Non-uniformly expanding dynamics in maps with singularities and criticalities

Stefano Luzzatto
Mathematics Institute
University of Warwick
Coventry CV4 7AL
United Kingdom
luzzatto@maths.warwick.ac.uk

Warwick Tucker
Department of Mathematics
Uppsala University
S-751 06 Uppsala
Sweden
warwick@math.uu.se

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Abstract

We investigate a one-parameter family of interval maps arising in the study of the geometric Lorenz flow for non-classical parameter values. Our conclusion is that for all parameters in a set of positive Lebesgue measure, the map has a positive Lyapunov exponent. Furthermore, this set of parameters has a density point which plays an important dynamic role. The presence of both singular and critical points introduces interesting dynamics, which have not yet been fully understood.

Contents

1 Introduction and statement of results

In this paper we study the dynamics of one-dimensional maps in which critical points and singular points (discontinuities) with infinite derivative coexist and have meaningful dynamical interaction. We consider the simplest setting of a one-parameter family with one singularity and two symmetric critical points, although the techniques are quite general and can easily be applied in the case of arbitrary numbers of critical points or singularities. The specific class of maps we consider are also motivated by models of the return map for the Lorenz flow for certain parameter values close to the classical ones, see [Spa82].

We prove that for a positive measure set Ω^* of parameters, the critical points satisfy a strong bound on the recurrence in both the critical and singular regions, and exhibit exponential growth of the norm of the derivative. We also show that there exists a parameter value which is a full Lebesgue density point of Ω^* for which on the contrary the critical points land on the singularity after a finite number of iterates. The arguments and estimates obtained in the proof can be used to prove the existence of an absolutely continuous (with respect to Lebesgue) invariant probability measure for all parameters in Ω^*. Under the additional hypothesis of topological mixing this measure exhibits exponential decay of correlation. The details will appear in a future paper by the same authors.

On a more technical level we develop further the work started in [LV98] and make significant progress in clarifying the effect of the interaction of the critical and singular points on the dynamics. In particular we uncover a remarkable duality between criticalities and singularities as far as control of the distortion and the dynamics is concerned. The infinite expansion at the singularity allows us to obtain somewhat stronger statements as compared to analogous results in the smooth case but it creates non trivial technical difficulties which require strong control of the recurrence. It turns out that the bounded recurrence condition which is needed is exactly the same as that needed to control the recurrence near the critical points. Consequently the arguments for estimating the parameter exclusions need to be significantly generalized compared to those used (for example by Benedicks and Carleson [BC91]) for the quadratic family which take advantage of certain characteristics of the critical point (in particular the notion of *binding*) which cannot be used in the context of singularities. We have set up the statistical argument in a more general way which allows us to treat the singularities and criticalities simultaneously and also clarifies the principal steps in the smooth case.

We will consider one-parameter families of interval maps $\{f_a\}$, $a \in \mathbb{R}$ of the form

$$f_a(x) = \mathrm{sgn}(x)(f(|x|) - a),$$

where $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ is C^1 and satisfies:

1. $f(x) = \mathcal{O}(x^{\ell_s}), \qquad 0 < \ell_s < 1;$
2. $f(|x-c|) = f(c) + \mathcal{O}(|x-c|^{\ell_c}), \qquad 1 < \ell_c < 2;$
3. $f''(x) < 0, \qquad x \neq c,$
4. $0 < f_c(c) < x_{\sqrt{2}}$, where $f'(x_{\sqrt{2}}) = \sqrt{2}$,
5. $|(f_c^2)'(x)| > 2$ for all $x \in [-c, c] \setminus \{0\}$ such that $f_c(x) \in [x_{\sqrt{2}} c]$.

Conditions 4-5 are open conditions used in [LV98] to guarantee some uniform expansion estimates outside the critical region; they will only be used in section 3. We also require the orders of the critical points to satisfy

$$0 < \ell_s \ell_c < 1 \qquad \text{and} \qquad \ell_s + \ell_c < 2.$$

There are intrinsic reasons for these conditions and we will point out the exact place in the proof where they are required.

It was proved in [LV98] that for parameters $a < c$ the dynamics is uniformly expanding and we have a class of maps commonly known as (expanding) *Lorenz-like maps*. The parameter value $a = c$ is a bifurcation point at which the critical point enters the domain of the maps, and therefore for $a > c$ there will occur parameters for which the critical point is periodic. Thus there will exist many open sets of parameters for which an attracting periodic orbit exists. Nevertheless the expanding dynamics is persistent in the sense that there is also a set of parameters, with c being a Lebesgue density point, for which the dynamics continues to be of an expanding nature. The proof uses the fact that for $a = c$ the singularity maps to the critical point. Here we will impose quite different starting conditions on our maps: we will start bifurcating with the critical points well inside the domains under consideration, and we consider the situation in which the critical points are mapped into the singularity after some finite number of iterations, thus making no assumptions on the orbits of the singularity. More precisely, we suppose that $f^k_{c+\epsilon}(\pm c) = 0$ for some small $\epsilon > 0$ and $k \geq 2$. Our bifurcation parameter will be η, and we will study the bifurcation

$$f_{c+\epsilon} \to f_{c+\epsilon+\eta}.$$

Figure 1 illustrates an example of the situation we are considering.

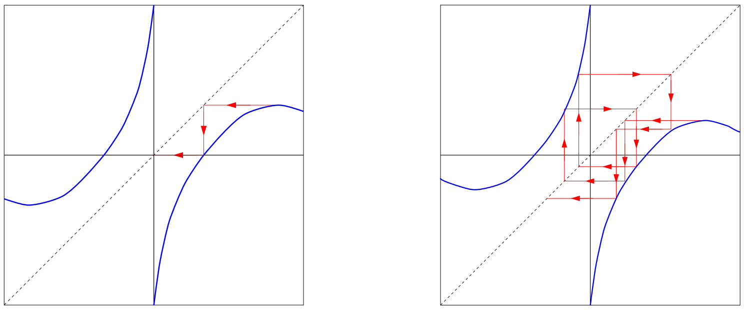

Figure 1: (a) At the bifurcation we have $f^2_{c+\epsilon}(\pm c) = 0$. (b) $f^2_{c+\epsilon+\eta}(\pm c) \neq 0$.

Theorem 1.1 *Let $\{f_a\}$ be a Lorenz-like family of maps satisfying conditions 1-5 and $f^k_{c+\epsilon}(c) = 0$ for some small $\epsilon > 0$ and $k \geq 2$. Then there exists $\lambda > 0$ and a set $\Omega^* \subseteq \mathbb{R}$ of positive measure such that, for all $a \in \Omega^*$ and for a.e. $x \in [-a, a]$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log |(f^n_a)'(x)| \geq \lambda.$$

Also, the parameter $a = c + \epsilon$ is a two-sided Lebesgue density point of Ω^*, i.e.,

$$\lim_{\eta\to 0} \frac{m(\Omega^* \cap [c+\epsilon-\eta, c+\epsilon+\eta])}{2\eta} = 1.$$

The proof actually gives a much stronger technical statement about the recurrence of the critical points in the singular and critical regions. It also shows that each parameter value $a \in \Omega^*$ is accumulated by parameters not belonging to Ω^* for which the conclusions of the theorem hold, i.e., which satisfy the properties of our map $f_{c+\epsilon}$, and which can be used as starting points for the entire construction.

Before giving an outline of the proof, we make a final remark on the relation between the families of maps we have defined and the so called *Lorenz Equations* [Lor63]. This system of ordinary differential equations in $\mathbb{R}^3$ has been enormously influential in Dynamics, providing inspiration for the definition of a variety of examples including the geometric models of [ABS77, GW79] and Hénon maps [Hén76, HP76]. The one-dimensional families of maps we study are a simplified model for two-dimensional return maps associated to the flow of the Lorenz Equations. Here we attempt to describe the transition between the essentially hyperbolic dynamics for $r \approx 28$, and the dynamics for $r \approx 32$ in which the geometry changes through the appearance of *folds*, modeled by critical points. In joint work with M. Viana [LV], the first author is generalizing the one-dimensional techniques presented here and the two dimensional techniques of [BC91, MV93] to study the dynamics of a two-dimensional model of the return maps . For the so called classical parameter value $r = 28$ the second author has recently solved a more than 30 year old open problem by showing that the actual Lorenz equations exhibit a global non-trivial attractor whose first return map satisfies strong hyperbolic properties [Tuc98].

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2 Outline of the proof and remarks

2.1 Notation

In what follows, we will use the positive constants θ, θ' and κ, satisfying the following conditions:

$$1 < \theta < \theta' < \frac{1-\ell_s}{\ell_c - 1} \qquad \text{and} \qquad 1 < \kappa < \frac{1}{\ell_s \ell_c}.$$

We also fix some small $\varepsilon > 0$ and $\alpha > 0$. Our bifurcation parameter η is always small compared to ϵ, i.e. $\epsilon \gg \eta$.

We define the parameter space $\Omega_\eta = [c+\epsilon-\eta, c+\epsilon+\eta]$, and its associated dynamical space $I_\eta = [-(c+\epsilon+\eta), c+\epsilon+\eta]$.

We will also be using a few neighbourhoods in our proofs. Let us define the following *critical* neighbourhoods

$$\Delta^\epsilon_{\pm c} = (\pm c - \epsilon^\kappa, \pm c + \epsilon^\kappa), \quad \Delta^\theta_{\pm c} = (\pm c - \eta^\theta, \pm c + \eta^\theta), \quad \Delta^{\theta'}_{\pm c} = (\pm c - \eta^{\theta'}, \pm c + \eta^{\theta'}),$$

and the following *singular* neighbourhoods

$$\Delta^\epsilon_s = (-\epsilon^{2\kappa}, \epsilon^{2\kappa}), \quad \Delta^\theta_s = (-\eta^\theta, \eta^\theta), \quad \Delta^{\theta'}_s = (-\eta^{\theta'}, \eta^{\theta'}).$$

Observe that we have $\Delta^\epsilon_{\pm c} \supset \Delta^\theta_{\pm c} \supset \Delta^{\theta'}_{\pm c}$, and $\Delta^\epsilon_s \supset \Delta^\theta_s \supset \Delta^{\theta'}_s$. We will also define unions of these neighbourhoods:

$$\Delta^\epsilon = \Delta^\epsilon_{-c} \cup \Delta^\epsilon_s \cup \Delta^\epsilon_c,$$

and similarly for θ and θ'.

We will define a *dynamic* distance as follows:

$$\mathfrak{D}(x) = \min\{|x|, \big||x|-c\big|\} \qquad \mathfrak{D}(\omega) = \inf_{x\in\omega} \mathfrak{D}(x),$$

where ω is a non-empty interval. Throughout the text, we will also use the notation $x_j = x_j(a) = f^j_a(x)$ for all $x \in [-a, a]$ and for all $j \geq 0$. In particular, we will study

$$c_j(a) = f^j_a(c),$$

and view this as a function from parameter space to dynamical space. Finally we introduce the notion of *bounded recurrence.*

Definition 2.1 (Bounded Recurrence) *If $a \in \Omega_\eta$ and if $1 < \nu_1 < \dots < \nu_s \leq n$ are the times at which $c_{\nu_i} \in \Delta^{\theta'}$, we say that the parameter a satisfies $(BR)_n = (BR)_n(\alpha, \eta)$ if for all $k \in [1, n]$ we have*

$$\mathfrak{D}(c_k) \geq \eta^\theta e^{-\alpha k} \quad \text{and} \quad \prod_{\nu_i \leq k} \frac{\mathfrak{D}(c_{\nu_i})}{\eta^\theta} \geq e^{-\alpha k}.$$

Notice that the first condition becomes redundant after time $N = \frac{\theta'-\theta}{\alpha}\log\eta^{-1}$.

2.2 Outline

In section 3 we discuss the dynamics and in particular the derivative estimates outside the critical region $\Delta^{\theta'}$ (from now on we shall often talk about critical region when referring to the union of the corresponding critical and singular regions). We state some estimates from [LV98] which show that the dynamics is essentially expanding outside Δ^{ϵ}: the small derivative at points near $\Delta^{\epsilon}_{\pm c}$ is *pre-compensated* by the large derivative at their preimages which lie very close to the singularity. The neighbourhoods $\Delta^{\theta'}_{\pm c}$, however, are much smaller and this is not longer true for points in $\Delta^{\epsilon}_{\pm c} \setminus \Delta^{\theta'}_{\pm c}$. Thus the main idea here is to use the fact that, at the bifurcation, points near the critical points are mapped very close to the singularity. This suggests that the contraction near the critical points can be compensated almost immediately, and in fact result in expansion. For these points the derivative is *post-compensated* by the large derivative at their images which lie near the singularity. We also show that condition $(BR)_n$ implies exponential derivative growth along the critical orbit at rates which are bounded above and below by constants independent of n.

On both a heuristic and technical level these expansion estimates are the fundamental and essentially the only ingredients of the entire proof. Combined with a lemma comparing derivatives with respect to the parameter to derivatives in the phase space they imply a degree of randomness in the distribution of high iterates of the critical points (for different parameter values). Therefore iterates of the critical points at certain given times are essentially equally likely to be anywhere in phase space. Consequently there is a very small probability of them being very close to the critical points or the singularity. The fact that our bounded recurrence condition implies an exponential growth of the derivatives allows us to continue to take advantage of this principle for those parameters which are good at a given stage in the iteration. The fact that Ω^* is not an open neighbourhood of η reflects the fact that the uniform expansion estimates only hold a priori outside a neighbourhood of the critical points. On the other hand, the fact that η is a full Lebesgue density point of Ω^* reflects the fact that the expansion estimates hold on a set which tends to the entire phase space as $\eta \to 0$.

The precise formulation and formalization of these ideas involves the construction of a special family of partitions satisfying some combinatorial and analytic estimates which allow us to show that *sufficiently often* the distribution in phase space of high iterates of critical points is *sufficiently close* to that of a random process. In section 4 we give the complete proof of the Theorem modulo two technical estimates which are proved in the remaining sections. We start with the inductive definition of a nested sequence of sets $\Omega^{(n)} \subseteq \cdots \subseteq \Omega^{(1)} = \Omega_\eta$ and associated partitions $\mathcal{P}^{(n)} \ldots \mathcal{P}^{(1)}$ satisfying certain properties, maybe the most important one being that the *recurrence function*

$$\mathcal{R}^{(k)}(a) = -\sum_{\nu_i \le k} \log \frac{\mathfrak{D}(c_{\nu_i})}{\eta^\theta}.$$

is essentially constant on elements on $\mathcal{P}^{(k)}$, $k \le n$ (we note that the definition of $\mathcal{R}^{(k)}$ given below differs slightly from this one) and satisfies $\mathcal{R}^{(k)} \le \alpha n$, i.e., condition $(BR)_k$ is satisfied. The precise definition is by induction and the general inductive step will be proved in sections 7.1 and 7.2.

We then define the set Ω^* as the intersection of all $\Omega^{(n)}$, and show that $|\Omega^*| > 0$ and that it has $c + \epsilon$ as a Lebesgue density point. To achieve this, we show that the measure of $\Omega^{(n-1)} \setminus \Omega^{(n)}$ is uniformly exponentially small in n and that there exists $N(\eta) \to \infty$ as $\eta \to 0$ such that no exclusions need to be made before time N, i.e., $\Omega^{(j)} = \Omega^{j-1}$ for all $j \le N$. The second part of this statement depends on the specific combinatorics of the maps; the first part amounts to estimating, in probabilistic language, the conditional probability of *not satisfying* $(BR)_n$ given that $(BR)_{n-1}$ *is satisfied.* The approach is the following: we consider the n:th iterates of maps in $\Omega^{(n-1)}$ and refine $\mathcal{P}^{(n-1)}$ to a partition $\tilde{\mathcal{P}}^{(n)}$ of $\Omega^{(n-1)}$ based on the dynamics up to time n, and such that $\mathcal{R}^{(n)}$ is essentially constant on elements of $\tilde{\mathcal{P}}^{(n)}$. A key technical estimate (whose proof is postponed to sections 6.1 and 6.2) allows us to link the *size* and *number* of elements of $\tilde{\mathcal{P}}^{(n)}$ to their recurrence, and in particular to show that the elements with large recurrence are *small* and *not too many*. Therefore the *average value* of $\mathcal{R}^{(n)}$ over $\Omega^{(n-1)}$ is very low and it follows by a standard large deviation argument that for most parameters $\mathcal{R}^{(n)}$ will be small and so they will belong to $\Omega^{(n)}$.

2.3 Remarks

The kind of results and techniques presented here can be traced back to the fundamental paper of Jakobson [Jak81] in which he proved the (measure-theoretical) persistence of maps with absolutely continuous invariant probability measures (acip's) for the quadratic family of one-dimensional maps.

There exist today several proofs of this result, including [BC85, Ryc88, Tsu93b, Yoc] (see also [Luz] for a proof in the spirit of the proof given here and for a discussion of the similarities and differences between some of these approaches). Generalizations of this result includes [dMvS93] , where smooth maps satisfying a non-recurrence condition of the critical points are shown to be Lebesgue density points of maps with acip's for generic families, and [Tsu93a], where this is further generalized to smooth maps satisfying a bounded recurrence condition similar to our condition (BR) and some other technical assumptions. In [Ree86] an analogous result was proved for *rational maps*; in [Rov93] a family of so called *contracting Lorenz maps* was studied where a discontinuity coincides with the critical point; the case of coexisting critical points and singularities with infinite derivative was first studied in [LV98]; in [PRV98] a class of maps with an infinite number of critical points and in [Thu98] a family of maps with completely degenerate (flat) critical points are considered. There exist also some higher dimensional generalizations of the methods and results presented here, see [BC91, MV93, LV, PRV, Cos98].

We remark that the existence of acip's is a consequence of some more basic geometrical/analytical properties. Thus the kind of results we are discussing can be conveniently split into a first stage in which such conditions are shown to be persistent in parameter space and a second stage in which they are shown to imply the existence of an acip. Several of the references mentioned above only deal with the first part of the work although remarkably the invariant measures can usually be constructed using essentially the same arguments and estimates as those used in parameter space, see for example [You98]. In fact this is a very deep aspect of the theory whose complete significance is not yet completely clear. Notice also that in the higher dimensional context the invariant measures are not usually absolutely continuous with respect to Lebesgue since they are supported on zero measure sets but they satisfy the so called SRB property: conditional measures on unstable manifolds are absolutely continuous with respect to Lebesgue measure restricted to these manifolds.

It is also worth pointing out that notwithstanding the extensive amount of research work mentioned above, the feeling is that we are just breaking into the field and do not yet have a complete and clear understanding of the basic principles at work. In particular, specific examples still have to be approached on an *ad hoc* basis and it is not clear how general the methods used so far really are. The proofs of most of the results mentioned are unfortunately rather long and technical. This appears to be due in part to the nature of the subject, which in itself is quite technical, and in part to the fact that it can still be considered pioneering work. We have made a special effort here to break up the proof as much as possible into small steps so as to clarify the extent to which the various arguments hold more generally and the extent to which they are tied to our specific example.

Finally we give a few words of motivation and some open questions which arise naturally out of the present work. Maps with discontinuities are extremely natural and important, arising for example in billiards or as return maps for flows with equilibrium points, and very often in modeling and applications. Discontinuities bring with them a significant amount of technical problems even in the uniformly expanding or uniformly hyperbolic situations, and thus so far must research has been essentially restricted to these cases. It is to be expected, however, that critical points, or in the higher dimensional context folds and homoclinic tangencies, will also occur quite naturally and therefore examples of systems in which both phenomena occur are bound to constitute an important object of study in the nearest future. Techniques such as the ones presented here will hopefully play a significant role.

Clearly it would be particularly interesting to identify and study phenomena which do not occur in the smooth context and which arise specifically from the interaction between criticalities and singularities. For example we mention the question of the existence of renormalizable maps in which the renormalization domain includes both the critical points and the singularity. Are there such infinitely renormalizable maps? What are their dynamical and ergodic properties? Are there fixed points of renormalization? Another question is the existence (and persistence) of maps having strictly non-uniform negative Lyapunov exponents, i.e., maps in which the derivative along certain orbits is asymptotically *decreasing* exponentially but which accumulate the singularity. This problem might be related to the existence of wandering intervals since by Pesin theory such a point would have a local stable manifold which would amount essentially to a wandering interval.

3 Expansion Estimates

In this section, we will prove that points satisfying certain bounded recurrence properties (see Definition 2.1) have positive Lyapunov exponents. The rest of the paper will be devoted to proving that these points form a set of positive measure.

In the proofs of the two first propositions below, we will use the following, slightly modified version of Lemma 2.3 from [LV98]:

Lemma 3.1 *Let $\rho \in (0,1)$. There exists $\epsilon_0, \hat{\lambda} > 0$ such that for any $a \in [c+\rho\epsilon, c+\epsilon]$, where $\epsilon \in (0,\epsilon_0)$, and for any $x \in [-a,a]$ we have the following:*

1. if $x = x_0, ..., x_{n-1} \notin \Delta_c^\epsilon$, then $|(f_a^n)'(x)| \geq \min\{e^{\hat{\lambda}}, |f_a'(x)|\}e^{\hat{\lambda}(n-1)}$;

2. if, in addition, $x_n \in \Delta_c^\epsilon$, then $|(f_a^n)'(x)| \geq e^{\hat{\lambda}n}$.

The two following propositions improve Lemma 3.1 by allowing much closer returns. Our first proposition states that we have expansion along pieces of orbits that stay outside the critical region $\Delta_c^{\theta'}$ for a while, and then fall into it.

Proposition 3.2 *Let $a = c + \epsilon + \eta$, where ϵ and η are taken sufficiently small. Then, for any $x \in [-a,a]$, we have the following:*

if $x = x_0, ..., x_{n-1} \notin \Delta_c^{\theta'}$, but $x_n \in \Delta_c^{\theta'}$, we have $|(f_a^n)'(x)| \geq e^{\hat{\lambda}n}$, for some $\hat{\lambda} > 0$.

Proof. Case 1: Suppose that $x_j \notin \Delta_c^\epsilon$ for all $0 \leq j \leq n-1$. Then the second part of Lemma 3.1 directly gives the result by taking $\epsilon + \eta \in (0,\epsilon_0)$ and $\eta \ll \epsilon$.

Case 2: Suppose that there exists $k \in [0, n-1]$ such that $x_k \in \Delta_c^\epsilon \setminus \Delta_c^{\theta'}$. Then Lemma 3.1 is not applicable, but by using the property $f_{c+\epsilon}^2(c) = 0$, we can compensate for the small derivative at x_k. This is how it works: By our choice of Δ_c^ϵ and $\Delta_c^{\theta'}$, we have that $\eta^{\theta'} \leq |x_k - c| \leq \epsilon^\kappa$. As $|(f_a^3)'(x_k)| \geq C_1|x_k - c|^{\ell_c - 1}(|x_k - c|^{\ell_c} + \eta)^{\ell_s - 1}$, we have two cases:

(1) if $|x_k - c|^{\ell_c} \geq \eta$, we have

$$\begin{aligned}|x_k - c|^{\ell_c-1}(|x_k - c|^{\ell_c} + \eta)^{\ell_s-1} &\geq |x_k - c|^{\ell_c-1}(2|x_k - c|^{\ell_c})^{\ell_s-1} \geq \\ &\geq C_2|x_k - c|^{\ell_c-1+\ell_c\ell_s-\ell_c} = C_2|x_k - c|^{\ell_c\ell_s-1} \geq C_2\epsilon^{\kappa(\ell_c\ell_s-1)}.\end{aligned}$$

This can be made arbitrarily large by taking ϵ small, since $\ell_c\ell_s - 1 < 0$.
(2) if $|x_k - c|^{\ell_c} \leq \eta$, we have

$$\begin{aligned}&|x_k - c|^{\ell_c - 1}(|x_k - c|^{\ell_c} + \eta)^{\ell_s - 1} \geq \\ &|x_k - c|^{\ell_c - 1}(2\eta)^{\ell_s - 1} \geq C_3\eta^{\theta'(\ell_c - 1)}\eta^{\ell_s - 1} = C_3\eta^{\theta'(\ell_c - 1) + \ell_s - 1}.\end{aligned}$$

Using the fact that $\theta'(\ell_c - 1) + \ell_s - 1 < 0$, this can be made arbitrarily large by taking η small.
Furthermore, if $x_k \in \Delta_c^\epsilon \setminus \Delta_c^{\theta'}$, the next two iterates will stay outside $\Delta_c^{\theta'}$ if ϵ is taken small enough. Thus the small derivative at x_k will be compensated before any further iterate of x_k can re-enter the critical region. Combining this with case 1, we get the desired result. □

The next proposition deals with points that have just left the critical region.

Proposition 3.3 *Let $a = c + \epsilon + \eta$, where ϵ and η are taken sufficiently small. Then, for any $x \in [-a, a]$, we have the following:*
if $x = x_0, ..., x_{n-1} \notin \Delta_c^{\theta'}$, but $x_{-1} \in \Delta_c^{\theta'}$, we have $|(f_a^n)'(x)| \geq e^{\hat{\lambda} n}$, for some $\hat{\lambda} > 0$.

Proof. Case 1: Suppose that $x_j \notin \Delta_c^\epsilon$ for all $0 \leq j \leq n-1$. Then, if $\epsilon + \eta \in (0, \epsilon_0)$ and $\eta \ll \epsilon$, the first part of Lemma 3.1 gives

$$|(f_a^n)'(x)| \geq \min\{e^{\hat{\lambda}}, |f_a'(x)|\}e^{\hat{\lambda}(n-1)}.$$

This directly gives the result by noting that $x_{-1} \in \Delta_c^{\theta'} \Rightarrow |f_a'(x_0)| > \sqrt{2}$.
Case 2: If $x_{\mu_j} \in \Delta_c^\epsilon \setminus \Delta_c^{\theta'}$, $1 < \mu_1 < ... < \mu_k \leq n$, then $C_1\eta^{\theta'(\ell_c - 1)} \leq |f_a'(x_{\mu_j})| \leq C_2\epsilon^{\kappa(\ell_c - 1)}$. Now, by case 2 in the proof of Proposition 3.2, the small derivatives at x_{μ_j} are compensated by the large derivatives at $x_{\mu_j + 2}$. The worst thing that can happen is that the final return to $\Delta_c^\epsilon \setminus \Delta_c^{\theta'}$, x_{μ_k}, is not compensated. This happens if $n \in \{\mu_k + 1, \mu_k + 2\}$. However, we will have compensated for this loss of expansion already at x_1, since $x_{-1} \in \Delta_c^{\theta'} \Rightarrow |f_a'(x_1)| > C_3\eta^{\ell_s - 1}$. As $|f_a'(x_{\mu_k})| \geq C_1\eta^{\theta'(\ell_c - 1)}$, we have

$$|(f_a^{\mu_k + 1})'(x)| \geq C_4\eta^{\ell_s - 1}e^{\lambda(\mu_k - 1)}\eta^{\theta'(\ell_c - 1)} = C_4 e^{\lambda(\mu_k - 1)}\eta^{\theta'(\ell_c - 1) - (1 - \ell_s)} \geq e^{\lambda(\mu_k + 1)},$$

since $\theta'(\ell_c - 1) < (1 - \ell_s)$. Finally, if $n = \mu_k + 2$, we observe that $|f_a'(x_{n-1})| > \sqrt{2}$. □

Hence, given a member f of our family of maps, we know that Propositions 3.2 and 3.3 hold for all f_a, where a is sufficiently close to the bifurcation parameter $c+\epsilon$.

The next proposition tells us that if $a \in \Omega_\eta$ satisfies $(BR)_n$ for all $n \geq 1$, then its associated critical orbit has positive Lyapunov exponent.

Proposition 3.4 *If $a \in \Omega_\eta$ satisfies $(BR)_n$, then $|(f_a^n)'(c_1)| \geq e^{\lambda n}$, for some $\lambda \in (0, \hat{\lambda})$.*

In the proof we will need the following simple lemma:

Lemma 3.5 *If $a \in \Omega_\eta$ satisfies $(BR)_n$ and s is the number of returns of the critical orbit to $\Delta^{\theta'}$ before time n, then $s \leq -\alpha n / \log \eta^{\theta'-\theta}$.*

Proof. If $c_{\nu_i} \in \Delta^{\theta'}$, we have $\mathfrak{D}(c_{\nu_i}) \leq \eta^{\theta'}$ and therefore, from the second part of $(BR)_n$, we have

$$-s \log \frac{\eta^{\theta'}}{\eta^{\theta}} \leq -\sum_{\nu_i \leq n} \log \frac{\mathfrak{D}(c_{\nu_i})}{\eta^{\theta}} \leq \alpha n,$$

which directly gives the result. □

This implies that s can be made small compared to n by taking η or α small.

Proof. (Proposition 3.4) Let $0 < \nu_1 < \ldots < \nu_s \leq n$ be the times at which $c_{\nu_i} \in \Delta^{\theta'}$. Then , by the chain rule, we have

$$|(f_a^n)'(c_1)| = |(f_a^{\nu_1})'(c_1) f_a'(c_{\nu_1}) (f_a^{\nu_2-\nu_1-1})'(c_{\nu_1+1}) \ldots f_a'(c_{\nu_s}) (f_a^{n-\nu_s-1})'(c_{\nu_s+1})|.$$

By Proposition 3.2 and Proposition 3.3, we have

$$\begin{aligned} |(f_a^{\nu_1})'(c_1)| &\geq e^{\hat{\lambda}\nu_1}, \\ |(f_a^{\nu_i-\nu_{i-1}-1})'(c_{\nu_{i-1}+1})| &\geq e^{\hat{\lambda}(\nu_i-\nu_{i-1}-1)} \qquad \text{for } i = 2, \ldots, s, \\ |(f_a^{n-\nu_s-1})'(c_{\nu_s+1})| &\geq e^{\hat{\lambda}(n-\nu_s-1)}. \end{aligned}$$

This means that we have expansion *between* returns to the neighbourhood $\Delta^{\theta'}$. However, we must also take into account the small derivatives from the returns to the critical regions, $\Delta^{\theta'}_{\pm c}$. Since the critical point is of order ℓ_c, there exists a constant $C \in (0,1)$ such that $|f_a'(x)| \geq C\big||x|-c\big|^{\ell_c-1} \geq C\big||x|-c\big|$

for all $x \in \Delta^{\theta'}_{\pm c}$. Therefore, assuming the worst case, i.e. that all returns to $\Delta^{\theta'}$ in fact are returns to $\Delta^{\theta'}_{\pm c}$, we get the following:

$$|f'_a(c_{\nu_i})| \geq C\mathfrak{D}(c_{\nu_i}) \quad \text{for } i = 1, ..., s,$$

which gives

$$|(f^n_a)'(c_1)| \geq e^{\hat{\lambda}(n-s)} \prod_{i=1}^{s} |f'_a(c_{\nu_i})|.$$

Taking logarithms, we get

$$\begin{aligned}
\log|(f^n_a)'(c_1)| &\geq \hat{\lambda}(n-s) + \sum_{i=1}^{s} \log|f'_a(c_{\nu_i})| \geq \hat{\lambda}(n-s) + \sum_{i=1}^{s} \log(C\mathfrak{D}(c_{\nu_i})) \\
&= \hat{\lambda}(n-s) + s\log C + \sum_{i=1}^{s} \log\mathfrak{D}(c_{\nu_i}) \\
&= \hat{\lambda}(n-s) + s\log C + \sum_{i=1}^{s} \big(\log\mathfrak{D}(c_{\nu_i}) - \log\eta^{\theta} + \log\eta^{\theta}\big) \\
&\geq \hat{\lambda}(n-s) + s\log(C\eta^{\theta}) + \sum_{i=1}^{s} \log\frac{\mathfrak{D}(c_{\nu_i})}{\eta^{\theta}} \\
&\geq \hat{\lambda}(n-s) + s\log(C\eta^{\theta}) - \alpha n \\
&\geq (\hat{\lambda} - \alpha)n + (\log(C\eta^{\theta}) - \hat{\lambda})s.
\end{aligned}$$

Using Lemma 3.5, we get

$$\begin{aligned}
|(f^n_a)'(c_1)| &\geq e^{(\hat{\lambda}-\alpha)n} e^{-(\log(C\eta^{\theta})-\hat{\lambda})\frac{\alpha n}{\log\eta^{\theta'-\theta}}} \\
&= e^{\left(\hat{\lambda}-\alpha-\alpha\frac{\log(C\eta^{\theta})-\hat{\lambda}}{\log\eta^{\theta'-\theta}}\right)n} \geq e^{\lambda n},
\end{aligned}$$

if we take $0 < \lambda < \hat{\lambda} - \alpha(1 + \frac{\log(C\eta^{\theta})-\hat{\lambda}}{\log\eta^{\theta'-\theta}})$. This completes the proof. $\square$

Observe that by taking η small, we can get λ arbitrarily close to $\hat{\lambda} - \alpha\big(1 + \frac{\theta}{\theta'-\theta}\big)$, and by taking both η and α small, we can get λ arbitrarily close to $\hat{\lambda}$. In fact, we will use the following proposition, which enables us to bound the growth of iterated intervals from both above and below:

Proposition 3.6 *If $a \in \Omega_{\eta}$ satisfies $(BR)_n$, then $e^{\lambda n} \leq |(f^n_a)'(c_1)| \leq e^{\Lambda n}$, for some $0 < \lambda < \Lambda < \infty$, where $\Lambda = \Lambda(\eta)$ and $\lim_{\eta\to 0}\Lambda(\eta) = \infty$.*

Proof. Since we already have the lower bound, we only have to prove the upper bound, $|(f_a^n)'(c_1)| \le e^{\Lambda n}$. As in the proof of proposition 3.4, we split the orbit into returns, c_{ν_i}, and non-returns to $\Delta^{\theta'}$. We will also assume the worst case, which this time means that all returns to $\Delta^{\theta'}$ in fact are returns to $\Delta_s^{\theta'}$.

When just considering returns, the accumulated derivative is therefore bounded as follows:

$$\prod_{i=1}^{s} |f_a'(c_{\nu_i})| \le \prod_{i=1}^{s} C_1 \mathfrak{D}(c_{\nu_i})^{\ell_s - 1}.$$

Taking logarithms, we get

$$\begin{aligned}
\sum_{i=1}^{s} \log |f_a'(c_{\nu_i})| &\le \sum_{i=1}^{s} \log C_1 \mathfrak{D}(c_{\nu_i})^{\ell_s-1} = s \log C_1 + (\ell_s - 1) \sum_{i=1}^{s} \log \mathfrak{D}(c_{\nu_i}) \\
&= s \log C_1 + (\ell_s - 1) \sum_{i=1}^{s} \big(\log \mathfrak{D}(c_{\nu_i}) - \log \eta^\theta + \log \eta^\theta \big) \\
&= s \big(\log C_1 + (\ell_s - 1) \log \eta^\theta \big) + (\ell_s - 1) \sum_{i=1}^{s} \log \frac{\mathfrak{D}(c_{\nu_i})}{\eta^\theta} \\
&\le s \big(\log C_1 + (\ell_s - 1) \log \eta^\theta \big) + (\ell_s - 1) \alpha n \\
&\le \frac{\alpha n}{\log \eta^{\theta - \theta'}} \big(\log C_1 + (\ell_s - 1) \log \eta^\theta \big) + (\ell_s - 1) \alpha n \\
&= \big(\frac{\log C_1}{\log \eta^{\theta - \theta'}} + \frac{\log \eta^{\theta(\ell_s - 1)}}{\log \eta^{\theta - \theta'}} + \ell_s - 1 \big) \alpha n = \Lambda_1(\eta) n.
\end{aligned}$$

Observe that $\lim_{\eta \to 0} \Lambda_1(\eta) = \frac{\ell_s \theta (1 - \ell_s)}{\theta' - \theta} \alpha$.

Outside $\Delta_s^{\theta'}$, we have $|c_j| > \eta^{\theta'}$, so $|f_a'(c_j)| < C_2 \eta^{\theta'(\ell_s - 1)}$ for some $C_2 > 1$. This gives

$$\begin{aligned}
\prod_{c_j \notin \Delta_s^{\theta'}} |f_a'(c_j)| &< \prod_{j=1}^{n-s} C_2 \eta^{\theta'(\ell_s - 1)} < \prod_{j=1}^{n} C_2 \eta^{\theta'(\ell_s - 1)} \\
&= C_2^n \eta^{\theta'(\ell_s - 1) n} = e^{(\log C_2 + \theta'(\ell_s - 1) \log \eta) n} = e^{\Lambda_2(\eta) n}.
\end{aligned}$$

Observe that $\lim_{\eta \to 0} \Lambda_2(\eta) = \infty$.

Now, by defining $\Lambda(\eta) = \Lambda_1(\eta) + \Lambda_2(\eta)$, we clearly have proved the proposition. □

4 Partitions and Large Deviations

We give here the complete proof of the Theorem modulo two technical results (Theorems 4.1 and 4.3) which will be proved later. We begin section 4.1 by defining a sort of *Whitney partition* of Δ^θ with the property that an element of the partition is small compared to its distance to the critical point or the singularity. Then we give the inductive definition of sets and partitions in parameter space essentially based on the principle that two parameters a and $\tilde{a}$ belong to the same element of $\mathcal{P}^{(n)}$ if the orbits of their respective critical points are close up to time n in the sense that they have the same history of returns to the critical and singular regions and whenever such a return occurs they fall in the same element of the fixed partition of Δ^θ. This simple principle implies all sorts of strong properties such as uniformly bounded distortion.

In section 4.2 we give the main statistical part of the argument leading to the proof of the fact that Ω^* has positive measure. This relies first of all on a subtle reinterpretation of the combinatorial structure defined in 4.1. The same objects appear in both cases but now rather than defining a family of partitions based on the dynamics up to some fixed time we begin by constructing a kind of induced map in parameter space, i.e. a partition $\mathcal{Q}^{(1)}$ with a first stopping time $\eta_1 \le n$ associated to each element ω of $\mathcal{Q}^{(1)}$ with the property that either $\eta_1 = n$ or the images of the critical points of all parameters in ω at time η_1 is larger than some fixed length, i.e. large scale has been reached. This large scale is sufficient to give the required randomness in the distribution that was mentioned in the outline above. Moreover our main technical result says essentially that the stopping times decay exponentially fast: the set of points which does not belong to a component which has achieved large scale by time n is exponentially small in n. The details of the construction also require the definition of $\mathcal{Q}^{(2)}, \dots \mathcal{Q}^{(n)}$ but in essence these are just refinements in the sense that once an element of $\mathcal{Q}^{(1)}$ has reached large scale its future dynamics and therefore its combinatorial structure is essentially a scaled copy of the structure of the whole.

4.1 The combinatorial structure

Partitions in Dynamical Space Suppose, without loss of generality, that η, θ and θ' are chosen such that

$$r_{\theta'} = (\theta' - \theta) \log \eta^{-1} + 1$$

is an integer. Then we can write

$$\Delta_s^\theta = (-\eta^\theta, \eta^\theta) = \{0\} \cup \bigcup_{|r|\geq 1} I_r^s \quad \text{and} \quad \Delta_s^{\theta'} = (-\eta^{\theta'}, \eta^{\theta'}) = \{0\} \cup \bigcup_{|r|\geq r_{\theta'}} I_r^s,$$

where

$$I_{|r|}^s = \eta^\theta [e^{-|r|}, e^{-|r|+1}) \quad \text{and} \quad I_{-r}^s = -I_r^s$$

for all $r \geq 1$. Subdividing each I_r^s into r^2 subintervals of equal lengths, we get partitions $\mathcal{I}_s$ and $\mathcal{I}_s'$ of Δ_s^θ and $\Delta_s^{\theta'}$, respectively. A generic element in either one of these partitions is of the form $I_{r,m}^s$, with $m \in [1, r^2]$. We let $I_{r,m}^{s,\ell}$ and $I_{r,m}^{s,\rho}$ denote the elements of $\mathcal{I}_s$ adjacent to $I_{r,m}^s$, and define $\hat{I}_{r,m}^s = I_{r,m}^{s,\ell} \cup I_{r,m}^s \cup I_{r,m}^{s,\rho}$. If $I_{r,m}^s$ happens to be one of the extreme subintervals of $\mathcal{I}_s$ we just let $I_{r,m}^{s,\ell}$ or $I_{r,m}^{s,\rho}$, depending on whether $I_{r,m}^s$ is a left or a right extreme, denote the interval $(-2\eta^\theta, -\eta^\theta]$ or $[\eta^\theta, 2\eta^\theta)$, respectively. Analogously, we define partitions of $\Delta_{\pm c}^\theta$ and $\Delta_{\pm c}^{\theta'}$, by translation, and set $\mathcal{I} = \mathcal{I}_{-c} \cup \mathcal{I}_s \cup \mathcal{I}_c$.

Partitions in Parameter Space In this section we define inductively a sequence of subsets of $\Omega = \Omega_\eta$ and their corresponding partitions. First let $\Omega^{(0)} = \Omega$ and $\mathcal{P}^{(0)} = \{\Omega\}$. Then fix $n \geq 1$ and suppose that a nested sequence

$$\Omega^{(n-1)} \subseteq \Omega^{(n-2)} \subseteq \cdots \subseteq \Omega^{(1)} \subseteq \Omega^{(0)} = \Omega$$

has been defined as well as a corresponding family of partitions

$$\mathcal{P}^{(n-1)}, \mathcal{P}^{(n-2)} \ldots\ldots, \mathcal{P}^{(1)}, \mathcal{P}^{(0)}$$

satisfying the following properties: For each $0 \leq k \leq n-1$

1. Each $a \in \Omega^{(k)}$ satisfies $(BR)_k$;
2. $\mathcal{P}^{(k)}$ is a partition of $\Omega^{(k)}$ into a finite number of intervals, and for each $\omega \in \mathcal{P}^{(k)}$, the map

   $$c_{k+1} : \omega \to \omega_{k+1} := \{c_{k+1}(a) : a \in \omega\}$$

   is a diffeomorphism (i.e., $\{-c, 0, c\} \cap \{\omega_i\}_{i=1}^k = \emptyset$);

3. Each $\omega \in \mathcal{P}^{(k)}$ has an associated *itinerary* constituted by the following information. A sequence

$$0 = \eta_0 < \eta_1 < \cdots < \eta_s \leq k, \qquad s = s(\omega) \geq 0$$

of *escape times*. Between any two escape times η_{i-1} and η_i (and between η_s and k) there is a sequence

$$\eta_{i-1} < \nu_1 < \cdots < \nu_t < \eta_i \qquad t = t(\omega, i) \geq 0$$

of *essential return times* (or *essential returns*), and between any two essential returns ν_{j-1} and ν_j (and between ν_t and η_i) there is a sequence

$$\nu_{j-1} < \mu_1 < \cdots < \mu_u < \nu_j \qquad u = u(\omega, i, j) \geq 0$$

of *inessential return times* (or *inessential returns*). Following every essential (resp. inessential) return there is a time interval

$$[\nu_j + 1, \nu_j + p_j] \quad (\text{resp. } [\mu_j + 1, \mu_j + p_j])$$

with $p_j > 0$ called the *binding period* associated to the return time ν_j (resp. μ_j). By definition a binding period cannot contain any essential or inessential return times although it may contain a sequence of *bound return times* (or *bound returns*). Associated to each essential, inessential and bound return time is a positive integer r called the *return depth* which is roughly the logarithm of the distance from the singular or critical points. We define the integer-valued functions

$$\mathcal{R}^{(k)} \colon \Omega^{(k)} \to \mathbb{N} \quad \text{and} \quad \mathcal{E}^{(k)} \colon \Omega^{(k)} \to \mathbb{N},$$

both constant on elements of $\mathcal{P}^{(k)}$, which assign to each $a \in \omega \in \mathcal{P}^{(k)}$ the total sum of all return depths, and the total sum of all essential return depths associated to the orbit of ω up to time k, respectively;

4. the distortion is uniformly bounded in the following sense: There exists a constant $D > 0$ independent of k or of $\omega \in \mathcal{P}^{(k)}$ such that

$$\frac{|c_j'(a)|}{|c_j'(b)|} \leq D \qquad \forall a, b \in \omega, \quad \forall j \leq \nu + p + 1,$$

where ν is the last essential or inessential return of ω before time k. Moreover if $\nu + p + 1 < k$, the same statement holds for all $\nu + p + 1 < j \leq k+1$ replacing ω by any subinterval $\omega' \subseteq \omega$ which satisfies $\omega_j' \subseteq \Delta^{\theta}$.

Definition of $\Omega^{(n)}$ We first define a partition $\hat{\mathcal{P}}^{(n)}$ of $\Omega^{(n-1)}$ which refines $\mathcal{P}^{(n-1)}$. Recall that induction assumption 2 implies the following: For every $\omega \in \mathcal{P}^{(n-1)}$, the map $c_n\colon \omega \to \omega_n$ is a diffeomorphism, and in particular a bijection.

1. **Non-chopping times:** We say that n is a non-chopping time for $\omega \in \mathcal{P}^{(n-1)}$ if one of the following situations occur:

   (a) $\omega_n \cap \Delta^\theta = \emptyset$;

   (b) n belongs to a binding period associated to some return time $\nu < n$ of ω;

   (c) $\omega_n \cap \Delta^\theta \neq \emptyset$ but ω_n does not intersect more than two elements of the partition $\mathcal{I}$.

   In all three cases we let $\omega \in \hat{\mathcal{P}}^{(n)}$. If $\omega_n \cap \Delta^{\theta'} \neq \emptyset$ in cases (b) or (c), we say that n is a bound return time or an inessential return time, respectively, for $\omega \in \hat{\mathcal{P}}^{(n)}$ and we define the corresponding return depth by
   $$r = \max\{|\rho|\colon \omega_n \cap I_\rho^* \neq \emptyset\}.$$

2. **Chopping times:** In all remaining cases, i.e., if $\omega_n \cap \Delta^\theta \neq \emptyset$ and ω_n intersects at least three elements of $\mathcal{I}$, we say that n is a chopping time for $\omega \in \mathcal{P}^{(n-1)}$. We define the natural subdivision
   $$\omega = \omega^\ell \cup \bigcup_{(r,m)} \omega^{(r,m)} \cup \omega^\rho,$$
   so that each $\omega_n^{(r,m)}$ fully contains a unique element of $\mathcal{I}$ (though possibly extending to intersect adjacent elements) and ω_n^ℓ and ω_n^ρ are components of $\omega_n \setminus (\Delta^\theta \cap \omega_n)$ with $|\omega_n^\ell| \geq \eta^\theta$ and $|\omega_n^\rho| \geq \eta^\theta$.

   Remark. If the connected components of $\omega_n \setminus (\Delta^\theta \cap \omega_n)$ fail to satisfy the above condition on their length, we just glue them to the adjacent interval of the form $\omega_n^{(r,m)}$.

   By definition we let each of the resulting subintervals of ω be elements of $\hat{\mathcal{P}}^{(n)}$. The intervals ω^ℓ, ω^ρ and $\omega^{(r,m)}$ with $|r| < r_{\theta'}$ are called *escape components* and are said to have an *escape* at time n. All other intervals are said to have an *essential return* at time n and the corresponding values of $|r|$ are the associated *essential return depths*.

This defines the refining partition $\hat{\mathcal{P}}^{(n)}$ of $\Omega^{(n-1)}$. Now let

$$\Omega^{(n)} = \bigcup \{\omega \in \hat{\mathcal{P}}^{(n)} \colon \mathcal{E}^{(n)}(\omega) \leq \alpha n/(2\mathfrak{C})\} \quad \text{and} \quad \mathcal{P}^{(n)} = \hat{\mathcal{P}}^{(n)}|\Omega^{(n)}.$$

Here $\mathfrak{C} = \mathfrak{C}(\alpha, \lambda, \ell_s)$ is a constant which will be defined in a later section.

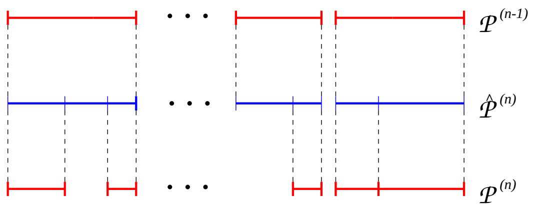

Figure 2: Creating $\Omega^{(n)}$ from $\Omega^{(n-1)}$.

Theorem 4.1 *All the inductive assumptions are satisfied for $\Omega^{(n)}$ and $\mathcal{P}^{(n)}$.*

Assumptions 2 and 3 follow immediately from the definition of $\Omega^{(n)}$ and $\mathcal{P}^{(n)}$ given above. The bounded distortion property and the fact that all $a \in \Omega^{(n)}$ satisfy $(BR)_n$ will be proved in subsequent sections.

4.2 The Probabilistic Argument

With Theorem 4.1 and Proposition 3.4 in mind, we define the set of "good" parameters

$$\Omega^* = \bigcap_{n \geq 1} \Omega^{(n)},$$

the point being that all parameters in Ω^* satisfy $(BR)_\infty$, and hence give rise to a positive Lyapunov exponent. This fact will be proven in a subsequent section. In this section, we will prove the following theorem:

Theorem 4.2 *Ω^* has positive Lebesgue measure, and the parameter value $a = c + \epsilon$ is a two-sided Lebesgue density point of Ω^*.*

Escape times will play a crucial role in this argument.

¿From the above construction, each $\omega \in \hat{\mathcal{P}}^{(n)}$ has an associated sequence $0 = \eta_0 < \eta_1 < \cdots < \eta_s \leq n, \; s = s(\omega) \geq 0$ of escape times and a corresponding sequence of escaping components

$$\omega \subseteq \omega^{(\eta_s)} \subseteq \cdots \subseteq \omega^{(\eta_0)} \quad \text{with} \quad \omega^{(\eta_i)} \subseteq \Omega^{(\eta_i)}.$$

Setting
$$\omega^{(\eta_i)} = \omega \quad \text{for all} \;\; s+1 \le i \le n$$
gives a well defined parameter interval $\omega^{(\eta_i)}$ associated to $\omega \in \hat{\mathcal{P}}^{(n)}$ for each $0 \le i \le n$. Notice that for two intervals $\omega, \tilde{\omega} \in \hat{\mathcal{P}}^{(n)}$ and any $0 \le i \le n$, the corresponding intervals $\omega^{(\eta_i)}$ and $\tilde{\omega}^{(\eta_i)}$ are either disjoint or coincide. Next, we define
$$\mathcal{Q}_n^{(i)} = \bigcup_{\omega \in \hat{\mathcal{P}}^{(n)}} \omega^{(\eta_i)}$$
where the natural partition into intervals of the form $\omega^{(\eta_i)}$ is preserved. Notice that $\mathcal{Q}_n^{(i)}$ is a set of the i:th escaping components, and thus differs tremendously from $\hat{\mathcal{P}}^{(n)}$ since time no longer is preserved. Furthermore, $\Omega^{(n-1)} = \mathcal{Q}_n^{(n)} \subseteq \cdots \subseteq \mathcal{Q}_n^{(0)} = \Omega^{(0)}$ since the number s of escape times is always strictly less than n, and therefore in particular $\omega^{(\eta_n)} = \omega$ for all $\omega \in \hat{\mathcal{P}}^{(n)}$ and so $\mathcal{Q}_n^{(n)} = \hat{\mathcal{P}}^{(n)}$.
For a given $\omega \in \mathcal{Q}_n^{(i-1)}$, $1 \le i \le n$, we let $\mathcal{Q}_n^{(i)}(\omega)$ denote all the elements of $\mathcal{Q}_n^{(i)}$ which are contained in ω. We also define
$$\Delta\mathcal{E}^{(i)} \colon \mathcal{Q}_n^{(i)}(\omega) \to \mathbb{N} \;\; \text{by} \;\; \Delta\mathcal{E}^{(i)}(a) = \mathcal{E}^{(\eta_i)}(a) - \mathcal{E}^{(\eta_{i-1})}(a).$$
This gives the total sum of all essential return depths associated to the itinerary of the element $\tilde{\omega} \in \mathcal{Q}_n^{(i)}(\omega)$ containing a, between the escape at time η_{i-1} and the escape at time η_i. Finally, we let
$$\mathcal{Q}_n^{(i)}(\omega, R) = \{\tilde{\omega} \in \mathcal{Q}_n^{(i)}(\omega) \colon \Delta\mathcal{E}^{(i)}(\tilde{\omega}) = R\}.$$
We now state a theorem, whose proof will be postponed. It contains all we need to prove Theorem 4.2.

Theorem 4.3 *Given any $\omega \in \mathcal{Q}_n^{(i-1)}$, $1 \le i \le n$, and any $R \in \mathbb{N}$, we have*

1. *(Combinatorial estimate)*
$$\#\mathcal{Q}_n^{(i)}(\omega, R) \le e^{5R/\sqrt{r_{\theta'}}};$$
2. *(Metric estimate) There exists $\alpha_1 > 0$ (independent of ω) such that, given any $\tilde{\omega} \in \mathcal{Q}_n^{(i)}(\omega, R)$,*
$$|\tilde{\omega}| \le |\omega| e^{-\alpha_1 R}.$$

In particular, for η sufficiently small,

$$|\mathcal{Q}_n^{(i)}(\omega, R)| \le |\omega| e^{(5/\sqrt{r_{\theta'}} - \alpha_1)R} \le |\omega| e^{-\frac{\alpha_1}{2}R}.$$

As a direct consequence, we have

Corollary 4.4 *Given any $\omega \in \mathcal{Q}_n^{(i-1)}$, $1 \le i \le n$,*

$$\sum_{\tilde{\omega} \in \mathcal{Q}_n^{(i)}(\omega)} |\tilde{\omega}| e^{\frac{\alpha_1}{4}\Delta\mathcal{E}^{(i)}(\tilde{\omega})} \le |\omega| e^{4/(\alpha_1 r_{\theta'})}.$$

Proof. Using the fact that $\Delta\mathcal{E}^{(i)}$ is constant on elements of $\mathcal{Q}_n^{(i)}(\omega)$, we can write

$$\begin{aligned}
\sum_{\tilde{\omega} \in \mathcal{Q}_n^{(i)}(\omega)} |\tilde{\omega}| e^{\frac{\alpha_1}{4}\Delta\mathcal{E}^{(i)}(\tilde{\omega})} &= |\mathcal{Q}_n^{(i)}(\omega, 0)| + \sum_{R \ge r_{\theta'}} |\mathcal{Q}_n^{(i)}(\omega, R)| e^{\frac{\alpha_1}{4}R} \\
&\le |\omega| + \sum_{R \ge r_{\theta'}} |\omega| e^{-\frac{\alpha_1}{2}R} e^{\frac{\alpha_1}{4}R} \\
&\le |\omega| \Big(1 + \sum_{R \ge r_{\theta'}} e^{-\frac{\alpha_1}{4}R}\Big) \le |\omega| \left(1 + 2e^{-\frac{\alpha_1}{4}r_{\theta'}}\right).
\end{aligned}$$

Using the fact that $1 + 2e^{-x} \le e^{1/x}$ for large x, gives the result. □

Next, we show that we remove only exponentially small sets of parameters when going from $\Omega^{(n-1)}$ to $\Omega^{(n)}$.

Lemma 4.5 *For η sufficiently small, there exists $\alpha_2 > 0$ such that*

$$\left|\Omega^{(n-1)} \setminus \Omega^{(n)}\right| \le \left|\Omega^{(0)}\right| e^{-\alpha_2 n}.$$

Proof. By the definition of $\Omega^{(n)}$, we have

$$\begin{aligned}
\left|\Omega^{(n-1)} \setminus \Omega^{(n)}\right| &= \left|\left\{a \in \Omega^{(n-1)} \colon \mathcal{E}^{(n)}(a) > \alpha n/(2\mathfrak{C})\right\}\right| \\
&= \left|\left\{a \in \Omega^{(n-1)} \colon e^{\frac{\alpha_1}{4}\mathcal{E}^{(n)}(a)} > e^{\frac{\alpha_1}{8\mathfrak{C}}\alpha n}\right\}\right| \le e^{-\frac{\alpha_1}{8\mathfrak{C}}\alpha n} \int_{\Omega^{(n-1)}} e^{\frac{\alpha_1}{4}\mathcal{E}^{(n)}(a)} da \\
&= e^{-\frac{\alpha_1}{8\mathfrak{C}}\alpha n} \sum_{\omega \in \hat{\mathcal{P}}^{(n)}} |\omega| e^{\frac{\alpha_1}{4}\mathcal{E}^{(n)}(\omega)}.
\end{aligned}$$

Now we use the fact that $\mathcal{E}^{(n)}(\omega) = \sum_{i=1}^{n} \Delta\mathcal{E}^{(i)}(\omega)$:

$$\begin{aligned}
\sum_{\omega\in\hat{\mathcal{P}}^{(n)}} |\omega| e^{\frac{\alpha_1}{4}\mathcal{E}^{(n)}(\omega)} &= \sum_{\omega\in\hat{\mathcal{P}}^{(n)}} |\omega| e^{\frac{\alpha_1}{4}\sum_{i=1}^{n}\Delta\mathcal{E}^{(i)}(\omega)} = \sum_{\omega\in\mathcal{Q}_n^{(n)}} |\omega| e^{\frac{\alpha_1}{4}\sum_{i=0}^{n-1}\Delta\mathcal{E}^{(i)}(\omega)} \\
&= \sum_{\omega\in\mathcal{Q}_n^{(n-1)}} \sum_{\tilde{\omega}\in\mathcal{Q}_n^{(n)}(\omega)} |\tilde{\omega}| e^{\frac{\alpha_1}{4}\left(\Delta\mathcal{E}^{(n)}(\tilde{\omega})+\sum_{i=1}^{n-1}\Delta\mathcal{E}^{(i)}(\tilde{\omega})\right)} \\
&= \sum_{\omega\in\mathcal{Q}_n^{(n-1)}} e^{\frac{\alpha_1}{4}\sum_{i=1}^{n-1}\Delta\mathcal{E}^{(i)}(\omega)} \sum_{\tilde{\omega}\in\mathcal{Q}_n^{(n)}(\omega)} |\tilde{\omega}| e^{\frac{\alpha_1}{4}\Delta\mathcal{E}^{(n)}(\tilde{\omega})} \\
&\leq \sum_{\omega\in\mathcal{Q}_n^{(n-1)}} e^{\frac{\alpha_1}{4}\sum_{i=1}^{n-1}\Delta\mathcal{E}^{(i)}(\omega)} |\omega| e^{4/(\alpha_1 r_{\theta'})} \\
&= e^{4/(\alpha_1 r_{\theta'})} \sum_{\omega\in\mathcal{Q}_n^{(n-1)}} |\omega| e^{\frac{\alpha_1}{4}\sum_{i=1}^{n-1}\Delta\mathcal{E}^{(i)}(\omega)} \leq \cdots \leq e^{4n/(\alpha_1 r_{\theta'})} \left|\Omega^{(0)}\right|.
\end{aligned}$$

Hence, for η sufficiently small, we arrive at

$$\left|\Omega^{(n-1)} \setminus \Omega^{(n)}\right| \leq \left|\Omega^{(0)}\right| e^{n\left(4/(\alpha_1 r_{\theta'}) - \frac{\alpha_1}{8\mathfrak{C}}\alpha\right)} = \left|\Omega^{(0)}\right| e^{-\alpha_2 n},$$

for some $\alpha_2 > 0$. □

It remains to prove that $a = c + \epsilon$ is a Lebesgue density point for Ω_η^* (we now indicate the dependence of η). This will immediately imply that Ω_η^* has positive measure. Up to time $N = \frac{\theta'-\theta}{\alpha}\log\eta^{-1}$ we will exclude all parameters in Ω_η giving rise to returns closer than η^θ, i.e., returns to Δ^θ. Furthermore, if after an exclusion, we are left with components having size less than η^θ, we exclude those pieces as well. This gives at most an extra factor two of the size of the excluded parameters. Another factor three comes from the fact that Δ^θ is made up of three components. Assuming a "worse than possible" scenario, that the expansion between returns is identically one, and that each interval is split in two at every iteration, we get a very blunt estimate of the size of set of excluded parameters, $E(\eta, N)$, at time N:

$$\begin{aligned}
|E(\eta, N)| \leq 6\sum_{i=0}^{N} 2^i\eta^\theta < 12\cdot 2^N\eta^\theta &= 12\eta^\theta e^{N\log 2} = 12\eta^\theta e^{\frac{\log 2}{\alpha}\log\eta^{-(\theta'-\theta)}} \\
&= 12\eta^\theta\eta^{-\frac{\log 2}{\alpha}(\theta'-\theta)} = 12\eta^{(1+\frac{\log 2}{\alpha})\theta - \frac{\log 2}{\alpha}\theta'}.
\end{aligned}$$

By imposing the restriction $\alpha > \frac{\theta'-\theta}{\theta-1}\log 2$, we get $|E(\eta, N)| \leq \eta^{1+\gamma}$, for some $\gamma > 0$. This means that the set of non-excluded parameters satisfies

$|\Omega_\eta^{(N)}| \geq (1-\eta^\gamma)|\Omega_\eta^{(0)}|$, and since we have

$$|\Omega_\eta^{(n)}| \geq \big(1-\sum_{i=N}^{n} e^{-\alpha_2 n}\big)|\Omega_\eta^{(N)}| \quad \text{and} \quad |\Omega_\eta^{*}| \geq \big(1-\sum_{i=N}^{\infty} e^{-\alpha_2 n}\big)|\Omega_\eta^{(N)}|,$$

it follows that

$$\begin{aligned}|\Omega_\eta^{*}| &\geq \big(1-\sum_{i=N}^{\infty} e^{-\alpha_2 n}\big)(1-\eta^\gamma)|\Omega_\eta^{(0)}| \\ &\geq \big(1-Ce^{-\alpha_2 N}\big)(1-\eta^\gamma)|\Omega_\eta^{(0)}| \geq \big(1-C\eta^{\frac{\alpha_2}{\alpha}(\theta'-\theta)}\big)(1-\eta^\gamma)|\Omega_\eta^{(0)}|.\end{aligned}$$

It is now plain to see that

$$\lim_{\eta\to 0}\frac{|\Omega_\eta^{*}|}{|\Omega_\eta^{(0)}|} = 1,$$

which concludes the proof of Theorem 4.2.

5 Fundamental results

5.1 Parameter Dependence

In this section we will consider the family of maps

$$\varphi^n\colon \Omega_\eta \to I_\eta \quad \text{defined by} \quad \varphi^n(a) = f_a^n(c_1),$$

where c_1 is the image of the critical point under f_a. Observe that $\varphi^n(a)$ is just the n:th iterate of the critical image under the map f_a. We will prove that if the derivative along the critical orbit $(f_a^n)'(c_1)$ is growing exponentially, then the derivative is growing exponentially along almost every orbit in I_η. In particular this means that the map f_a has positive Lyapunov exponent.

By the definition of Ω_η, we know that f_a satisfies some uniform expansion estimates outside the critical regions whenever $a \in \Omega_\eta$. By considering $\varphi^n(\Omega_\eta)$, we are taking care of all of these maps simultaneously. However, the orbits of the critical point must also satisfy some bounded recurrence (condition (BR)) to ensure that the maps f_a have positive Lyapunov exponents. This means that whenever $\varphi^n(\Omega_\eta)$ comes to close to the singular or critical points, we must discard some parameters. To make sure that we do not throw away all parameters, we need (to start with) to show that the images $\varphi^n(\omega)$ of small intervals $\omega \subset \Omega_\eta$ in parameter space are growing exponentially as long as the derivatives along the corresponding critical orbits are.

The first result tells us that the growth rate in parameter space is comparable to that in dynamical space under certain conditions.

Proposition 5.1 *There exists a constant $P > 1$ such that if $|(f_a^k)'(c_1)| \geq e^{\lambda k}$ for all $1 \leq k \leq n$ and η is sufficiently small, then for all $a \in \Omega_\eta$*

$$P^{-1} \leq \frac{|(\varphi^n)'(a)|}{|(f_a^n)'(c_1)|} \leq P$$

for all $1 \leq k \leq n$.

Proof. By explicit calculation we have

$$\frac{(\varphi^n)'(a)}{(f_a^n)'(c_1)} = -1 + \sum_{i=1}^{n} \frac{\partial_a f_a(c_i)}{(f_a^i)'(c_1)}$$

(notice that we do not include modulus signs in the expression). Thus we need to show that the sum on the right hand side is bounded away from 1. By definition, the partial derivatives $\partial_a f_a(c_i)$ only take on values ± 1, and thus by the assumption on the growth of the derivative and using the fact that λ can be chosen close to $\log \sqrt{2}$ we have

$$\left| \sum_{4}^{n} \frac{\partial_a f_a(c_i)}{(f_a^i)'(c_1)} \right| \leq \frac{1}{2}.$$

Thus we just need to consider the first three terms and show that

$$\sum_{1}^{3} \frac{\partial_a f_a(c_i)}{(f_a^i)'(c_1)}$$

is bounded away from $1/2$. This follows easily from the definition of f_a since for the first term we have $\partial_a f_a(c_1)/f_a'(c_1) < 0$ and for the second two terms we have $|(f_a^i)'(c_1)|$ large. $\square$

In particular, this means that as long as the space derivatives $|(f_a^n)'(c_1)|$ are growing exponentially for all $a \in \omega \subset \Omega_\eta$, the maps $\varphi^n \colon \omega \to \varphi^n(\omega)$ are diffeomorphisms, since $|(\varphi^n)'(a)| \geq P^{-1}|(f_a^n)'(c_1)| \neq 0$. As an important consequence we have the following Parameter Mean Value Theorem:

Corollary 5.2 *Let $\omega \subset \Omega_\eta$ and suppose that $|(f_a^k)'(c_1)| \geq e^{\lambda k}$ for all $1 \leq k \leq n$ and for all $a \in \omega$. Then for every pair of integers $1 \leq i \leq j \leq n$ there exists some $\zeta \in \omega$ such that*

$$P^2|(f_\zeta^{j-i})'(\varphi^i(\zeta))| \leq \frac{|\omega_j|}{|\omega_i|} \leq P^{-2}|(f_\zeta^{j-i})'(\varphi^i(\zeta))|,$$

where $\omega_j = \varphi^j(\omega)$ and $\omega_i = \varphi^i(\omega)$.

Proof. Consider the map $\tilde{\varphi}\colon \omega_i \to \omega_j$ given by $\tilde{\varphi}(\varphi^i(a)) = \varphi^j(a) = \varphi^j(\varphi^{-i}(\varphi^i(a)))$. By the Mean Value Theorem there exists $\zeta \in \omega$ such that $|\omega_j| = |\tilde{\varphi}'(\varphi^i(\zeta))||\omega_i|$. Then, by the chain rule and Proposition 5.1, we have

$$\begin{aligned}|\tilde{\varphi}'(\varphi^i(\zeta))| &= |(\varphi^j)'(\zeta)(\varphi^{-i})'(\varphi^i(\zeta))| \\ &\leq P^{-1}|(f_\zeta^j)'(c_1)|P^{-1}|(f_\zeta^{-i})'(\varphi^i(\zeta))| = P^{-2}|(f_\zeta^{j-i})'(\varphi^i(\zeta))|.\end{aligned}$$

The lower bound is obtained in the same way. □

5.2 Binding

In this section we will show that if $\omega_n = \varphi^n(\omega) \subset \Delta^\theta$, and if all parameters in ω satisfy $(BR)_n$, then a large expansion can be guaranteed after a certain time p. The point is that the parameter intervals grow sufficiently large, so that when they intersect Δ^θ, only a very small proportion of the intervals are excluded due to deep returns.

Definition 5.3 *Given $x \in \Delta_c^\theta$, we define the binding period of x as*

$$p = p(x) = \max\{n \in \mathbb{N}\colon |x_j - c_j| \leq \eta^\theta e^{-2\alpha j} \text{ for } 0 \leq j \leq n\}.$$

In particular, this means that $|x_{p+1} - c_{p+1}| > \eta^\theta e^{-2\alpha(p+1)}$. For notational convenience, we will define the binding period to be zero for returns to Δ_s^θ. We will now list some consequences of this definition.

The first consequence is that we have bounded distortion during binding periods. Given a small interval, $\omega = [x, y] \subseteq \Delta_c^\theta$ we define $\omega^\star$ to be the convex hull of ω and c: if $y \leq c$, we have $\omega^\star = \omega_0^\star = [x, c]$ and $\omega_j^\star = f_a^j(\omega^\star) = [x_j, c_j]$. Hence we have $|\omega_j^\star| \leq \eta^\theta e^{-2\alpha j}$ for $0 \leq j \leq p$ and $|\omega_{p+1}^\star| > \eta^\theta e^{-2\alpha(p+1)}$.

Proposition 5.4 *There exists a constant $D = D\alpha)$ such that*

$$\operatorname{Dist}(f_a^k, \omega_1^\star) = \sup_{\xi_1, \zeta_1 \in \omega_1^\star} \left| \frac{(f_a^k)'(\xi_1)}{(f_a^k)'(\zeta_1)} \right| \le D \quad \forall k \in [1, p].$$

Before we prove this proposition, we will prove two simple lemmas:

Lemma 5.5 *Given an interval ω, such that $f_a^j|_{\omega_j}$ is a diffeomorphism for $0 \le j \le k-1$, we have*

$$\operatorname{Dist}(f_a^k, \omega) \le e^{\sum_{j=0}^{k-1} D_j} \quad \textit{where} \quad D_j = \sup_{\xi_j, \zeta_j \in \omega_j} \left| \frac{f_a''(\xi_j)}{f_a'(\zeta_j)} \right| |\omega_j|.$$

Proof. Use the chain rule to write

$$\begin{aligned} \left| \frac{(f_a^k)'(\xi)}{(f_a^k)'(\zeta)} \right| &= \left| \prod_{j=0}^{k-1} \frac{f_a'(\xi_j)}{f_a'(\zeta_j)} \right| \\ &= \left| \prod_{j=0}^{k-1} \left(1 + \frac{f_a'(\xi_j) - f_a'(\zeta_j)}{f_a'(\zeta_j)} \right) \right| \le \prod_{j=0}^{k-1} \left(1 + \frac{|f_a'(\xi_j) - f_a'(\zeta_j)|}{|f_a'(\zeta_j)|} \right). \end{aligned}$$

By the Mean Value Theorem we have $|f_a'(\xi_j) - f_a'(\zeta_j)| = |f_a''(z_j)||\xi_j - \zeta_j| \le |f_a''(z_j)||\omega_j|$ for some $z_j \in [\xi_j, \zeta_j]$. Therefore

$$\begin{aligned} \operatorname{Dist}(f_a^k, \omega) &\le \sup_{\xi_j, \zeta_j \in \omega_j} \prod_{j=0}^{k-1} \left(1 + \left| \frac{f_a''(\xi_j)}{f_a'(\zeta_j)} \right| |\omega_j| \right) \le \sup_{\xi_j, \zeta_j \in \omega_j} e^{\sum_{j=0}^{k-1} \log\left(1 + \left| \frac{f_a''(\xi_j)}{f_a'(\zeta_j)} \right| |\omega_j| \right)} \\ &\le \sup_{\xi_j, \zeta_j \in \omega_j} e^{\sum_{j=0}^{k-1} \left| \frac{f_a''(\xi_j)}{f_a'(\zeta_j)} \right| |\omega_j|} = e^{\sum_{j=0}^{k-1} D_j}, \end{aligned}$$

where we used $\log(1+x) \le x$ in the final inequality. $\square$

Lemma 5.6 *Given an interval ω such that $|\omega| < \mathfrak{D}(\omega)$, there exists C such that*

$$\sup_{\xi, \zeta \in \omega} \left| \frac{f_a''(\xi)}{f_a'(\zeta)} \right| \le C \sup_{z \in \omega} \frac{1}{\mathfrak{D}(z)} = \frac{C}{\mathfrak{D}(\omega)}.$$

Proof. We may assume that $\omega \cap \Delta^\epsilon \neq \emptyset$, otherwise the lemma is trivial. So suppose first that $\omega \cap \Delta_s^\epsilon \neq \emptyset$. Then $\omega \subseteq 2\Delta_s^\epsilon$, and by **M1**, we have

$$|f_a''(\xi)| \leq C_2 \mathfrak{D}(\omega)^{\ell_s - 2} \quad \text{for all } \xi \in \omega,$$

and

$$|f_a'(\zeta)| \geq C_3(\mathfrak{D}(\omega) + |\omega|)^{\ell_s - 1} \geq C_4 \mathfrak{D}(\omega)^{\ell_s - 1} \quad \text{for all } \zeta \in \omega.$$

Hence, by taking quotients and supreme, the result follows. In the remaining case, $\omega \cap \Delta_c^\epsilon \neq \emptyset$, the bound on the second derivative is just like the previous case, using **M2** instead of **M1**. To bound the first derivative from below, we just observe that

$$|f_a'(\zeta)| \geq C_5 \mathfrak{D}(\omega)^{\ell_c - 1} \quad \text{for all } \zeta \in \omega,$$

which gives the desired result. □

We are now ready to prove Proposition 5.4.

Proof. (Proposition 5.4) By Lemma 5.5 we only need to estimate $\sum_{j=0}^{k-1} D_j \quad \forall k \in [1,p]$. Due to binding, we have $|\omega_j^\star| \leq \eta^\theta e^{-2\alpha j} \quad \forall j \in [1,p]$, and due to $(BR)_n$, we have $\mathfrak{D}(c_j) \geq \eta^\theta e^{-\alpha j}$. Combining these two inequalities gives $\mathfrak{D}(\omega_j^\star) \geq \eta^\theta(e^{-\alpha j} - e^{-2\alpha j})$. Thus

$$\mathfrak{D}(\omega_j^\star) \geq \eta^\theta(e^{-\alpha j} - e^{-2\alpha j}) > \eta^\theta e^{-2\alpha j} \geq |\omega_j^\star|,$$

so by Lemma 5.6,

$$\left| \frac{f_a''(\xi_j)}{f_a'(\zeta_j)} \right| \leq C_1 \sup_{z_j \in \omega_j^\star} \frac{1}{\mathfrak{D}(z_j)} \leq C_1 \eta^{-\theta}(e^{-\alpha j} - e^{-2\alpha j})^{-1} \leq C_2 \eta^{-\theta} e^{\alpha j},$$

which means that

$$D_j \leq C_2 \eta^{-\theta} e^{\alpha j} \eta^\theta e^{-2\alpha j} = C_3 e^{-\alpha j}.$$

Hence, $\forall k \in [1,p]$, we finally have

$$\sum_{j=0}^{k-1} D_j \leq C_2 \sum_{j=0}^{\infty} e^{-\alpha j} \leq C_4 = C_4(\alpha),$$

which completes the proof. □

The next consequence is that we can get an idea of how long a binding period is.

Lemma 5.7 *Let $x \in \Delta_c^\theta$ with $\mathfrak{D}(x) \geq \eta^\theta e^{-\alpha n}$. Then, if η is sufficiently small, we have*

$$-\frac{\ell_c}{2\Lambda}\log\frac{\mathfrak{D}(x)}{\eta^\theta} \leq p(x) \leq -\frac{\ell_c}{\lambda}\log\frac{\mathfrak{D}(x)}{\eta^\theta}.$$

Furthermore, we also have $p(x) < n$.

Proof. As usual, let $\omega^\star = [x, c]$ and set $\hat{p} = \min\{p, n\}$. By the binding condition, we have $|\omega^\star_{\hat{p}}| \leq \eta^\theta e^{-2\alpha\hat{p}}$. Combining this with the distortion estimates from Proposition 5.4, the estimates of the growth of the derivative from Proposition 3.4, and the PMVT gives

$$\eta^\theta e^{-2\alpha\hat{p}} \geq |\omega^\star_{\hat{p}}| \geq C_1|\omega^\star_1|e^{\lambda(\hat{p}-1)} \geq C_2\mathfrak{D}(x)^{\ell_c}e^{\lambda(\hat{p}-1)},$$

which gives

$$\hat{p} \leq \frac{\ell_c\log\mathfrak{D}(x)^{-1} + \theta\log\eta + \log C_3}{\lambda + 2\alpha} \leq -\frac{\ell_c}{\lambda}\log\frac{\mathfrak{D}(x)}{\eta^\theta},$$

as long as $\mathfrak{D}(x) \leq \eta^\theta$ is small enough (which can be achieved by taking η small). By $(BR)_n$, $\log\mathfrak{D}(x)^{-1} + \theta\log\eta \leq \alpha n$, and since $\log\mathfrak{D}(x)^{-1} < \log\mathfrak{D}(x)^{-\ell_c}$, we also get

$$\hat{p} \leq \frac{\ell_c\alpha n}{\lambda} < n$$

as long as α is sufficiently small ($\alpha < \frac{\lambda}{\ell_c}$ works). Hence we always have $\hat{p} = p$, i.e. the binding finishes before time n.

To get a lower bound on p, we use the estimates on $|\omega^\star_{p+1}|$ ($|\omega^\star_{\hat{p}+1}| \geq \eta^\theta e^{-2\alpha(\hat{p}+1)}$) and, once again, the distortion estimates from Proposition 5.4, the estimates of the growth of the derivative from Proposition 3.6, and the PMVT:

$$\eta^\theta e^{-2\alpha(p+1)} \leq |\omega^\star_{p+1}| \leq C_4|\omega^\star_1|e^{\Lambda p} \leq C_5\mathfrak{D}(x)^{\ell_c}e^{\Lambda p},$$

which gives

$$p \geq \frac{\ell_c\log\mathfrak{D}(x)^{-1} + \theta\log\eta + \log C_6}{\Lambda + 2\alpha} \geq -\frac{\ell_c}{2\Lambda}\log\frac{\mathfrak{D}(x)}{\eta^\theta},$$

as long as $\mathfrak{D}(x) \leq \eta^\theta$ is small enough. This completes the proof. □

We also want to know how much a small interval has grown during the binding period. If $\omega \subseteq 2\Delta_c^\theta$ is an interval, we define the length of the binding period associated to ω as

$$p = p(\omega) = \min\{p(x)\colon x \in \omega\}.$$

Lemma 5.8 *Suppose that $I^*_{r,m} \subseteq \omega \subseteq \hat{I}^*_{r,m}$ for some $|r| \le \alpha n$, and $* \in \{-c, s, c\}$. Then*

$$|\omega_{p+1}| \ge C\eta^\theta e^{-\frac{1+\ell_s}{2}|r|} \gg |\omega|$$

for α sufficiently small. Furthermore, for all $x \in \omega$, we have

$$|(f_a^{p+1})'(x)| \ge e^{\frac{1-\ell_s}{2}r}.$$

Proof. We split the proof in two parts:

$* = s$: Here we use the fact that, by definition, $p = 0$, and so

$$|\omega_{p+1}| = |\omega_1| \ge C_1|\omega|^{\ell_s} \ge C_2\eta^{\ell_s\theta}e^{-\ell_s|r|}/|r|^{2\ell_s}.$$

Now we have to be a bit careful: Since $r_{\theta'} \le |r| \le \alpha n$, we do not have an upper bound (depending on η) for $|r|$. Continuing the estimate, and recalling that $\ell_s < 1$, we get

$$|\omega_{p+1}| = |\omega_1| \ge C_3\eta^{\ell_s\theta}e^{-\frac{1+\ell_s}{2}|r|}\left(e^{\frac{1-\ell_s}{2}|r|}/|r|^{2\ell_s}\right) \ge C_4\eta^\theta e^{-\frac{1+\ell_s}{2}|r|}$$

for η sufficiently small (i.e. $|r|$ sufficiently large), and α sufficiently small.

$* = c$: First, observe that $\omega \subseteq \hat{I}^c_{r,m} \Rightarrow \mathfrak{D}(\omega) \ge \eta^\theta e^{-(|r|+1)}$. Hence, by Lemma 5.7,

$$p = p(\omega) \le -\frac{\ell_c}{\lambda}\log\frac{\mathfrak{D}(\omega)}{\eta^\theta} \le \frac{\ell_c}{\lambda}(|r|+1) \le 4|r|/\lambda,$$

so, by Definition 5.3,

$$|\omega^\star_{p+1}| \ge \eta^\theta e^{-2\alpha(p+1)} \ge \eta^\theta e^{-2\alpha}e^{-8\alpha|r|/\lambda} \ge \eta^\theta e^{-12\alpha|r|/\lambda}.$$

By **M2** there exists a constant C_1 such that

$$\frac{|\omega_1|}{|\omega_1^\star|} \ge C_1\frac{e^{-\ell_c|r|}/r^2}{e^{-\ell_c|r|}} \ge \frac{C_1}{r^2}.$$

Thus we can use the distortion estimate from Proposition 5.4 to get

$$\frac{|\omega_{p+1}|}{|\omega^\star_{p+1}|} \ge \frac{C_2}{r^2},$$

which gives us

$$|\omega_{p+1}| \ge C_2|\omega^\star_{p+1}|r^{-2} \ge C_2\eta^\theta e^{-12\alpha|r|/\lambda}r^{-2} \gg C_2\eta^\theta e^{-\frac{1+\ell_s}{2}|r|} \gg |\omega|$$

by taking α small enough, and recalling that $|\omega| \sim \eta^\theta e^{-|r|}/r^2$.

The second estimate follows immediately using the bounded distortion and Corollary 5.2. □

6 Proof of Theorem 4.3

6.1 Combinatorial Estimates

The following lemma is one of the main motivations behind the method of construction of the partitions $\mathcal{Q}_n^{(i)}$.

Lemma 6.1 *Given an element $\omega \in \mathcal{Q}_n^{(i-1)}$ and a sequence $(r_1, m_1), ..., (r_t, m_t)$ with $|r_j| \geq r_{\theta'}$ and $m_j \in [1, r_j^2]$, there exists at most $7r_{\theta'}^3$ elements $\tilde{\omega} \in \mathcal{Q}_n^{(i)}$, $\tilde{\omega} \subseteq \omega$ with essential return times $\nu_j \in (\eta_{i-1}, \eta_i)$ such that $\tilde{\omega}_{\nu_j} \cap I^*_{r_j, m_j} \neq \emptyset$ for $j = 1, ..., t$.*

Proof. We simply have to calculate how many ways a set $\omega^{(r,m)}$ can become an escape as in case (3a). This number is clearly bounded above by the number of elements of the partition $\mathcal{I}|_{\Delta^\theta \setminus \Delta^{\theta'}}$ plus two elements for each $\mathcal{I}^*$ which can escape by falling outside Δ^θ. The number of such elements is at most $3(2r_{\theta'}^3 + 2) \leq 7r_{\theta'}^3$. □

We now define a family of integer-valued functions $\mathcal{E}_n^{(i)} \colon \Omega_\eta \to \mathbb{N}$, constant on elements of $\mathcal{P}_n^{(i)}$, which assign to each $a \in \omega \in \mathcal{P}_n^{(i)}$ the total sum of the absolute values of all essential return depths associated to the orbit of ω up to time i. For $i \in [1, n]$, take $\omega \in \mathcal{Q}_n^{(i-1)}$, and consider the partition $\mathcal{Q}_n^{(i)}$ restricted to ω. For each $\tilde{\omega} \in \mathcal{Q}_n^{(i)}|_\omega$ there are escape times $\eta_{i-1}(\omega)$ and $\eta_i(\tilde{\omega})$. Let

$$\Delta\mathcal{E}_n^{(i)} = \mathcal{E}_n^{(\eta_i)} - \mathcal{E}_n^{(\eta_{i-1})}.$$

Then $\Delta\mathcal{E}_n^{(i)}(\tilde{\omega})$ simply adds the essential return depths of $\tilde{\omega}$ occuring between the escape times $\eta_{i-1}(\omega)$ and $\eta_i(\tilde{\omega})$. We have the following proposition:

Proposition 6.2 *For all η sufficiently small, $\omega \in \mathcal{Q}_n^{(i)}$, and $R \geq r_{\theta'}$, we have*

$$\#\{\tilde{\omega} \in \mathcal{Q}_n^{(i)}|_\omega \colon \Delta\mathcal{E}_n^{(i)}(\tilde{\omega}) = R\} \leq e^{5R/\sqrt{r_{\theta'}}}.$$

Proof. The proof is purely combinatorial. By Lemma 6.1, it is sufficient to estimate the number of possible sequences $(r_1, m_1), ..., (r_t, m_t)$ for any $t \geq 1$ with $|r_1| + ... + |r_t| = R$. We begin by estimating the number of integer solutions to

$$\Delta\mathcal{E}_n^{(i)}(\tilde{\omega}) = |r_1| + ... + |r_t| = R$$

for a fixed choice of $t \geq 1$. The number of such solutions corresponds to the number of ways one can partition R objects into t disjoint subsets, which is bounded above by the number of ways one can pick t balls out of a row of $R+t$ balls. This is seen by observing that any choice of t balls out of this row will uniquely determine a partition of the remaining R balls into at most t disjoint subsets. Using Stirling's approximation formula for factorials, $\sqrt{2\pi k}k^k e^{-k} \leq k! \leq (1+\frac{1}{4k})\sqrt{2\pi k}k^k e^{-k}$, we get

$$\binom{R+t}{t} = \frac{(R+t)!}{R!t!} \leq C_1 \frac{(R+t)^{r+t}}{R^R t^t} = C_1 \left(\frac{R+t}{R}\right)^R \left(\frac{R+t}{t}\right)^t.$$

Now, by repeatedly using the fact that $t \leq R/r_{\theta'}$, we get

$$\left(\frac{R+t}{R}\right)^R \leq \left(\frac{R(1+\frac{1}{r_{\theta'}})}{R}\right)^R = \left(1+\frac{1}{r_{\theta'}}\right)^R = e^{R\log(1+\frac{1}{r_{\theta'}})} \leq e^{R/r_{\theta'}}.$$

We also have

$$\left(\frac{R+t}{t}\right)^t = \left[\left(\frac{R+t}{t}\right)^{\frac{t}{R}}\right]^R \leq \left[\left(\frac{R(1+\frac{1}{r_{\theta'}})}{t}\right)^{\frac{t}{R}}\right]^R = \left[\left(\frac{R}{t}\right)^{\frac{t}{R}}\left(1+\frac{1}{r_{\theta'}}\right)^{\frac{t}{R}}\right]^R.$$

Now, by taking η sufficiently small (this makes $r_{\theta'}$ large), and recalling that $\frac{R}{t} \geq r_{\theta'}$, we have that $\frac{t}{R}\log\frac{R}{t} \leq \frac{1}{r_{\theta'}}\log r_{\theta'} \leq \frac{1}{\sqrt{r_{\theta'}}}$.
Hence

$$\begin{aligned}\left(\frac{R+t}{t}\right)^t &\leq \left[e^{\frac{t}{R}\log\frac{R}{t}+1/r_{\theta'}\log(1+1/r_{\theta'})}\right]^R \leq \left[e^{\frac{t}{R}\log\frac{R}{t}+1/r_{\theta'}^2}\right]^R \\ &\leq \left[e^{1/\sqrt{r_{\theta'}}+1/r_{\theta'}}\right]^R \leq e^{2R/\sqrt{r_{\theta'}}}.\end{aligned}$$

Concluding these estimates, we have

$$\binom{R+t}{t} \leq e^{3R/\sqrt{r_{\theta'}}},$$

if we take η sufficiently small. Finally, we have to take into account that the r_j's may be negative, and that we can have $3r_i^2$ distinct sets sharing the same depth r_i (this is because $\Delta^\theta = \Delta^\theta_{-c} \cup \Delta^\theta_s \cup \Delta^\theta_c$).

Furthermore, by Lemma 6.1, we can have $7r_{\theta'}^3$ sets sharing the same sequence $(r_1, m_1), ..., (r_t, m_t)$. Therefore, summing over all possible values of t, and keeping in mind that $t \le R/r_{\theta'}$ and $R \ge r_{\theta'}$, we get

$$\begin{aligned}\#\{\tilde\omega \in \mathcal{Q}_n^{(i)}|_\omega \colon \Delta\mathcal{E}_n^{(i)}(\tilde\omega) = R\} &\le 7r_{\theta'}^3 \sum_{t \le R/r_{\theta'}} 2^t \sum_{j=1}^{t} 3r_j^2 e^{3R/\sqrt{r_{\theta'}}} \\ &\le 7r_{\theta'}^3 \sum_{t \le R/r_{\theta'}} e^{4R/\sqrt{r_{\theta'}}} \le 7r_{\theta'}^3 R e^{4R/\sqrt{r_{\theta'}}} \le e^{5R/\sqrt{r_{\theta'}}},\end{aligned}$$

if η is chosen sufficiently small. □

6.2 Metric Estimates

In Proposition 6.2, we showed that the number of intervals $\tilde\omega \in \mathcal{Q}_n^{(i)}$ such that $\tilde\omega \subseteq \omega \in \mathcal{Q}_n^{(i-1)}$ and $\Delta\mathcal{E}_n^{(i)}(\omega) = R > 0$ was exponentially bounded with respect to R. Even more, we proved that the exponential constant could be made arbitrarily small by taking η small. In this section, we will turn our attention towards the sizes of the intervals $\tilde\omega$ with respect to their host intervals ω.

Proposition 6.3 *There exists $\alpha_1 > 0$ such that for all α and η sufficiently small, and for every interval $\tilde\omega \in \mathcal{Q}_n^{(i)}|_\omega$, $\omega \in \mathcal{Q}_n^{(i-1)}$ with $\Delta\mathcal{E}_n^{(i)}(\omega) = R > 0$, we have*

$$|\tilde\omega| \le e^{-\alpha_1 R}|\omega|.$$

Proof. By the construction of the partition, there is a nested sequence of intervals, $\tilde\omega \subseteq \omega^{(s)} \subseteq ... \subseteq \omega^{(1)} \subseteq \omega^{(0)} = \omega$, created due to their essential returns at times $\nu_1, ..., \nu_s$. Write

$$\frac{|\tilde\omega|}{|\omega|} = \frac{|\omega^{(1)}|}{|\omega^{(0)}|} \cdots \frac{|\omega^{(s)}|}{|\omega^{(s-1)}|}\frac{|\tilde\omega|}{|\omega^{(s)}|}.$$

Clearly $|\tilde\omega|/|\omega^{(s)}| \le 1$, since $\tilde\omega \subseteq \omega^{(s)}$. The remaining factors can be estimated as below:

Lemma 6.4 *With the same notation as above, we have*

$$\frac{|\omega^{(j+1)}|}{|\omega^{(j)}|} \le De^{-|r_{j+1}| + \frac{1+\ell_s}{2}|r_j|} \qquad j = 1, ..., s-1.$$

Proof. By the construction $\omega^{(j)}$ has an essential return at time ν_j. In particular $I^*_{r_j,m_j} \subseteq \omega^{(j)}_{\nu_j} \subseteq \hat{I}^*_{r_j,m_j}$ for some $|r_j| \geq r_{\theta'}$, so $|\omega^{(j)}_{\nu_j}| \geq \eta^\theta e^{-|r_j|}/r_j^2$. This scenario splits into two different cases: $* = \pm c$ and $* = s$.

$* = \pm c$: By Lemma 5.8, we have $|\omega^{(j)}_{\nu_j+p_j+1}| \geq C_1 \eta^\theta e^{-\frac{1+\ell_s}{2}|r_j|}$, and by Proposition 3.2 we have $|\omega^{(j+1)}_{\nu_{j+1}}| \geq e^{\lambda(\nu_{j+1}-\nu_j-p_j-1)}|\omega^{(j+1)}_{\nu_j+p_j+1}|$, so

$$|\omega^{(j+1)}_{\nu_j+p_j+1}| \leq e^{-\lambda(\nu_{j+1}-\nu_j-p_j-1)}|\omega^{(j+1)}_{\nu_{j+1}}| \leq |\omega^{(j+1)}_{\nu_{j+1}}| \leq \eta^\theta e^{-|r_{j+1}|}.$$

Since the bounded distortion holds up to time $\nu_j + p_j + 1$, we get

$$\begin{aligned} \frac{|\omega^{(j+1)}|}{|\omega^{(j)}|} &\leq D_1 \frac{|\omega^{(j+1)}_{\nu_j+p_j+1}|}{|\omega^{(j)}_{\nu_j+p_j+1}|} \leq D_1 \frac{\eta^\theta e^{-|r_{j+1}|}}{C_1 \eta^\theta e^{-\frac{1+\ell_s}{2}|r_j|}} \\ &\leq D e^{-|r_{j+1}|+\frac{1+\ell_s}{2}|r_j|} \leq D e^{-|r_{j+1}|+\frac{1+\ell_s}{2}|r_j|}, \end{aligned}$$

for α sufficiently small.

$* = s$: Here we use the fact that, by definition, $p_j = 0$, and so

$$|\omega^{(j)}_{\nu_j+p_j+1}| \geq C_2 |\omega^{(j)}_{\nu_j}|^{\ell_s} \geq C_3 \eta^{\ell_s \theta} e^{-\ell_s |r_j|}/|r_j|^{2\ell_s}.$$

Now we have to be a bit careful: since $|r_j| \geq r_{\theta'}$, we do not have an upper bound (depending on η) for $|r_j|$. Continuing the estimate, and recalling that $\ell_s < 1$, we get

$$|\omega^{(j)}_{\nu_j+p_j+1}| = |\omega^{(j)}_{\nu_j+1}| \geq C_3 \eta^{\ell_s \theta} e^{-\frac{1+\ell_s}{2}|r_j|}\left(e^{\frac{1-\ell_s}{2}|r_j|}/|r_j|^{2\ell_s}\right) \geq C_4 \eta^\theta e^{-\frac{1+\ell_s}{2}|r_j|},$$

for η sufficiently small (i.e. r_j sufficiently large). Treating $|\omega^{(j+1)}_{\nu_j+p_j+1}|$ as in the first case, we get

$$\frac{|\omega^{(j+1)}|}{|\omega^{(j)}|} \leq D_1 \frac{\eta^\theta e^{-|r_{j+1}|}}{C_4 \eta^\theta e^{-\frac{1+\ell_s}{2}|r_j|}} \leq D e^{-|r_{j+1}|+\frac{1+\ell_s}{2}|r_j|}.$$

□

Lemma 6.5 *With the same notation as above, we have*

$$\frac{|\omega^{(1)}|}{|\omega^{(0)}|} \leq D e^{-|r_1|+\frac{1+\ell_s}{2} r_{\theta'}}.$$

Proof. Recall that $\omega^{(0)} = \omega \in \mathcal{Q}_n^{(i-1)}$, which means that $\omega^{(0)}$ has an escape time at time $\mu_{i-1} = \nu_0$.

First, suppose that $\omega_{\nu_1}^{(0)} \subseteq \Delta^\theta$. Then, by Proposition 7.1, we have bounded distortion up to time ν_1, and we thus have

$$\frac{|\omega^{(1)}|}{|\omega^{(0)}|} \leq D\frac{|\omega_{\nu_1}^{(1)}|}{|\omega_{\nu_1}^{(0)}|} \leq D\eta^\theta e^{-|r_1|}|\omega_{\nu_1}^{(0)}|^{-1}.$$

Hence, we just have to show that $|\omega_{\nu_1}^{(0)}| \geq C\eta^\theta e^{-\frac{1+\ell_s}{2}r_{\theta'}}$. We have two cases:

(1) $\omega_{\nu_0}^{(0)} \cap \Delta^\theta \neq \emptyset$: Then $I_{r,m}^* \subseteq \omega_{\nu_0}^{(0)} \subseteq \hat{I}_{r,m}^*$ for some $|r| \in [1, r_{\theta'}]$, and we have a binding period of length p_0 following ν_0. If $* = \pm c$, Lemma 5.8 and Proposition 3.2 give

$$|\omega_{\nu_1}^{(0)}| \geq |\omega_{\nu_0+p_0+1}^{(0)}| \geq C_1\eta^\theta e^{-\frac{1+\ell_s}{2}|r|},$$

for α sufficiently small.

If $* = s$, we have defined the binding period as $p_0 = 0$, so Proposition 3.2 gives

$$\begin{aligned}|\omega_{\nu_1}^{(0)}| &\geq |\omega_{\nu_0+1}^{(0)}| \geq C_2|\omega_{\nu_0}^{(0)}|^{\ell_s} \geq C_2\eta^{\ell_s\theta}e^{-\ell_s|r|}/|r|^{2\ell_s} \geq C_2\eta^\theta e^{-\ell_s|r|}/|r|^2 \\ &= C_2\eta^\theta e^{-\frac{1+\ell_s}{2}|r|}e^{\frac{1-\ell_s}{2}|r|}/|r|^2 \geq C_2\eta^\theta e^{-\frac{1+\ell_s}{2}|r|},\end{aligned}$$

for η sufficiently small (i.e., for $r_{\theta'}$ sufficiently large).

(2) $\omega_{\nu_0}^{(0)} \cap \Delta_*^\theta = \emptyset$: Then, by construction, $|\omega_{\nu_0}^{(0)}| \geq \eta^\theta$ and if $* = \pm c$, using Proposition 3.2 gives

$$|\omega_{\nu_1}^{(0)}| \geq |\omega_{\nu_0+3}^{(0)}| \geq C_4\eta^{\ell_c\theta+\ell_s-1} = C_4\eta^\theta\eta^{(\ell_c-1)\theta+\ell_s-1} \gg C_4\eta^\theta e^{-\ell_s r_{\theta'}} \geq C_4\eta^\theta e^{-\frac{1+\ell_s}{2}r_{\theta'}},$$

for η sufficiently small, since $(\ell_c-1)\theta + \ell_s - 1 < 0$. If $* = s$, we immediately have

$$|\omega_{\nu_1}^{(0)}| \geq |\omega_{\nu_0+1}^{(0)}| \geq C_5\eta^{\ell_s\theta} = C_5\eta^\theta\eta^{(\ell_s-1)\theta} \gg C_5\eta^\theta e^{-\ell_s r_{\theta'}} \geq C_5\eta^\theta e^{-\frac{1+\ell_s}{2}r_{\theta'}},$$

since $\ell_s - 1 < 0$.

Finally, if $\omega_{\nu_1}^{(0)} \not\subseteq \Delta^\theta$, we cannot apply Proposition 7.1 to the whole of $\omega^{(0)}$ up to time ν_1. However, by restricting ourselves to a maximal subinterval $\bar{\omega}^{(0)} \subset \omega^{(0)}$ such that $\bar{\omega}_{\nu_1}^{(0)} \subseteq \Delta^\theta$, we have

$$\frac{|\omega^{(1)}|}{|\omega^{(0)}|} \leq \frac{|\omega^{(1)}|}{|\bar{\omega}^{(0)}|} \leq D\frac{|\omega_{\nu_1}^{(1)}|}{|\bar{\omega}_{\nu_1}^{(0)}|} \leq D\eta^\theta e^{-|r_1|}|\bar{\omega}_{\nu_1}^{(0)}|^{-1}.$$

Hence it suffices to show that $|\bar{\omega}_{\nu_1}^{(0)}| \geq C\eta^\theta e^{-\frac{1+\ell_s}{2} r_{\theta'}}$. As $\bar{\omega}_{\nu_1}^{(0)}$ intersects $\Delta^{\theta'}$ (remember that $\bar{\omega}_{\nu_1}^{(0)} \supseteq \omega_{\nu_1}^{(1)}$ and $\omega_{\nu_1}^{(1)} \subseteq \Delta^{\theta'}$) and extends all the way to the boundary of Δ^θ (we assumed that $\omega_{\nu_1}^{(0)} \not\subseteq \Delta^\theta$), we have

$$|\bar{\omega}_{\nu_1}^{(0)}| \geq C_6 \eta^\theta \geq C_6 \eta^\theta e^{-\frac{1+\ell_s}{2} r_{\theta'}},$$

by the same argument as above. □

Returning to the proof of Proposition 6.3, the above two lemmas give

$$\begin{aligned}\frac{|\tilde{\omega}|}{|\omega|} &\leq D^s e^{-\sum_{i=1}^s |r_i| + \frac{1+\ell_s}{2}(r_{\theta'} + \sum_{i=1}^{s-1} |r_i|)} = D^s e^{(\frac{1+\ell_s}{2} - 1)\sum_{i=1}^s |r_i| + \frac{1+\ell_s}{2}(r_{\theta'} - |r_s|)} \\ &= D^s e^{(\frac{1+\ell_s}{2} - 1)R + \frac{1+\ell_s}{2}(r_{\theta'} - |r_s|)}.\end{aligned}$$

Since each $|r_i| \geq r_{\theta'}$, we have that $sr_{\theta'} \leq |r_1| + ... + |r_s| = R$, and so $s \leq R/r_{\theta'}$. Hence, $D^s = e^{s \log D} \leq e^{\frac{R \log D}{r_{\theta'}}} = e^{\beta R}$. Taking η small, i.e., $r_{\theta'}$ large, we can make β arbitrarily small. Moreover, $e^{\frac{1+\ell_s}{2}(r_{\theta'} - |r_s|)} \leq 1$, and we have the desired result with $\alpha_1 = 1 - \frac{1+\ell_s}{2} - \beta$. Inserting the expression for β, and taking α small gives

$$\frac{1+\ell_s}{2} + \beta = \frac{1+\ell_s}{2} + \frac{\log D}{r_{\theta'}} < \sqrt{\frac{1+\ell_s}{2}},$$

where the inequality is valid for small η. This gives

$$\alpha_1 = 1 - \frac{1+\ell_s}{2} - \beta > 1 - \sqrt{\frac{1+\ell_s}{2}} > 0.$$

□

7 Proof of Theorem 4.1

7.1 Bounded Distortion

Proposition 7.1 *If $\omega \in \mathcal{P}^{(n)}$, then*

$$\frac{|(f_a^k)'(c_1)|}{|(f_b^k)'(c_1)|} \leq e^{1/r_\delta} = D \quad \textit{and} \quad \frac{|(\varphi^k)'(a)|}{|(\varphi^k)'(b)|} \leq e^{1/r_\delta} = D$$

for all $a, b \in \omega$ and all $k \leq \nu_q + p_q + 1$, where ν_q is the last essential or inessential return of ω before time n, and p_q is the length of the corresponding binding period. If $n > \nu_q + p_q + 1$, then the same statement holds for all $k \leq n$ restricted to any subinterval $\bar{\omega} \subseteq \omega$ such that $\bar{\omega}_k \subseteq \Delta^\theta$.

Proof. By Lemma 5.5, we can write

$$\frac{|(f_a^k)'(c_1)|}{|(f_b^k)'(c_1)|} \leq e^{\sum_{j=1}^k D_j} \quad \text{where} \quad D_j = |\omega_j| \sup_{a,b\in\omega} \left| \frac{f''(c_j(a))}{f'(c_j(b))} \right|.$$

Suppose first that $k \leq \nu_q + p_q$. We will show that $\sum_{j=1}^k D_j \leq C/r_{\theta'}^2$, which immediately gives

$$\frac{|(f_a^k)'(c_1)|}{|(f_b^k)'(c_1)|} \leq e^{\sum_{j=1}^k D_j} \leq e^{C/r_{\theta'}^2} \ll e^{1/r_{\theta'}} \leq D.$$

Let $0 < \nu_1 < \cdots < \nu_q < k$ be all the times for which $\omega_{\nu_i} \subset \hat{\Delta}^\theta$, and which do not belong to any binding periods. By our construction of $\mathcal{I}$, there is a unique element I_{r_i,m_i} in $\mathcal{I}$ associated to each ν_i. For notational convenience, we define ν_0 and p_0 such that $\nu_0 + p_0 = 0$. Then we can write

$$\sum_{j=1}^{\nu_q+p_q} D_j = \sum_{i=0}^{q-1} \sum_{j=\nu_i+p_i+1}^{\nu_{i+1}+p_{i+1}} D_j = \sum_{i=0}^{q-1} \left(\sum_{j=\nu_i+p_i+1}^{\nu_{i+1}-1} D_j + D_{\nu_{i+1}} + \sum_{j=\nu_{i+1}+1}^{\nu_{i+1}+p_{i+1}} D_j \right),$$

i.e., we split the orbit into free times, non-bound return times, and bound times, respectively. The point of this splitting is that each of the three terms on the right hand side can be bounded above by some multiple of $|\omega_{\nu_{i+1}}|\eta^{-\theta} e^{r_{i+1}}$. We will prove this in three consecutive lemmas.
We start with the free times:

Lemma 7.2

$$\sum_{j=\nu_i+p_i+1}^{\nu_{i+1}-1} D_j \leq C(\epsilon)|\omega_{\nu_{i+1}}|\eta^{-\theta}.$$

Proof. Since $\omega_{\nu_{i+1}} \subseteq \hat{I}^*_{r_{i+1},m_{i+1}} \subset \hat{\Delta}^\theta$, condition $(UE)_\epsilon$ immediately gives

$$|(f^{\nu_{i+1}-j})'(c_j(a))| \geq C_1(\epsilon)e^{\lambda(\nu_{i+1}-j)},$$

which means that, by the PMVT, we have

$$|\omega_j| \leq C_2(\epsilon) P^{-2} e^{-\lambda(\nu_{i+1}-j)} |\omega_{\nu_{i+1}}|.$$

Moreover, for $j \in [\nu_i + p_i + 1, \nu_{i+1} - 1]$, ω_j stays out of Δ^θ, which implies

$$\sup_{a,b\in\omega} \left| \frac{f''(c_j(a))}{f'(c_j(b))} \right| \leq C_3 \eta^{-\theta}.$$

This gives

$$\sum_{j=\nu_i+p_i+1}^{\nu_{i+1}-1} D_j \leq C_4(\epsilon) P^{-2} |\omega_{\nu_{i+1}}| \eta^{-\theta} \sum_{j=\nu_i+p_i+1}^{\nu_{i+1}-1} e^{-\lambda(\nu_{i+1}-j)}$$

$$\leq C_5(\epsilon) |\omega_{\nu_{i+1}}| \eta^{-\theta} \sum_{j=0}^{\infty} e^{-\lambda j} \leq C_6(\epsilon) |\omega_{\nu_{i+1}}| \eta^{-\theta}$$

□

Next, we deal with the non-bound return times:

Lemma 7.3

$$D_{\nu_{i+1}} \leq C |\omega_{\nu_{i+1}}| \eta^{-\theta} e^{r_{i+1}}.$$

Proof. Since $\omega_{\nu_{i+1}} \subseteq \hat{I}^*_{r_{i+1},m_{i+1}} \subset \hat{\Delta}^\theta$, we have

$$\mathfrak{D}(\omega_{\nu_{i+1}}) \geq \eta^\theta e^{-(r_{i+1}+1)} \gg 3\eta^\theta e^{-(r_{i+1})}/r_{i+1}^2 \geq |\omega_{\nu_{i+1}}|.$$

Hence

$$D_{\nu_{i+1}} \leq |\omega_{\nu_{i+1}}| \sup_{a,b\in\omega} \left| \frac{f''(c_{\nu_{i+1}}(a))}{f'(c_{\nu_{i+1}}(b))} \right| \leq \frac{C_1 |\omega_{\nu_{i+1}}|}{\mathfrak{D}(\omega_{\nu_{i+1}})} \leq C_2 |\omega_{\nu_{i+1}}| \eta^{-\theta} e^{r_{i+1}}.$$

□

Finally, we take care of the bound times:

Lemma 7.4

$$\sum_{j=\nu_{i+1}+1}^{\nu_{i+1}+p_{i+1}} D_j \leq C(\epsilon) |\omega_{\nu_{i+1}}| \eta^{-\theta} e^{r_{i+1}}.$$

Proof. If $\omega_{\nu_{i+1}}$ is a return to $\hat{\Delta}_s^\theta$, the binding period is empty, and there is nothing to prove. Hence, we may assume that $\omega_{\nu_{i+1}} \subseteq \hat{I}^*_{r_{i+1},m_{i+1}} \subset \hat{\Delta}^\theta_c$. By the PMVT and bounded distortion during binding periods, we know that, for $j \in [1, p_{i+1}]$, we have

$$|\omega_j| \le C_1 |\omega_{\nu_{i+1}}| \sup_{a\in\omega} |(f_a^{\nu_{i+1}+j})'(c_{\nu_{i+1}}(a))|.$$

By the chain rule, we have

$$|(f_a^{\nu_{i+1}+j})'(c_{\nu_{i+1}}(a))| = |f_a'(c_{\nu_{i+1}+1}(a))(f_a^{\nu_{i+1}+j-1})'(c_{\nu_{i+1}+1}(a))|.$$

Without loss of generality, we may assume that $c_{\nu_{i+1}}(a) < c$. Then we let $\omega^\star = [c_{\nu_{i+1}}(a), c]$, and $\omega_j^\star = [c_{\nu_{i+1}+j}(a), c_j(a)]$. Since we are in a binding period, we know that $|\omega_j^\star| \le \eta^\theta e^{-2\alpha j}$, and by the order of the critical point, we also have $|\omega_1^\star| \ge C_2\big(\eta^\theta e^{-r_{i+1}}\big)^{\ell_c}$. Once again, using the PMVT and bounded distortion during binding periods, we know that, for $j \in [1, p_{i+1}]$, we have

$$|(f_a^{\nu_{i+1}+j-1})'(c_{\nu_{i+1}+1}(a))| \le C_3 \frac{|\omega_j^\star|}{|\omega_1^\star|} \le C_4 \eta^\theta e^{-2\alpha j}\big(\eta^\theta e^{-r_{i+1}}\big)^{-\ell_c},$$

and since $\mathfrak{D}(c_{\nu_{i+1}}(a)) \le C_5 \eta^\theta e^{-r_{i+1}}$, we have $|f_a'(c_{\nu_{i+1}+1}(a))| \le C_6\big(\eta^\theta e^{-r_{i+1}}\big)^{\ell_c - 1}$. Combining these estimates gives

$$|\omega_j| \le C_7 |\omega_{\nu_{i+1}}| \big(\eta^\theta e^{-r_{i+1}}\big)^{\ell_c-1} \eta^\theta e^{-2\alpha j}\big(\eta^\theta e^{-r_{i+1}}\big)^{-\ell_c} = C_7 |\omega_{\nu_{i+1}}| e^{-2\alpha j} e^{r_{i+1}}.$$

Now, by $(BD)_n$, $\mathfrak{D}(c_j(a)) \ge C_8 \eta^\theta e^{-\alpha j}$, and combining this with $|\omega_j^\star| \le \eta^\theta e^{-2\alpha j}$, gives

$$\mathfrak{D}(\omega_j^\star) \ge \eta^\theta e^{-\alpha j}(1 - e^{-\alpha j}) \ge \eta^\theta e^{-2\alpha j} \ge |\omega_j^\star|.$$

By Lemma 5.6,

$$\sup_{a,b\in\omega} \left| \frac{f''(c_j(a))}{f'(c_j(b))} \right| \le \frac{C_9}{\mathfrak{D}(\omega_j^\star)} \le C_8 \eta^{-\theta} e^{\alpha j}(1 - e^{-\alpha j})^{-1} \le C_9 \eta^{-\theta} e^{\alpha j}.$$

Therefore, $D_j \le C_7 |\omega_{\nu_{i+1}}| e^{-2\alpha j} e^{r_{i+1}} C_9 \eta^{-\theta} e^{\alpha j} = C_{10} |\omega_{\nu_{i+1}}| e^{-\alpha j} e^{r_{i+1}} \eta^{-\theta}$, and summing gives the result. □

Returning to the proof of the proposition, we have

$$\sum_{j=0}^{\nu_q + p_q} D_j = \sum_{i=0}^{q-1} \sum_{j=\nu_i+p_i+1}^{\nu_{i+1}+p_{i+1}} D_j \le C(\epsilon)\eta^{-\theta} \sum_{i=0}^{q-1} |\omega_{\nu_i}| e^{r_i}.$$

The right-most sum can be split up into partial sums corresponding to return times with the same return depth:

$$\sum_{i=0}^{q-1} |\omega_{\nu_i}| e^{r_i} = \sum_{R \geq r_{\theta'}} e^R \sum_{i:\, r_i = R} |\omega_{\nu_i}|.$$

Lemma 7.5 *For any $R \geq r_{\theta'}$, we have*

$$\sum_{i:\, r_i = R} |\omega_{\nu_i}| \leq C\eta^\theta e^{-R}/R^2.$$

This lemma immediately gives

$$\begin{aligned}\sum_{j=0}^{\nu_q+p_q} D_j &\leq C(\epsilon)\eta^{-\theta} \sum_{R \geq r_{\theta'}} e^R \sum_{i:\, r_i = R} |\omega_{\nu_i}| \\ &\leq C(\epsilon)\eta^{-\theta} \sum_{R \geq r_{\theta'}} e^R C\eta^\theta e^{-R}/R^2 \leq C_1(\epsilon) \sum_{R \geq r_{\theta'}} \frac{1}{R^2} \leq C_2(\epsilon)/r_{\theta'}^2,\end{aligned}$$

which is exactly what we wanted to show.

Proof. (of Lemma 7.5) Let $\mu_j = \nu_{i_j}$, $j = 1, \ldots, m$ be the subsequence of returns with return depth R. By construction, we know that $|\omega_{\mu_m}| \leq C_1\eta^\theta e^{-R}/R^2$. Using the binding period estimates and uniform expansion, we have for all parameters $a \in \omega$ and all $j = 1, \ldots, m-1$,

$$|(f_a^{\mu_{j+1}-\mu_j})'(c_{\mu_j}(a))| \geq |(f_a^{\mu_j+p_j+1})'(c_{\mu_j}(a))| \geq e^{\frac{1-\ell_s}{2}R} \geq e^{\frac{1-\ell_s}{2}r_{\theta'}} \geq C_2\eta^{-(\theta'-\theta)\frac{1-\ell_s}{2}}.$$

The PMVT gives $|\omega_{\mu_j}| \leq C_3\eta^{(\theta'-\theta)\frac{1-\ell_s}{2}}|\omega_{\mu_{j+1}}|$, so

$$\sum_{i:\, r_i = R} |\omega_{\nu_i}| = \sum_{j=1}^{m} |\omega_{\mu_j}| \leq C_3 \sum_{j=0}^{m-1} \eta^{j(\theta'-\theta)\frac{1-\ell_s}{2}} |\omega_{\mu_m}| \leq C_4|\omega_{\mu_m}| \leq C_5\eta^\theta e^{-R}/R^2.$$

□

This takes care of the first part of the proposition. However, if $k > \nu_q + p_q + 1$, we need to consider the additional terms

$$\sum_{j=\nu_q+p_q+1}^{k-1} D_j,$$

restricting ourselves to a subinterval $\bar{\omega} \subset \omega$ with $\bar{\omega}_k \subseteq \Delta^\theta$. It is clear that the previous estimates are unaffected (actually they are improved) by this restriction. Using the uniform expansion estimates, we have

$$|\bar{\omega}_j| \le e^{-\hat{\lambda}(k-j)}|\bar{\omega}_k| \le 2e^{-\hat{\lambda}(k-j)}\eta^\theta.$$

Therefore, using the fact that $|f'(c_j(a))| \ge C_1\eta^{(\ell_c-1)\theta}$ since $\bar{\omega}_j \cap \Delta^\theta = \emptyset$, we get

$$\sum_{j=\nu_q+p_q+1}^{k-1} D_j \le C_2 \sum_{j=\nu_q+p_q+1}^{k-1} e^{-\hat{\lambda}(k-j)} \le C_3.$$

This completes the proof in this case also. □

7.2 Condition $(BR)_n$

In this section, we will prove the following theorem, as promised:

Theorem 7.6 *All parameters in $\Omega^{(n)}$ satisfy $(BR)_n$.*

The proof relies heavily on the fact that the sum of all return depths is proportional to the sum of all essential return depths. Consider the partition $\mathcal{Q}_n^{(n)} = \hat{\mathcal{P}}^{(n)}$ of $\Omega^{(n-1)}$. By our construction, each element of $\mathcal{Q}_n^{(n)}$ has associated sequences of escape and return times, and corresponding return depths. Recall the integer-valued functions

$$\mathcal{R}^{(n)}\colon \Omega^{(n)} \to \mathbb{N} \quad \text{and} \quad \mathcal{E}^{(n)}\colon \Omega^{(n)} \to \mathbb{N},$$

both constant on elements of $\hat{\mathcal{P}}^{(n)}$, which assign to each $a \in \omega \in \hat{\mathcal{P}}^{(n)}$ the total sum of all return depths, and the total sum of all essential return depths associated to the orbit of ω up to time n, respectively. Recall also that, by definition, $\Omega^{(n)}$ is formed by all components $\omega \in \hat{\mathcal{P}}^{(n)}$ satisfying $\mathcal{E}^{(n)}(\omega) \le \alpha n/(2\mathfrak{C})$.

Although one could expect $\mathcal{R}$ to be much larger than $\mathcal{E}$, this is simply not the case.

Proposition 7.7 *For any $\omega \in \mathcal{Q}_n^{(n)}$,*

$$\mathcal{R}^{(n)}(\omega) \le \mathfrak{C}\mathcal{E}^{(n)}(\omega),$$

where

$$\mathfrak{C} = \big(1 + \frac{4\alpha}{\lambda}\big)\frac{4}{1-\ell_s}.$$

Before proving the proposition, we show how it implies the statement in Theorem 7.6.

Proof. (Theorem 7.6) Take a parameter $a \in \omega \in \mathcal{Q}_n^{(n)}$. Then, by construction, we have $\mathcal{E}^{(n)}(a) \leq \alpha n/(2\mathfrak{C})$. By Proposition 7.7, it immediately follows that

$$\mathcal{R}^{(n)}(a) \leq \mathfrak{C}\alpha n/(2\mathfrak{C}) \leq \alpha n/2.$$

Now, if ν_i is a return for ω, and its associated return depth is r_i, then all parameters in ω satisfy $\mathfrak{D}(c_{\nu_i}(a)) \geq \eta^\theta e^{-2r_i}$ (in fact, we have $\mathfrak{D}(c_{\nu_i}(a)) \geq \eta^\theta e^{-(r_i+2)}$). Hence, taking all returns up to time n into account, we have

$$\prod_{\nu_i \leq n} \frac{\mathfrak{D}(c_{\nu_i}(a))}{\eta^\theta} \geq \prod_{\nu_i \leq n} \frac{\eta^\theta e^{-2r_i}}{\eta^\theta} = \prod_{\nu_i \leq n} e^{-2r_i} = e^{-2\sum_{\nu_i \leq n} r_i} = e^{-2\mathcal{R}^{(n)}(a)} \geq e^{-\alpha n},$$

and this is exactly condition $(BR)_n$ for $n \geq N = \frac{\theta' - \theta}{\alpha} \log \eta^{-1}$. As we excluded all parameters giving rise to returns to Δ^θ before time N, it is clear that all parameters in $\Omega^{(n)}$ satisfy $(BR)_n$. □

Let us now return to the proof of Proposition 7.7:

Proof. (Proposition 7.7) The idea of the proof is the following. Suppose $\nu := \mu_0$ is an essential return for some $\omega \in \mathcal{Q}_n^{(n)}$, and its corresponding return depth is r_0. There follows a binding period during which bound returns can occur, and a sequence of inessential returns each one followed by its own binding period. To each inessential return μ_i and bound return $\zeta_{i,j}$, we associate the return depth r_i and $\rho_{i,j}$, respectively.

$$\mu_0 < \zeta_{0,1} < \cdots < \zeta_{0,t_0} < \mu_0 + p_0 < \mu_1 < \cdots < \mu_k < \zeta_{k,1} < \cdots < \zeta_{k,t_k} < \mu_k + p_k$$

We want to show that the total sum of all these inessential and bound return depths is bounded from above by $\mathfrak{C} r_0$. We start by showing that the sum of all inessential return depths is proportional to r_0:

Lemma 7.8 *Let $\omega \in \mathcal{P}^{(\nu)}$, $\nu \in [1, n]$ and suppose that ν is an essential return for ω: $I^*_{r_0,m} \subseteq \omega_\nu \subseteq \hat{I}^*_{r_0,m}$. Set $\nu := \mu_0$ and let $\mu_1 < \cdots < \mu_k$ be a maximal sequence of inessential returns occurring after time ν and before any subsequent chopping time, all with the corresponding inessential return depths $r_0, \ldots, r_k$. Then*

$$\sum_{i=0}^{k} r_i \leq \frac{4}{1 - \ell_s} r_0.$$

Proof. To each return, we associate binding periods $p_0, \ldots, p_k$. By Proposition 5.8, we have

$$|(f_a^{p_i+1})'(c_{\mu_i}(a))| \geq e^{\frac{1-\ell_s}{2} r_i},$$

for all $a \in \omega$, and by our expansion estimates, valid outside $\Delta^{\theta'}$, we know that no loss of expansion occurs between the end of a binding period and the following return, i.e.,

$$|(f_a^{\mu_{i+1}-(\mu_i+p_i+1)})'(c_{\mu_i+p_i+1}(a))| \geq 1.$$

Combining these two estimates gives

$$|(f_a^{\mu_k+p_k+1-\mu_0})'(c_{\mu_0}(a))| \geq e^{\frac{1-\ell_s}{2}\sum_{i=0}^k r_i}.$$

Since no interval can grow larger than the whole dynamical space I_η, we can use the Parameter Mean Value Theorem to get

$$3c \geq |I_\eta| \geq |\omega_{\mu_k+p_k+1}| \geq P^{-2}|\omega_{\mu_0}|e^{\frac{1-\ell_s}{2}\sum_{i=0}^k r_i} \geq \frac{P^{-2}}{r_0^2}e^{-r_0}e^{\frac{1-\ell_s}{2}\sum_{i=0}^k r_i}.$$

A little rearranging gives

$$\sum_{i=0}^k r_i \leq \frac{2}{1-\ell_s}\left(r_0 + \log 3P^2 c r_0^2\right) \leq \frac{4}{1-\ell_s} r_0,$$

for η sufficiently small, i.e., for r_0 sufficiently large. □

Next, we take care of the sum of all bound return depths:

Lemma 7.9 *Let $\omega \in \mathcal{P}^{(\mu)}$, $\mu \in [1, n]$ and suppose that μ is an inessential return for ω: $\omega_\mu \subseteq \hat{I}^*_{r,m}$. Let p be the length of the binding period following time μ, and let $\mu < \zeta_1 < \cdots < \zeta_t < \mu + p$ be the bound returns of ω, (i.e., those times for which $\omega_{\zeta_i} \cap \Delta^{\theta'} \neq \emptyset$), all with the corresponding bound return depths $\rho_1, \ldots, \rho_t$. Then*

$$\sum_{i=1}^t \rho_i \leq \frac{4\alpha}{\lambda} r.$$

Proof. During the binding periods, the intervals ω_j, $j \in [\mu+1, \mu+p]$ are very close to the intervals $\omega_{j-\mu}$, i.e., ω is almost retracing part of its initial itinerary. Since none of the bound returns are chopping times, we have

$$\eta^\theta e^{-\rho_i} \leq \inf_{a\in\omega} \mathfrak{D}(c_{\zeta_i}(a)) \leq \sup_{a\in\omega} \mathfrak{D}(c_{\zeta_i}(a)) \leq \eta^\theta e^{-\rho_i+1}.$$

By the definition of binding, and since a satisfies $(BR)_\mu$, we have

$$|c_{\zeta_i}(a) - c_{\zeta_i-\mu}(a)| \le \eta^\theta e^{-2\alpha(\zeta_i-\mu)} \quad \text{and} \quad \mathfrak{D}(c_{\zeta_i-\mu}(a)) \ge \eta^\theta e^{-\alpha(\zeta_i-\mu)},$$

respectively. Therefore

$$\begin{aligned}\mathfrak{D}(c_{\zeta_i}(a)) &\ge \mathfrak{D}(c_{\zeta_i-\mu}(a)) - \eta^\theta e^{-2\alpha(\zeta_i-\mu)} = \mathfrak{D}(c_{\zeta_i-\mu}(a))\left(\frac{1-\eta^\theta e^{-2\alpha(\zeta_i-\mu)}}{\mathfrak{D}(c_{\zeta_i-\mu}(a))}\right)\\ &\ge \mathfrak{D}(c_{\zeta_i-\mu}(a))(1-\eta^\theta e^{-\alpha(\zeta_i-\mu)}) \ge \mathfrak{D}(c_{\zeta_i-\mu}(a))(1-\eta^\theta e^{-\alpha}).\end{aligned}$$

This immediately gives

$$\eta^\theta e^{-\rho_i+1} \ge \mathfrak{D}(c_{\zeta_i-\mu}(a))(1-\eta^\theta e^{-\alpha}),$$

and after a little rearranging, we have

$$e^{-\rho_i} \ge e^{-1}(1-\eta^\theta e^{-\alpha})\frac{\mathfrak{D}(c_{\zeta_i-\mu}(a))}{\eta^\theta}.$$

Taking all bound returns into account gives

$$\prod_{i=1}^t e^{-\rho_i} \ge e^{-t}(1-\eta^\theta e^{-\alpha})^t \prod_{i=1}^t \frac{\mathfrak{D}(c_{\zeta_i-\mu}(a))}{\eta^\theta} \ge \big(e^{-1}(1-\eta^\theta e^{-\alpha})\big)^t e^{-\alpha p},$$

since, by $(BR)_\mu$, the right-most product is bounded below by $e^{-\alpha p}$ (yes, $p < \mu$). By Lemma 3.5, we have $t \le \alpha p / r_{\theta'}$ so by taking logarithms, we arrive at

$$\sum_{i=1}^t \rho_i \le t\big(1-\log(1-\eta^\theta e^{-\alpha})\big) + \alpha p \le \Big(\frac{1-\log(1-\eta^\theta e^{-\alpha})}{r_{\theta'}} + 1\Big)\alpha p \le 2\alpha p,$$

for η sufficiently small. Now, by our binding period estimates, we have

$$p \le -\frac{\ell_c}{\lambda}\log\frac{\mathfrak{D}(c_\mu(a))}{\eta^\theta} \le \frac{\ell_c}{\lambda} r,$$

which finally gives

$$\sum_{i=1}^t \rho_i \le \frac{2\alpha\ell_c}{\lambda} r \le \frac{4\alpha}{\lambda} r.$$

$\square$

Returning to the proof of Proposition 7.7, we just have to sum over all return depths:

$$\begin{aligned}\sum_{i=0}^{k}\Big(r_i+\sum_{j=1}^{t_i}\rho_{i,j}\Big) &\le \sum_{i=0}^{k}\Big(r_i+\frac{4\alpha}{\lambda}r_i\Big)\\ &= \big(1+\frac{4\alpha}{\lambda}\big)\sum_{i=0}^{k} r_i \le \big(1+\frac{4\alpha}{\lambda}\big)\frac{4}{1-\ell_s}r_0,\end{aligned}$$

which completes the proof. □

8 Appendix

By our definition of f_a, we have the following inequalities:

M1: For all $x \neq 0$ close enough to the origin,

$$\begin{aligned}|f_a(x)+\operatorname{sgn}(x)a| &\sim |x|^{\ell_s},\\ f_a'(x) &\sim |x|^{\ell_s-1}.\end{aligned}$$

M2: For all x close enough to one of the critical points, $\pm c$,

$$\begin{aligned}|f_a(c)-f_a(x)| &\sim \big||x|-c\big|^{\ell_c},\\ |f_a'(x)| &\sim \big||x|-c\big|^{\ell_c-1}.\end{aligned}$$

Here $f_a(x)+\operatorname{sgn}(x)a \sim |x|^{\ell_s}$ means $f_a(x)+\operatorname{sgn}(x)a = \mathcal{O}(|x|^{\ell_s})$. We shall use this notation in the remainder of the paper. We will now state some necessary estimates for the forward and backward iterates of our previously defined neighbourhoods. Since the functions we are studying are odd, we will only state results for the positive critical point, c.

Proposition 8.1 *For all $\epsilon, \eta > 0$ sufficiently small, we have*

$$\begin{aligned}f^{-1}_{c+\epsilon+\eta}(c) &\sim -\epsilon^{\frac{1}{\ell_s}} \quad &&\textit{and} \quad |f^{-1}_{c+\epsilon+\eta}(\Delta_c^\theta)| \sim \eta^\theta \ll \eta \ll \epsilon^{\frac{1}{\epsilon ll_s}},\\ f^{2}_{c+\epsilon+\eta}(c) &\sim -\eta \quad &&\textit{and} \quad \eta^{2\theta} \ll |f^{2}_{c+\epsilon+\eta}(\Delta_c^\theta)| \sim \eta^{\ell_c\theta} \ll \eta,\\ f^{3}_{c+\epsilon+\eta}(c) &\sim c+\epsilon \quad &&\textit{and} \quad \eta^{\theta} \ll |f^{3}_{c+\epsilon+\eta}(\Delta_c^\theta)| \sim \eta^{\ell_c\theta+\ell_s-1} \ll \epsilon.\end{aligned}$$

This proposition, although simple in its statement and proof, is crucial in order to obtain the expansion estimates valid outside $\Delta^{\theta'}$.

We will often refer to another, similar proposition:

Proposition 8.2 *For all $\epsilon, \eta > 0$ sufficiently small, we have*

$$
\begin{aligned}
&|f^{-1}_{c+\epsilon+\eta}(\Delta^{\epsilon}_{c})| \sim \epsilon^{\kappa+\frac{1}{\ell_s}-1} \ll \epsilon^{\frac{1}{\ell_s}},\\
&|f^{2}_{c+\epsilon+\eta}(\Delta^{\epsilon}_{c})| \sim \epsilon^{\ell_c\kappa} \gg \epsilon^{2\kappa} \sim |\Delta^{\epsilon}_{s}|,\\
&|f_{c+\epsilon+\eta}(\Delta^{\theta}_{s})| \sim \eta^{\ell_s\theta} \gg \eta^{\theta} \sim |\Delta^{\theta}_{s}|,\\
&|f_{c+\epsilon+\eta}(\Delta^{\epsilon}_{s})| \sim \epsilon^{2\kappa\ell_s} \gg \epsilon^{2\kappa} \sim |\Delta^{\epsilon}_{s}|.
\end{aligned}
$$

The essence of Propositions 8.1 and 8.2 is illustrated in the following figure:

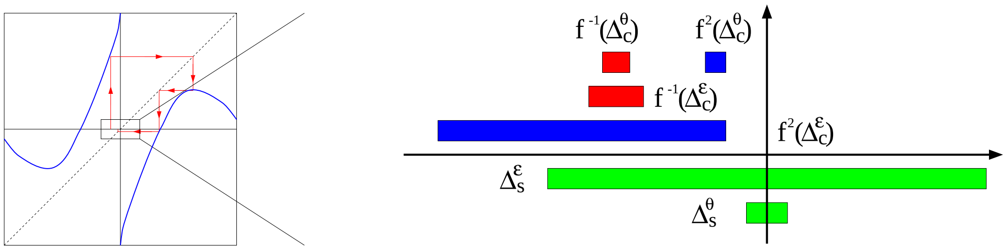

Figure 3: The forward and backward iterates of the neighbourhoods.

We will prove Proposition 8.1 and Proposition 8.2 by a sequence of simple lemmas.

Lemma 8.3 *There exists constants $0 < C_1 < C_2$ such that for all $\epsilon, \eta, r > 0$ sufficiently small, we have*

$$
-\Big(\frac{\epsilon+\eta+r}{C_1}\Big)^{\frac{1}{\ell_s}} \le f^{-1}_{c+\epsilon+\eta}([c-r, c+r]) \le -\Big(\frac{\epsilon+\eta-r}{C_2}\Big)^{\frac{1}{\ell_s}},
$$

Proof. By the first line in **M1**, there exists constants $0 < C_1 < C_2$ such that

$$
\begin{aligned}
-C_2|f^{-1}_{c+\epsilon+\eta}(c-r)|^{\ell_s} &\le f_{c+\epsilon+\eta}(f^{-1}_{c+\epsilon+\eta}(c-r)) - (c+\epsilon+\eta)\\
&= -(r+\epsilon+\eta) \le -C_1|f^{-1}_{c+\epsilon+\eta}(c-r)|^{\ell_s}.
\end{aligned}
$$

Rearranging this gives

$$-\Big(\frac{\epsilon+\eta+r}{C_1}\Big)^{\frac{1}{\ell_s}} \le f^{-1}_{c+\epsilon+\eta}(c-r) \le -\Big(\frac{\epsilon+\eta+r}{C_2}\Big)^{\frac{1}{\ell_s}},$$

which gives the lower bound on $f^{-1}_{c+\epsilon+\eta}([c-r,c+r])$. By considering $c+r$, we get the upper bound analogously. □

Lemma 8.4 *$f^{-1}_{c+\epsilon+\eta}(c) \sim -\epsilon^{\frac{1}{\ell_s}}$ and $|f^{-1}_{c+\epsilon+\eta}(\Delta^\theta_c)| \sim \eta^\theta \ll \eta \ll \epsilon^{\frac{1}{\ell_s}}$.*

Proof. The first part follows immediately from Lemma 8.3 by taking $r=0$ and noting that $\eta \ll \epsilon$.

To prove the second part, we take $r=\eta^\theta$, which gives

$$\begin{aligned}|f^{-1}_{c+\epsilon+\eta}(\Delta^\theta_c)| = |f^{-1}_{c+\epsilon+\eta}([c-\eta^\theta, c+\eta^\theta])| &= \Big(\frac{\epsilon+\eta+\eta^\theta}{C_1}\Big)^{\frac{1}{\ell_s}} - \Big(\frac{\epsilon+\eta-\eta^\theta}{C_2}\Big)^{\frac{1}{\ell_s}}\\ &\sim (\epsilon+\eta+\eta^\theta)^{\frac{1}{\ell_s}} - (\epsilon+\eta-\eta^\theta)^{\frac{1}{\ell_s}}\\ &= (\epsilon+\eta)^{\frac{1}{\ell_s}}\Big((1+\frac{\eta^\theta}{\epsilon+\eta})^{\frac{1}{\ell_s}} - (1-\frac{\eta^\theta}{\epsilon+\eta})^{\frac{1}{\ell_s}}\Big)\\ &\sim (\epsilon+\eta)^{\frac{1}{\ell_s}}\big((1+\frac{\eta^\theta}{\epsilon\ell_s}) - (1-\frac{\eta^\theta}{\epsilon\ell_s})\big) = \frac{2}{\ell_s}\epsilon^{\frac{1}{\ell_s}-1}\eta^\theta \sim \eta^\theta.\end{aligned}$$

Since $\theta > 1$ and $\eta \ll \epsilon$, the proof is complete. □

The remaining parts of Proposition 8.1 are even simpler to prove:

Lemma 8.5 *$f^2_{c+\epsilon+\eta}(c) \sim -\eta$ and $\eta^{2\theta} \ll |f^2_{c+\epsilon+\eta}(\Delta^\theta_c)| \sim \eta^{\ell_c\theta} \ll \eta$.*

Proof. The first part follows by using the fact that $0 = f^2_{c+\epsilon}(c) = f_{c+\epsilon}(f_{c+\epsilon}(c)) = f_{c+\epsilon}(c_1)$. We thus have

$$\begin{aligned}f^2_{c+\epsilon+\eta}(c) &= f_{c+\epsilon+\eta}(f_{c+\epsilon+\eta}(c)) = f_{c+\epsilon+\eta}(c_1-\eta)\\ &\sim f_{c+\epsilon+\eta}(c_1) - \eta f'_{c+\epsilon+\eta}(c_1) = 0 - C_1\eta \sim -\eta.\end{aligned}$$

The second part now follows since

$$|f^2_{c+\epsilon+\eta}(\Delta^\theta_c)| \sim |\Delta^\theta_c|(f^2_{c+\epsilon+\eta})'(c) = C_2\eta^\theta\eta^{\theta(\ell_c-1)} \sim \eta^{\ell_c\theta},$$

and by the fact that $1 < \ell_c < 2$. □

Lemma 8.6 $f^3_{c+\epsilon+\eta}(c) \sim c+\epsilon$ *and* $\eta^\theta \ll |f^3_{c+\epsilon+\eta}(\Delta^\theta_c)| \sim \eta^{\ell_c\theta+\ell_s-1} \ll \epsilon$.

Proof. The first part follows from the fact that

$$f^3_{c+\epsilon+\eta}(c) \sim f_{c+\epsilon+\eta}(-\eta) \sim f_{c+\epsilon+\eta}(0^+) = c+\epsilon+\eta \sim c+\epsilon.$$

Since $f^2_{c+\epsilon+\eta}(c) \sim -\eta$ and $f'_{c+\epsilon+\eta}(-\eta) \sim \eta^{\frac{1}{\ell_s}}$, it follows that

$$|f^3_{c+\epsilon+\eta}(\Delta^\theta_c)| \sim |f^2_{c+\epsilon+\eta}(\Delta^\theta_c)||f'_{c+\epsilon+\eta}(-\eta) \sim \eta^{\ell_c\theta}\eta^{\ell_s-1} = \eta^{\ell_c\theta+\ell_s-1}.$$

By definition,

$$\theta < \frac{1-\ell_s}{\ell_c-1} \iff \theta(\ell_c-1) < 1-\ell_s \iff \ell_c\theta+\ell_s-1 < \theta,$$

which completes the proof. □

Thus Proposition 8.1 is proved. We now proceed to the proof of Proposition 8.2:

We will prove this proposition in a sequence of lemmas:

Lemma 8.7 $|f^{-1}_{c+\epsilon+\eta}(\Delta^\epsilon_c)| \sim \epsilon^{\kappa+\frac{1}{\ell_s}-1} \ll \epsilon^{\frac{1}{\ell_s}}$.

Proof. Using Lemma 8.3, we take $r=\epsilon^\kappa$, which gives

$$\begin{aligned}
|f^{-1}_{c+\epsilon+\eta}(\Delta^\epsilon_c)| &= |f^{-1}_{c+\epsilon+\eta}([c-\epsilon^\kappa, c+\epsilon^\kappa])| \\
&= \Big(\frac{\epsilon+\eta+\epsilon^\kappa}{C_1}\Big)^{\frac{1}{\ell_s}} - \Big(\frac{\epsilon+\eta-\epsilon^\kappa}{C_2}\Big)^{\frac{1}{\ell_s}} \\
&\sim (\epsilon+\epsilon^\kappa)^{\frac{1}{\ell_s}} - (\epsilon-\epsilon^\kappa)^{\frac{1}{\ell_s}} \\
&= \epsilon^{\frac{1}{\ell_s}}\Big(\big(1+\frac{\epsilon^\kappa}{\epsilon}\big)^{\frac{1}{\ell_s}} - \big(1-\frac{\epsilon^\kappa}{\epsilon}\big)^{\frac{1}{\ell_s}}\Big) \\
&\sim \epsilon^{\frac{1}{\ell_s}}\Big(\big(1+\frac{\epsilon^{\kappa-1}}{\ell_s}\big) - \big(1-\frac{\epsilon^{\kappa-1}}{\ell_s}\big)\Big) \\
&= \frac{2}{\ell_s}\epsilon^{\frac{1}{\ell_s}-1}\epsilon^{\kappa-1} \sim \epsilon^{\kappa+\frac{1}{\ell_s}-1}.
\end{aligned}$$

Since $\kappa > 1$, we have $\kappa+\frac{1}{\ell_s}-1 > \frac{1}{\ell_s}$, which completes the proof. □

Lemma 8.8 $|f^2_{c+\epsilon+\eta}(\Delta^\epsilon_c)| \sim \epsilon^{\ell_c\kappa} \gg \epsilon^{2\kappa} \sim |\Delta^\epsilon_s|$.

Proof. The left-most $\sim$ follows since

$$|f^2_{c+\epsilon+\eta}(\Delta^\epsilon_c)| \sim |\Delta^\epsilon_c|(f^2_{c+\epsilon+\eta})'(c+\epsilon^\kappa) = 2\epsilon^\kappa C_1 \epsilon^{\kappa(\ell_c-1)} \sim \epsilon^{\ell_c\kappa}.$$

The $\gg$ follows since $\ell_c < 2$. □

Lemma 8.9

$$|f_{c+\epsilon+\eta}(\Delta^\theta_s)| \sim \eta^{\ell_s\theta} \gg \eta^\theta \sim |\Delta^\theta_s|,$$
$$|f_{c+\epsilon+\eta}(\Delta^\epsilon_s)| \sim \epsilon^{2\kappa\ell_s} \gg \epsilon^{2\kappa} \sim |\Delta^\epsilon_s|.$$

Proof. The first statement follows since

$$|f_{c+\epsilon+\eta}(\Delta^\theta_s)| \sim |\Delta^\theta_s| f'_{c+\epsilon+\eta}(\eta^\theta) \sim \eta^\theta \eta^{\theta(\ell_s-1)} \sim \epsilon^{\ell_s\theta},$$

and from the fact that $\ell_s < 1$.
The second statement follows since

$$|f_{c+\epsilon+\eta}(\Delta^\epsilon_s)| \sim |\Delta^\epsilon_s| f'_{c+\epsilon+\eta}(\epsilon^{2\kappa}) \sim \epsilon^{2\kappa}\epsilon^{2\kappa(\ell_s-1)} \sim \epsilon^{2\kappa\ell_s},$$

and again from the fact that $\ell_s < 1$. □
This completes the proof of Proposition 8.2.

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