Transitivity of Lorenz-like Maps and the Tired Baker's Map

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September 25, 1996

Abstract

Starting from the classical Baker's map, we define a family of "Tired Baker's" maps. It is proved that for this family, uniform expansion is not a sufficient condition for topological transitivity. As an application, we show that some of the one-dimensional Poincaré maps of the geometric Lorenz model are not good models of the real flow. We conclude the article by presenting a way to overcome these difficulties by studying a non-standard parameter range of the original Lorenz equations.

1 Introduction

Recall the Baker's map, $B : [-1,1] \to [-1,1]$, defined by

$$B(x) = \mathrm{sgn}(x)(2|x| - 1).$$

The name comes from the geometrical interpretation of its action on the interval $I = [-1,1]$ (which should be thought of as a piece of dough): the baker cuts I in half, he then stretches both halves by a factor 2, and finally puts one of the resulting pieces on top of the other. Repeating this process indefinitely gives a completely mixed dough. Mathematically speaking, we say that the map B is topologically transitive on I. The Baker's map is discussed in almost all introductory texts on dynamical systems, see e.g. [2].

Definition 1.1 *Consider a map, $f\colon I \to I$, where I is any compact interval. f is topologically transitive on I if for any two open sets $U, V \subset I$ there exists $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$.*

Now, suppose that our baker gets a bit tired stretching the dough all day long. What happens if, instead of stretching the two halves by a factor 2, he only stretches them by a factor s? First, let us define a new class of Baker's maps, see Figure 1.

Definition 1.2 $B_s(x) = \mathrm{sgn}(x)(s|x| - 1)$.

Strictly speaking, we are only interested of the parameter values $s \in (1, 2]$. The reason for this is that these parameter values are the only ones for which we can hope to have any interesting dynamics. When $s = 2$, we have the classical Baker's map, and when $s = 1$, the baker is so tired he can not bring himself to stretch the dough at all. We will call the family $\{B_s\}_{s>1}^2$ the family of Tired Baker's Maps (TBMs).

In the next section, we will show that for some values of s the map B_s is topologically transitive on I, and for some values it is not. Note that B_s is uniformly expanding on I if $s > 1$.

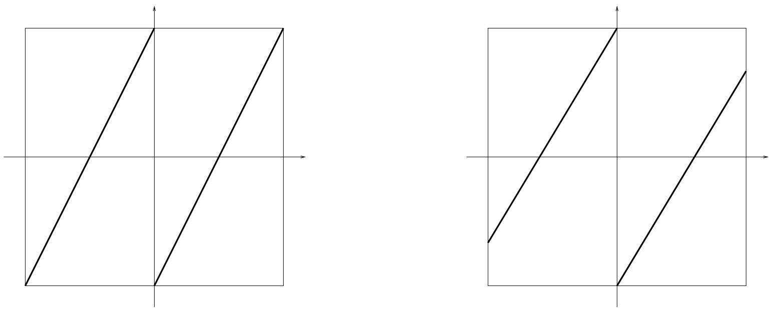

Figure 1: (a) The Baker's map. (b) A tired Baker's map.

2 Basic Results

Referring to the definitions in the previous section, we can now state our theorem which will be used as a basic model in the next section. Although the result should come as no surprise to mathematicians working in the area of one-dimensional dynamics, see [5], we will present a simple proof which suffices for our needs. Furthermore, we feel that the consequences of the theorem have not yet been completely dealt with.

Theorem 2.1 *B_s is topologically transitive on I iff $s \in [\sqrt{2}, 2]$.*

As invariance and expansion are trivially fulfilled, we have the following corollary:

Corollary 2.2 *B_s is chaotic on I iff $s \in [\sqrt{2}, 2]$.*

Proof. (Theorem 2.1) Suppose $s \in (\sqrt{2}, 2]$. Then B_s is eventually onto by an argument similar to that in [1], which is a stronger condition than being topologically transitive.
Now suppose that $s \in (1, \sqrt{2})$. We will construct two disjoint open subsets, J_1 and J_2, of I. The point is that these two subsets strictly contain each others inverse images, i.e. $B_s^{-1}(J_1) \subset J_2$ and $B_s^{-1}(J_2) \subset J_1$. This means that points outside $J = J_1 \cup J_2$ will remain outside J under iterations of B_s. If not, there would exist $x \in I \setminus J$ such that $B_s(x) \in J$. But this would imply that $x \in B_s^{-1}(J) \subset J$ which gives a contradiction.

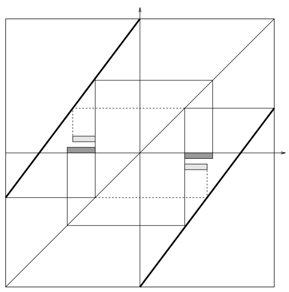

Figure 2: Dark intervals = J_1 and J_2. Lighter intervals = $B_s^{-1}(J_1)$ and $B_s^{-1}(J_2)$.

Set $J_1 = (B_s(1), B_s^2(-1)) = (s-1, 1+s-s^2)$ and $J_2 = (B_s^2(1), B_s(-1)) = -J_1$, see Figure 2. We will now prove that $B_s^{-1}(J_1) \subset J_2$ and $B_s^{-1}(J_2) \subset J_1$. It is clear that $B_s^{-1}(J_1) \subset (-1, 0)$ since 1 is mapped to the left of J_1. Hence J_1 and J_2 have unique inverse images. It is consequently a straight forward calculation: $B_s^{-1}(J_1) = B_s^{-1}(B_s(1), B_s^2(-1)) = (B_s^{-1}(B_s(1)^+), B_s(-1)) \subset (B_s^2(1), B_s(-1))$ if $B_s^2(1) < B_s^{-1}(B_s(1)^+)$. This condition is clearly fulfilled since $B_s^2(1) = s^2 - s - 1 - < \frac{1}{s}(s-2) = B_s^{-1}(B_s(1)^+)$ for $s \in (1, \sqrt{2})$.

We will now prove that $s = \sqrt{2}$ implies that B_s is eventually onto. For this parameter value the intervals J_1 and J_2 shrink to the points $\{\sqrt{2}-1\}$ and $\{1-\sqrt{2}\}$, resp. which are each others images, thus having period two. As $B_{\sqrt{2}}$ is expanding, the pre-images of $\{0\}$ are dense in I. Hence it is sufficient to show that the set $I_\epsilon = (0, \epsilon)$ eventually is mapped onto I. This is easily seen as, for $1 < k < k_0(\epsilon)$, we have

$$B^{2k}_{\sqrt{2}}(I_\epsilon) = (1-\sqrt{2}, B^{2k}_{\sqrt{2}}(\epsilon)) \quad \text{and} \quad B^{2k+1}_{\sqrt{2}}(I_\epsilon) = (\sqrt{2}-1, B^{2k+1}_{\sqrt{2}}(\epsilon)).$$

The iterates spin around the periodic orbit mentioned above, and since $B'_{\sqrt{2}} = \sqrt{2}$, we have $|B^{n+1}_{\sqrt{2}}(I_\epsilon)| = \sqrt{2}|B^{n}_{\sqrt{2}}(I_\epsilon)|$ until an iterate is sliced by the discontinuity. However, when this happens, the previous iterate must contain the segment $(1-\sqrt{2}, 0)$, and three more iterates of this interval hits every open set in I.

Finally, if $s > 2$, I is not invariant under B_s (however, there exists an unstable horseshoe $\Omega \subset I$ of measure zero which is invariant under B_s). If $a \in (0,1)$ intervals decrease in lengths under iterations, and if $s = 1$ all points are eventually periodic. Hence these parameter values will not give rise to topological transitivity. This completes the proof. □

Dynamically, we say that when $s \in (1, \sqrt{2})$, almost all points in the two sets, J_1 and J_2, belong to the basin of attraction, but not to the attractor of B_s itself. This is clear since $J_1 \cup J_2 \subset B_s(J_1 \cup J_2)$. In the critical case, $s = \sqrt{2}$, the set $J_1 \cup J_2 = \{\sqrt{2}-1, 1-\sqrt{2}\}$ becomes forward invariant. However, as it has Lebesgue measure zero, is is not relevant from our dynamical point of view.

3 Applications

One-dimensional Poincaré maps of the geometric Lorenz model have been studied in order to understand the complicated dynamics of the Lorenz equations. During these investigations, some simplifying assumptions have been made concerning these maps. In this section, we will describe a way of overcoming these simplifications and we find more truthful conditions to impose on the Poincaré maps. The article [1] and many other papers, require roughly the following conditions of the class of maps $f_a\colon [-a, a] \to [-a, a]$.

$f_a(x) = \operatorname{sgn}(x)(f(|x|) - a)$, where for all $x > 0$, f satisfies :

(1) $f(x) = \kappa x^\nu + o(x^\nu)$, where $\kappa > 0$ and $\nu \in (0, 1)$,

(2) $f''(x) < 0$,

(3) $f(c) = 2c$ for some $c > 0$,

(4) $f'(x) > \sqrt{2}$.

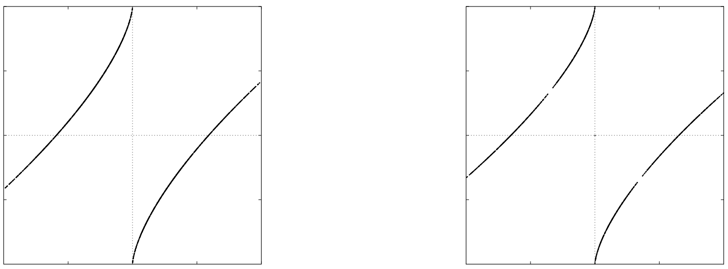

Figure 3: (a) Conditions (1)-(4) satisfied. (b) Conditions (1)-(3), but not (4), satisfied.

The classical parameter values modelled by these maps when $a > c$ are roughly $(\sigma, b, r) = (10, 8/3, 28)$. However, for these parameter values, numerical experiments indicate that condition (4) does not hold. As it is an open condition, the numerical results should carry over to the one-dimensional projection of the actual two-dimensional Poincaré map. A natural question is then if we can weaken (4), i.e. is it enough to assume that $f'(x) > s$ for some $s < \sqrt{2}$? The answer is in general no. The reason for this is that we have Poincaré maps C^0-close to TBMs. We know that the TBMs are not topologically transitive if $s < \sqrt{2}$, and as the proof in the previous section is stable under C^0-perturbations, this property will carry over to nearby Poincaré maps, thus making them very poor models of the Lorenz flow. In Figure 3, we show the "limit image" of two similar maps illustrating the problem. The limit image is just the attractor lifted up onto the two branches of the map. In Figure 3.b, we clearly see gaps which are created by the mechanism described in the previous section.

One could replace condition (4) with a slightly less restrictive one, say

$(\tilde{4})$ $(f^2)'(x) > 2$,

but according to numerical results, this also turns out to be false. In several articles, see e.g. [3], [6], milder conditions guaranteeing topological transitivity are presented. However, these conditions are not easily verifiable for a given map. Furthermore, they do not apply when the maps are not uniformly expanding.

4 A New Approach

A completely new way of dealing with this is to consider a different parameter range than the classical one. It seems to be a good choice to consider $(\sigma, b, r) \approx (10, 8/3, 24.06)$. This situation corresponds to when the invariant set of the geometric Lorenz system goes from being an unstable horseshoe to being a stable attractor, see Figure 4. The top (bottom) cusp is the first return of the left (right) half of the rectangle Σ. Here Σ is to be taken transversal to the flow. The invariant set of the return map becomes attracting just as the cusps leave the two cylinders representing the stable manifolds of the two unstable closed orbits, γ^+ and γ^-. For a certain small neighbourhood of these parameters, conditions (1) and (4) can be dropped almost entirely, without any loss of topological transitivity, see [7]. In this situation, we only require the following:

$f_a(x) = \mathrm{sgn}(x)(f(|x|) - a)$, where f satisfies :

$(\hat{1})$ $f(0) = 0$ and $f'(0^+) > 0$,

$(\hat{2})$ $f''(x) < 0$ for all $x > 0$,

$(\hat{3})$ $f(c) = 2c$ for some $c > 0$,

$(\hat{4})$ $f'(c) \geq 1$,

Thus we do not even require the derivative to tend to infinity as x approaches the discontinuity, although in the case of the Lorenz flow, this appears to be a plausible demand. Conditions $(\hat{1})$-$(\hat{3})$ force the derivative near the origin to be in the interval $(2, \infty)$, while condition $(\hat{4})$ enables us to avoid pathological cases that do no seem to occur in the Lorenz case for the parameters we are considering. Also note that we allow $f_a'(x) < 1$ for all x such that $|x| \in (c, a]$.

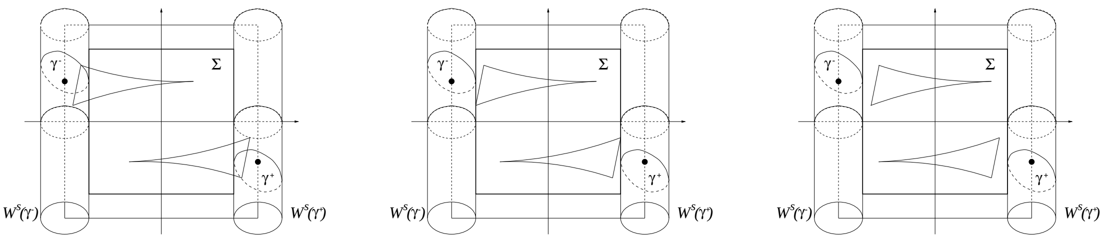

Figure 4: As the cusps leave the stable cylinders, the invariant set becomes attracting. Computer simulations indicate that this happens for $(\sigma, b, r) \approx (10, 8/3, 24.06)$.

It has been shown that the flow of the Lorenz equations exhibits a horseshoe, see [4], and computer simulations clearly indicate this turning into a stable attracting set, which in our one-dimensional model corresponds to a becoming larger than c. By a perturbation argument, we can turn f_a into a two-dimensional map, $R_{a,b}$, corresponding to a Poincaré map of a three-dimensional flow. In an open neighbourhood of the parameters (a, b) we are considering, this map induces an attracting set, $\mathcal{A}$, which is forward invariant and such that $R_{a,b}$ is topologically transitive on $\mathcal{A}$ and has positive Lyapounov exponents for a.e. $x \in \mathcal{A}$, see [7]. Due to this, this parameter range is probably more suitable to study than the classical one when searching for a strange attractor induced by the actual Lorenz equations.

Acknowledgments: This paper was written during my stay at IMPA (Instituto de Matemática Pura e Aplicada) in Rio, Brazil. I am very greatful for the financial support and the warm hospitality I received there.

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