(x,y,z), x\$2, y\$1) =$ *diff*$(g(x,y,z), x\$2, y\$1);$

$$\frac{\partial^3}{\partial y\,\partial x^2}\left(\left(\frac{e^{\frac{\sin(x)}{\cos(2y)}}}{\ln(x^2y^2+5)\,z}\right)e_x + \left(\frac{e^{\frac{\sin(x)}{\ln(x^2+1)yz}\,\tan(y)}}{x^2+z}\right)e_y + (\tan(z)\,y^2\cos(x))e_z\right) = \Bigg(\tag{2.3}$$

$$-\frac{2\sin(x)\,e^{\frac{\sin(x)}{\cos(2y)}}\sin(2y)}{\cos(2y)^2\ln(x^2y^2+5)\,z} - \frac{2\sin(x)^2\sin(2y)\,e^{\frac{\sin(x)}{\cos(2y)}}}{\cos(2y)^3\ln(x^2y^2+5)\,z}$$

$$+\frac{2\sin(x)\,e^{\frac{\sin(x)}{\cos(2y)}}\,x^2y}{\cos(2y)\ln(x^2y^2+5)^2\,z\,(x^2y^2+5)} + \frac{4\cos(x)^2\,e^{\frac{\sin(x)}{\cos(2y)}}\sin(2y)}{\cos(2y)^3\ln(x^2y^2+5)\,z}$$

$$+\frac{2\cos(x)^2\sin(x)\sin(2y)\,e^{\frac{\sin(x)}{\cos(2y)}}}{\cos(2y)^4\ln(x^2y^2+5)\,z} - \frac{2\cos(x)^2\,e^{\frac{\sin(x)}{\cos(2y)}}\,x^2y}{\cos(2y)^2\ln(x^2y^2+5)^2\,z\,(x^2y^2+5)}$$

$$-\frac{8\cos(x)\,e^{\frac{\sin(x)}{\cos(2y)}}\,xy^2\sin(2y)}{\cos(2y)^2\ln(x^2y^2+5)^2\,z\,(x^2y^2+5)} - \frac{8\cos(x)\sin(x)\sin(2y)\,e^{\frac{\sin(x)}{\cos(2y)}}\,xy^2}{\cos(2y)^3\ln(x^2y^2+5)^2\,z\,(x^2y^2+5)}$$

$$+\frac{16\cos(x)\,e^{\frac{\sin(x)}{\cos(2y)}}\,x^3y^3}{\cos(2y)\ln(x^2y^2+5)^3\,z\,(x^2y^2+5)^2} - \frac{8\cos(x)\,e^{\frac{\sin(x)}{\cos(2y)}}\,xy}{\cos(2y)\ln(x^2y^2+5)^2\,z\,(x^2y^2+5)}$$

$$+\frac{8\cos(x)\,e^{\frac{\sin(x)}{\cos(2y)}}\,x^3y^3}{\cos(2y)\ln(x^2y^2+5)^2\,z\,(x^2y^2+5)^2} + \frac{16\sin(x)\sin(2y)\,e^{\frac{\sin(x)}{\cos(2y)}}\,x^2y^4}{\cos(2y)^2\ln(x^2y^2+5)^3\,z\,(x^2y^2+5)^2}$$

$$- \frac{48\, e^{\frac{\sin(x)}{\cos(2y)}} x^4 y^5}{\ln(x^2 y^2 + 5)^4 z (x^2 y^2 + 5)^3} + \frac{40\, e^{\frac{\sin(x)}{\cos(2y)}} x^2 y^3}{\ln(x^2 y^2 + 5)^3 z (x^2 y^2 + 5)^2}$$

$$- \frac{48\, e^{\frac{\sin(x)}{\cos(2y)}} x^4 y^5}{\ln(x^2 y^2 + 5)^3 z (x^2 y^2 + 5)^3} - \frac{4 \sin(x) \sin(2y)\, e^{\frac{\sin(x)}{\cos(2y)}} y^2}{\cos(2y)^2 \ln(x^2 y^2 + 5)^2 z (x^2 y^2 + 5)}$$

$$- \frac{4\, e^{\frac{\sin(x)}{\cos(2y)}} y}{\ln(x^2 y^2 + 5)^2 z (x^2 y^2 + 5)} + \frac{20\, e^{\frac{\sin(x)}{\cos(2y)}} y^3 x^2}{\ln(x^2 y^2 + 5)^2 z (x^2 y^2 + 5)^2}$$

$$+ \frac{8 \sin(x) \sin(2y)\, e^{\frac{\sin(x)}{\cos(2y)}} x^2 y^4}{\cos(2y)^2 \ln(x^2 y^2 + 5)^2 z (x^2 y^2 + 5)^2} - \frac{16\, e^{\frac{\sin(x)}{\cos(2y)}} x^4 y^5}{\ln(x^2 y^2 + 5)^2 z (x^2 y^2 + 5)^3} \Bigg) e_x$$

$$+ \Bigg(\frac{1}{x^2 + z} \Bigg(\Bigg(- \frac{\sin(x) (1 + \tan(y)^2)}{\ln(x^2 + 1)\, y z} + \frac{\sin(x) \tan(y)}{\ln(x^2 + 1)\, y^2 z} - \frac{4 \cos(x) (1 + \tan(y)^2)\, x}{\ln(x^2 + 1)^2 y z (x^2 + 1)}$$

$$+ \frac{4 \cos(x) \tan(y)\, x}{\ln(x^2 + 1)^2 y^2 z (x^2 + 1)} + \frac{8 \sin(x) (1 + \tan(y)^2)\, x^2}{\ln(x^2 + 1)^3 y z (x^2 + 1)^2}$$

$$- \frac{8 \sin(x) \tan(y)\, x^2}{\ln(x^2 + 1)^3 y^2 z (x^2 + 1)^2} - \frac{2 \sin(x) (1 + \tan(y)^2)}{\ln(x^2 + 1)^2 y z (x^2 + 1)}$$

$$+ \frac{2 \sin(x) \tan(y)}{\ln(x^2 + 1)^2 y^2 z (x^2 + 1)} + \frac{4 \sin(x) (1 + \tan(y)^2)\, x^2}{\ln(x^2 + 1)^2 y z (x^2 + 1)^2}$$

$$- \frac{4 \sin(x) \tan(y)\, x^2}{\ln(x^2 + 1)^2 y^2 z (x^2 + 1)^2} \Bigg) e^{\frac{\sin(x) \tan(y)}{\ln(x^2 + 1) y z}} \Bigg) + \frac{1}{x^2 + z} \Bigg(\Bigg(- \frac{\sin(x) \tan(y)}{\ln(x^2 + 1)\, y z}$$

$$- \frac{4 \cos(x) \tan(y)\, x}{\ln(x^2 + 1)^2 y z (x^2 + 1)} + \frac{8 \sin(x) \tan(y)\, x^2}{\ln(x^2 + 1)^3 y z (x^2 + 1)^2} - \frac{2 \sin(x) \tan(y)}{\ln(x^2 + 1)^2 y z (x^2 + 1)}$$

$$+ \frac{4 \sin(x) \tan(y) x^2}{\ln(x^2+1)^2 y z (x^2+1)^2} \Bigg) \left(\frac{\sin(x) \left(1+\tan(y)^2\right)}{\ln(x^2+1) y z} \right.$$

$$- \left. \frac{\sin(x) \tan(y)}{\ln(x^2+1) y^2 z} \right) e^{\frac{\sin(x) \tan(y)}{\ln(x^2+1) y z}} \Bigg) + \frac{1}{x^2+z} \Bigg(2 \left(\frac{\cos(x) \tan(y)}{\ln(x^2+1) y z} \right.$$

$$- \left. \frac{2 \sin(x) \tan(y) x}{\ln(x^2+1)^2 y z (x^2+1)} \right) e^{\frac{\sin(x) \tan(y)}{\ln(x^2+1) y z}} \left(\frac{\cos(x) \left(1+\tan(y)^2\right)}{\ln(x^2+1) y z} - \frac{\cos(x) \tan(y)}{\ln(x^2+1) y^2 z} \right.$$

$$- \left. \frac{2 \sin(x) \left(1+\tan(y)^2\right) x}{\ln(x^2+1)^2 y z (x^2+1)} + \frac{2 \sin(x) \tan(y) x}{\ln(x^2+1)^2 y^2 z (x^2+1)} \right) \Bigg)$$

$$+ \frac{1}{x^2+z} \left(\left(\frac{\cos(x) \tan(y)}{\ln(x^2+1) y z} - \frac{2 \sin(x) \tan(y) x}{\ln(x^2+1)^2 y z (x^2+1)} \right)^2 \left(\frac{\sin(x) \left(1+\tan(y)^2\right)}{\ln(x^2+1) y z} \right. \right.$$

$$- \left. \frac{\sin(x) \tan(y)}{\ln(x^2+1) y^2 z} \right) e^{\frac{\sin(x) \tan(y)}{\ln(x^2+1) y z}} \Bigg) - \frac{1}{(x^2+z)^2} \left(4 \left(\frac{\cos(x) \left(1+\tan(y)^2\right)}{\ln(x^2+1) y z} \right. \right.$$

$$- \left. \frac{\cos(x) \tan(y)}{\ln(x^2+1) y^2 z} - \frac{2 \sin(x) \left(1+\tan(y)^2\right) x}{\ln(x^2+1)^2 y z (x^2+1)} + \frac{2 \sin(x) \tan(y) x}{\ln(x^2+1)^2 y^2 z (x^2+1)} \right)$$

$$e^{\frac{\sin(x) \tan(y)}{\ln(x^2+1) y z}} x \Bigg) - \frac{1}{(x^2+z)^2} \left(4 \left(\frac{\cos(x) \tan(y)}{\ln(x^2+1) y z} \right. \right.$$

$$- \left. \frac{2 \sin(x) \tan(y) x}{\ln(x^2+1)^2 y z (x^2+1)} \right) \left(\frac{\sin(x) \left(1+\tan(y)^2\right)}{\ln(x^2+1) y z} \right.$$

$$- \left. \frac{\sin(x) \tan(y)}{\ln(x^2+1) y^2 z} \right) e^{\frac{\sin(x) \tan(y)}{\ln(x^2+1) y z}} x \Bigg)$$

$$+\frac{8\left(\frac{\sin(x)\,(1+\tan(y)^2)}{\ln(x^2+1)\,y\,z}-\frac{\sin(x)\,\tan(y)}{\ln(x^2+1)\,y^2\,z}\right)\mathrm{e}^{\frac{\sin(x)\,\tan(y)}{\ln(x^2+1)\,y\,z}}\,x^2}{(x^2+z)^3}$$

$$\left.-\frac{2\left(\frac{\sin(x)\,(1+\tan(y)^2)}{\ln(x^2+1)\,y\,z}-\frac{\sin(x)\,\tan(y)}{\ln(x^2+1)\,y^2\,z}\right)\mathrm{e}^{\frac{\sin(x)\,\tan(y)}{\ln(x^2+1)\,y\,z}}}{(x^2+z)^2}\right)e_y$$

$$-2\tan(z)\,y\cos(x)e_z$$

Some derivatives:

/ feels fast /
∂^6 g / ∂x^3 ∂y^3 | x,y,z = 2,3,4

evalf(*subs*($x=2$, $y=3$, $z=4$, *diff*($g(x,y,z)$, $x\$3$, $y\$3$)));

$$(-24.27206757)e_x+(-0.01398227167)e_y+(0.)e_z \quad \textbf{(2.4)}$$

/ around 3s /
∂^10 g / ∂x^5 ∂y^5 | x,y,z = 2,3,4

evalf(*subs*($x=2$, $y=3$, $z=4$, *diff*($g(x,y,z)$, $x\$5$, $y\$5$)));

$$(24617.25406)e_x+(-0.3625226504)e_y+(0.)e_z \quad \textbf{(2.5)}$$

/ around 20s /
∂^14 g / ∂x^7 ∂y^7 | x,y,z = 2,3,4

evalf(*subs*($x=2$, $y=3$, $z=4$, *diff*($g(x,y,z)$, $x\$7$, $y\$7$)));

$$(-6.994923710\ 10^7)e_x+(-29.76406303)e_y+(0.)e_z \quad \textbf{(2.6)}$$

! WARNING: THIS COMMAND CAUSED MAPLE TO FREEZE ON THE TEST LAPTOP, CONSUMING TOO MUCH TIME AND RESOURCES !
∂^18 g / ∂x^9 ∂y^9 | x,y,z = 2,3,4

evalf(*subs*($x=2$, $y=3$, $z=4$, *diff*($g(x,y,z)$, $x\$9$, $y\$9$)));