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# A rigorous study of possible configurations of limit cycles bifurcating from a hyper-elliptic Hamiltonian of degree five

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We consider a hyper-elliptic Hamiltonian of degree five, chosen from a generic set of parameters, and study what configurations of limit cycles can bifurcate from the corresponding differential system under quartic perturbations. Perturbations of Lienard type are considered separately. Several different configurations with seven (four) limit cycles, bifurcating from the given system for general (Lienard type) quartic perturbations, are constructed. We also discuss how to construct perturbations yielding a given configuration, and how to validate the correctness of such a candidate perturbation.

**Keywords:** limit cycles; bifurcation theory; planar Hamiltonian systems; interval analysis

**AMS subject classifications:** Primary: 37G15; 37M20; Secondary: 34C07; 65G20



## 1. Introduction

Non-linear ordinary differential equations are one of the most common models used in any application of mathematical modelling. The most fundamental is perhaps the model of one-dimension mechanical motion,

$$\ddot{x} = f(x, \dot{x}). \tag{1}$$

In this article we study families of such equations on the form

$$\ddot{x} + \epsilon f(x, \dot{x}) + g(x) = 0, \tag{2}$$

where $g(x) = (\partial H/\partial x)(x)$ with $H$ a hyper-elliptic Hamiltonian[1] of degree five, $f(x, \dot{x})$ is a quartic polynomial, and the system depends on a small parameter $\epsilon$.

A fundamental question about such systems is to determine the number and location of limit cycles bifurcating from the corresponding planar vector field

$$\begin{cases} \dot{x} = -y \\ \dot{y} = \epsilon f(x, \dot{x}) + g(x) \end{cases} \tag{3}$$

as $\epsilon \to 0$, where $\epsilon = 0$ corresponds to the Hamiltonian system.

The aim of the present article is to perform a study of the configurations of limit cycles that can bifurcate from a planar polynomial Hamiltonian vector field.

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With a *configuration* we mean an invariant describing the number of limit cycles bifurcating from each annulus of periodic orbits of the unperturbed vector field.

In general, the question about the maximal number of limit cycles, and their location, of a planar polynomial vector field is the second part of Hilbert's 16th problem, which is unsolved even for quadratic polynomials. For an overview of the progress that has been made to solve this problem we refer to [1]. Results for the quadratic case, and a general introduction to the bifurcation theory of planar polynomial vector fields can be found in [2]. What is known, is that any given polynomial vector field can have only a finite number of limit cycles; this is proved in [3,4].

A restricted version of Hilbert's 16th problem, known as the *weak*, or sometimes the *tangential*, or the *infinitesimal*, Hilbert's 16th problem, asks for the number of limit cycles that can bifurcate from a perturbation of a Hamiltonian system, see e.g. [5]. The weak Hilbert's 16th problem has been solved for the quadratic case, see [6].

Special cases of Hamiltonian systems are the systems (2), which we study in this article. If one assumes that $f(x, \dot{x}) = f(x)\dot{x}$, (2) is known as a Lienard equation. Such equations have been thoroughly studied, and the case where $f$ and $g$ have degree three has been solved, see [7–10].

We use a rigorous, computer-aided method [11] to enclose the values of Abelian integrals; a different computer-aided method to determine the phase portraits of planar vector fields is described in [12–14]. From this knowledge we can establish the existence of various configurations of limit cycles bifurcating from the Hamiltonian system. In general, it is known that any configuration of limit cycles is realizable with a polynomial vector field [15]. To determine which configurations of limit cycles that are realisable as perturbations of a given Hamiltonian system is, however, a hard problem. In the present article we illustrate how to determine many such possibilities with a procedure, that simultaneously proves the correctness of the results. We stress that the approach is completely rigorous, seeing that all computations are done in interval arithmetic with directed rounding.



## 2. Abelian integrals

A classical method to prove the existence of limit cycles bifurcating from a continuous family of ovals of a Hamiltonian, $\Gamma_h \subset H^{-1}(h)$, depending continuously on $h$, is to study Abelian integrals, or, more generally, the Melnikov function, see e.g. [5,16]. Given a Hamiltonian system and a perturbation,

$$\begin{cases} \dot{x} = -H_y(x, y) + \epsilon f(x, y) \\ \dot{y} = H_x(x, y) + \epsilon g(x, y), \end{cases} \tag{4}$$

the Abelian integral, in general multiple-valued, is defined as

$$I(h) = \int_{\Gamma_h} f(x, y)\mathrm{d}y - g(x, y)\mathrm{d}x. \tag{5}$$

We denote the integrand $\omega$, and call it the one-form associated with the perturbation. In this article all perturbations are polynomial.

The most important property of Abelian integrals is described by the Poincaré-Pontryagin theorem.

**Theorem 2.1** (Poincaré-Pontryagin): *Let $P$ be the return map defined on some section transversal to the ovals of $H$, parametrized by the values $h$ of $H$, where $h$ is taken from some bounded interval $(a, b)$. Let $d(h) = P(h) - h$ be the displacement function. Then, $d(h) = \epsilon(I(h) + \epsilon\phi(h, \epsilon))$,* as $\epsilon \to 0$, *where $\phi(h, \epsilon)$ is analytic and uniformly bounded on a compact neighbourhood of $\epsilon = 0$, $h \in (a, b)$.*

**Proof:** See e.g. [5]. □

As a consequence of the above theorem, one can prove that simple zeros of $I(h)$ correspond to limit cycles bifurcating from the Hamiltonian system as $\epsilon \to 0$.

The strongest result concerning the number of zeros of Abelian integrals on ovals of hyper-elliptic Hamiltonians is [17], where it is proved that the number of zeros is finite and bounded by a certain tower function.

### 2.1. *Computer-aided proofs*

To prove mathematical statements on a computer, we need an arithmetic which gives guaranteed results. Many computer-aided proofs, including the results in this article, are based on interval analysis, e.g. [18–20]. Interval analysis yields rigorous results for continuous problems, taking both discretization and rounding errors into account. For a thorough introduction to interval analysis we refer to [21–25].

### 2.2. *Computer-aided computation of Abelian integrals*

We use the method developed in [11] to enclose the values of Abelian integrals. The method enables us to sample values of $I(h)$. If we can find two ovals $\Gamma_{h_1}$, and $\Gamma_{h_2}$, such that

$$I(h_1)I(h_2) < 0, \tag{6}$$

then there exists $h^* \in (h_1, h_2)$, such that $I(h^*) = 0$, and a neighbourhood of $\Gamma_{h^*}$ that is either attracting or repelling for the perturbed field.

Since $P_\epsilon$, the return map of the perturbed vector field, is analytic and non-constant, it has isolated fixed points. Thus, we have proved the existence of (at least) one limit cycle bifurcating from $\Gamma_{h^*}$.

In order to construct perturbations such that the associated Abelian integral has a given number of zeros, the perturbation has to be chosen in a careful manner. The heuristic approach we have used to construct such perturbations is described in Section 5.

## 3. The hyper-elliptic Hamiltonians of degree five

Hyper-elliptic Hamiltonian systems, and their perturbations, are studied in e.g. [17,26,27]. We study the hyper-elliptic Hamiltonians of degree five, using the normal form described in [27],

$$H(x, y) = \frac{1}{2}y^2 - \frac{\lambda\mu}{2}x^2 + \frac{\lambda + \mu + \lambda\mu}{3}x^3 - \frac{1 + \lambda + \mu}{4}x^4 + \frac{1}{5}x^5 \tag{7}$$

corresponding to the differential system,

$$\begin{cases} \dot{x} = -y \\ \dot{y} = -\lambda\mu x + (\lambda + \mu + \lambda\mu)x^2 - (1 + \lambda + \mu)x^3 + x^4 \end{cases}, \tag{8}$$

where $0 \le \mu \le \lambda \le 1$. There are two generic cases, see Figure 1 (left and right), corresponding to $\lambda \in (0, 1)$, $\mu \in (0, \lambda)$, and either (i) $\mu < (3\lambda^2 - 5\lambda)/5(\lambda - 2)$, or (ii) $\mu > (3\lambda^2 - 5\lambda)/5(\lambda - 2)$. The system has four equilibrium points at: 0 (saddle), $\mu$ (centre), $\lambda$ (saddle) and 1 (centre). In the first generic case the system has two homoclinic loops, and in the second it has a figure-eight loop surrounded by a homoclinic loop, see Figure 1.

We are interested in limit cycles bifurcating from the periodic solutions of (8), corresponding to integral curves of (7). The closed level curves of (7) are called *ovals*.

Many authors have studied the case of the Lienard[2] equation, e.g. [28,29],

$$\ddot{x} + \epsilon f(x)\dot{x} + g(x) = 0, \tag{9}$$

which in our case reads,

$$\ddot{x} + \epsilon(\alpha_{00} + \alpha_{10}x + \alpha_{20}x^2 + \alpha_{30}x^3)\dot{x} - \lambda\mu x + (\lambda + \mu + \lambda\mu)x^2 - (1 + \lambda + \mu)x^3 + x^4 = 0. \tag{10}$$

In addition to this, we also study the general case of a quartic perturbation of a hyper-elliptic Hamiltonian,

$$f(x, \dot{x}) = \left(\alpha_{00} + \alpha_{10}x + \alpha_{20}x^2 + \alpha_{30}x^3\right)y + \alpha_{02}\frac{y^3}{3} + \alpha_{12}x\frac{y^3}{3}. \tag{11}$$

## 4. Summary of the results

We study a generic Hamiltonian representing the second case described above, as in Figure 1 (right). We label the configurations $(l, r, o)$, where $l$, $r$, and $o$ denote the number of limit cycles in the left-hand side of the figure-eight loop, the right-hand side of the figure-eight loop, and outside of the figure-eight loop, respectively.

**Theorem 4.1** (Lienard case)**:** *Consider the Hamiltonian* (7) *with* $\lambda = 0.7$, *and* $\mu = 0.35$, *perturbed as in* (10). *Then one can choose* $\alpha_{i0}$, $0 \le i \le 3$, *such that, as* $\epsilon \to 0$, *the following configurations of four limit cycles appear*: (*i*) $(4, 0, 0)$, (*ii*) $(2, 1, 1)$, (*iii*) $(2, 0, 2)$, (*iv*) $(1, 1, 2)$.

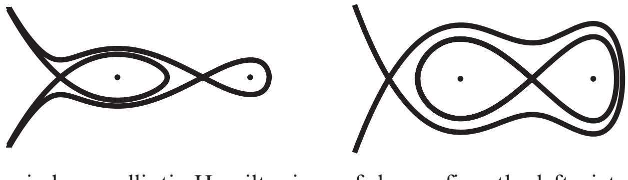

Figure 1. The generic hyper-elliptic Hamiltonians of degree five; the left picture illustrates case (i) and the right picture case (ii).

**Theorem 4.2** (General case): *Consider the Hamiltonian* (7) *with* $\lambda = 0.7$, *and* $\mu = 0.35$, *perturbed as in* (11). *Then one can choose* $\alpha_{i0}$, $0 \le i \le 3$, *and* $\alpha_{i2}$, $0 \le i \le 1$, *such that, as* $\epsilon \to 0$, *the following configurations of limit cycles appear*:

(*Seven limit cycles*) (i) $(2, 3, 2)$, (ii) $(1, 3, 3)$,

(*Six limit cycles*) (iii) $(4, 0, 2)$, (iv) $(3, 2, 1)$, (v) $(3, 1, 2)$, (vi) $(2, 1, 3)$, (vii) $(1, 1, 4)$.

In addition to the configurations above, any configuration with three (five) limit cycles, can always be constructed in the Lienard (general) case. This will be described in the next section.

## 5. Computational results

In this section, we apply the methods developed in [11] to a special case of a hyper-elliptical Hamiltonian of degree five. Using the notation of Section 3, we set $\lambda = 0.7$, and $\mu = 0.35$. The first part of our approach is to integrate monomial forms at some points, $h_1, \dots, h_N$, and then to specify the coefficients of

$$\omega = \left((\alpha_{00} + \alpha_{10}x + \alpha_{20}x^2 + \alpha_{30}x^3)y + \alpha_{02}\frac{y^3}{3} + \alpha_{12}\frac{xy^3}{3}\right)\mathrm{d}x, \tag{12}$$

such that

$$I(h_\ell) = \int_{\Gamma_{h_\ell}} \omega = 0, \qquad \ell = 1, \dots, N. \tag{13}$$

Therefore, let

$$I_{ij}(h) = -\int_{D_h} x^i y^j \, \mathrm{d}x \wedge \mathrm{d}y, \tag{14}$$

where $\partial D_h = \Gamma_h$. Then

$$I(h) = \alpha_{00}I_{00}(h) + \alpha_{10}I_{10}(h) + \alpha_{20}I_{20}(h) + \alpha_{30}I_{30}(h) + \alpha_{02}I_{02}(h) + \alpha_{12}I_{12}(h). \tag{15}$$

Given some candidate coefficients of the form $\omega$, we calculate the $I_{ij}(h)$ at intermediate ovals, $\tilde{h}_1 < h_1 < \tilde{h}_2 < \cdots < h_N < \tilde{h}_{N+1}$. If the linear combination (15) of the $I_{ij}(\tilde{h})$ has validated sign changes between the sample points we are done: it has been proved that the corresponding perturbation yields bifurcations with the given number of limit cycles as $\epsilon \to 0$.

We recall that, in general, the Abelian integral is multiple-valued, and the above-mentioned computations are done for each continuous family of ovals separately.

### 5.1. *Generating candidate coefficients*

Using the tools developed in [11], the computation of verified sign changes of the Abelian integral is automatic, once we have a set of proper coefficients. To choose such candidate coefficients, however, is non-trivial. The reason is that the regions in the parameter space yielding a large number of zeros is, typically, small. To sample the $I_{ij}$'s on a relatively tight grid using the validated integrator is not an option; it is too slow to be used to calculate a large number of samples. Instead, we need floating point approximations of the values.

Fortunately, in the case of hyper-elliptic Hamiltonians, it is easy to (non-rigorously) approximate the integrals using a floating point method.

We sample each of the monomials $I_{ij}^{k}$, where $k \in \{l, r, o\}$, at 1000 uniformly distributed points in the respective domains. For the case at hand, $H(0)=0$, $H(\mu)=-0.0031388$, $H(\lambda)=-0.0014006$ and $H(1)=-0.0033333$. We would like to choose the coefficients of $\omega$, such that the three branches of the Abelian integral oscillate together, utilizing the fact that $H(\mu)\approx H(1)$. The reason is that we wish to construct one-forms such that the total number of zeros on all branches is larger than what is generically possible on one branch. Generically, it is expected that the space of Abelian integrals on one branch should be Chebyshev, see e.g. [30], i.e. the number of zeros of a function in the space is one less than the dimension of the space.

The first step is to choose some ovals where we force the Abelian integral to be zero, by solving the corresponding linear system for the coefficients of $\omega$. Note, this method can automatically give any configuration of three (five) limit cycles for the Lienard (general) case. By choosing two ovals at (roughly) the same $H$-value for the left- and right-hand branches, and one additional oval for the left (right) branch, typically the Abelian integral on the right (left) branch oscillates together with the Abelian integral on the left (right) branch. By changing the coefficients a little, we can locate a good candidate form $\omega$.

The configurations in Theorem 4.1 (4.2), were located by first writing down a list of all possible configurations with four (six) limit cyles, and then trying to construct coordinates by forcing zeros on ovals as described above. To locate parameters with a higher probability of describing degenerate behaviour, some of the ovals were chosen to be close to either the loop or the centres. In Theorem 4.2 (i) and (ii), the results turned out to be even more degenerate. For three of the located configurations: $(4,0,0)$, $(2,0,2)$ and $(4,0,2)$, the domains of the Abelian integrals do not overlap. These configurations were found by testing if there could be a zero of higher order close to the figure-eight loop. To locate such higher order zeros we have experimented with two zeros close to the loop, and to each other, since if a higher order zero bifurcates from the loop the two limit cycles should, generically, persist for some time before coalescing.

All computations were performed on a Intel Xeon 2.0 GHz, 64 bit processor with 7970 Mb of RAM. The program was compiled with `gcc`, version 3.4.6. The software for interval arithmetic was provided by the `CXS-C` package, version 2.1.1, see [31,32]. The run time of the validated program [11] was between 1 and 24 h for each oval, depending on which accuracy that was needed.



### 5.2. *The Lienard case*

Recall the form of the Lienard equation

$$\ddot{x}+\epsilon f(x)\dot{x}+g(x)=0, \tag{16}$$

which corresponds to the following polynomial one-form $\omega$, where we have normalized $\omega$ by setting $\alpha_{30}=-1$:

$$\omega=\left(\alpha_{00}+\alpha_{10}x+\alpha_{20}x^2-x^3\right)y\,\mathrm{d}x. \tag{17}$$

Using the method described above to generate candidate coefficients, we get the result listed in Table 1.

Table 1. The generated coefficients for the Lienard case.

| Configuration | $\alpha_{00}$ | $\alpha_{10}$ | $\alpha_{20}$ |
|---|---|---|---|
| $(4, 0, 0)$ | $-0.090955087198520$ | $0.146673784978146$ | $0.673237332307929$ |
| $(2, 1, 1)$ | $0.527151835377033$ | $-2.412743372700620$ | $2.902506452438232$ |
| $(2, 0, 2)$ | $0.493218693940730$ | $-2.287498888223113$ | $2.814936399538578$ |
| $(1, 1, 2)$ | $0.335790039585681$ | $-1.666698134140512$ | $2.334707342050355$ |

Table 2. The computed enclosures of the Abelian integrals for the Lienard case.

| Configuration | Branch | $h$ | $I(h)$ |
|---|---|---|---|
| $(4, 0, 0)$ | Left | $-0.002845$ | $[5.50, 7.18]\times 10^{-8}$ |
| $(4, 0, 0)$ | Left | $-0.002168$ | $[-4.87, -1.26]\times 10^{-8}$ |
| $(4, 0, 0)$ | Left | $-0.001655$ | $[\ 2.44, 7.81]\times 10^{-8}$ |
| $(4, 0, 0)$ | Left | $-0.001440$ | $[-7.97, -1.14]\times 10^{-8}$ |
| $(4, 0, 0)$ | Left | $-0.001401$ | $[\ 4.56, 81.6]\times 10^{-9}$ |
| $(2, 1, 1)$ | Left | $-0.002583$ | $[\ 2.84, 2.87]\times 10^{-5}$ |
| $(2, 1, 1)$ | Left | $-0.001609$ | $[-2.30, -2.26]\times 10^{-5}$ |
| $(2, 1, 1)$ | Left | $-0.001404$ | $[\ 1.96, 2.02]\times 10^{-5}$ |
| $(2, 1, 1)$ | Right | $-0.002251$ | $[-1.50, -1.48]\times 10^{-4}$ |
| $(2, 1, 1)$ | Right | $-0.001404$ | $[\ 9.76, 9.85]\times 10^{-5}$ |
| $(2, 1, 1)$ | Out | $-0.001168$ | $[\ 2.92, 2.94]\times 10^{-4}$ |
| $(2, 1, 1)$ | Out | $-0.000400$ | $[-5.14, -5.11]\times 10^{-4}$ |
| $(2, 0, 2)$ | Left | $-0.002489$ | $[\ 3.90, 3.92]\times 10^{-5}$ |
| $(2, 0, 2)$ | Left | $-0.001578$ | $[-7.53, -7.20]\times 10^{-6}$ |
| $(2, 0, 2)$ | Left | $-0.001404$ | $[\ 2.45, 2.51]\times 10^{-5}$ |
| $(2, 0, 2)$ | Out | $-0.001398$ | $[-3.96, -3.85]\times 10^{-5}$ |
| $(2, 0, 2)$ | Out | $-0.001216$ | $[\ 6.82, 6.93]\times 10^{-5}$ |
| $(2, 0, 2)$ | Out | $-0.000840$ | $[-1.23, -1.21]\times 10^{-4}$ |
| $(1, 1, 2)$ | Left | $-0.002617$ | $[\ 2.64, 2.66]\times 10^{-5}$ |
| $(1, 1, 2)$ | Left | $-0.001435$ | $[-9.37, -9.33]\times 10^{-5}$ |
| $(1, 1, 2)$ | Right | $-0.002444$ | $[-2.67, -2.63]\times 10^{-5}$ |
| $(1, 1, 2)$ | Right | $-0.001439$ | $[\ 6.14, 6.12]\times 10^{-5}$ |
| $(1, 1, 2)$ | Out | $-0.001398$ | $[-10.4, -9.53]\times 10^{-6}$ |
| $(1, 1, 2)$ | Out | $-0.001305$ | $[\ 1.07, 1.16]\times 10^{-5}$ |
| $(1, 1, 2)$ | Out | $-0.000840$ | $[-1.76, -1.74]\times 10^{-4}$ |

The next step is to validate that the generated coefficients yield the expected behaviour. Therefore, we enclose the value of the corresponding Abelian integrals at intermediate ovals. As is shown in Table 2, the generated coefficients correspond to perturbations for which the claimed number of limit cycles bifurcate from the hyper-elliptic Hamiltonian.

### 5.3. *The general case*

We are now studying

$$\ddot{x} + \epsilon f(x, \dot{x}) + g(x) = 0, \tag{18}$$

Table 3. The generated coefficients for the seven limit-cycle configurations.

| Configuration | (2, 3, 2) | (1, 3, 3) |
|---|---|---|
| $\alpha_{00}$ | −0.014217512025472 | −0.014509471576202 |
| $\alpha_{10}$ | 0.074583260821912 | 0.076198084267168 |
| $\alpha_{20}$ | −0.118228516889348 | −0.120790364287194 |
| $\alpha_{30}$ | 1.047128604358265 | 1.044988054862947 |
| $\alpha_{02}$ | 0.057870420877864 | 0.059110998792988 |

Table 4. The generated coefficients for the six limit-cycle configurations.

| Configuration | (4, 0, 2) | (3, 2, 1) | (3, 1, 2) |
|---|---|---|---|
| $\alpha_{00}$ | 0.003467719369657 | −0.012579001706296 | −0.013396623733006 |
| $\alpha_{10}$ | −0.016936137126896 | 0.063426140322458 | 0.068949475891139 |
| $\alpha_{20}$ | 0.022925125136071 | −0.093641653584006 | −0.105227606694727 |
| $\alpha_{30}$ | 0.241782262869751 | 0.744102555169092 | 0.830973679932326 |
| $\alpha_{02}$ | −0.008117967534894 | 0.042925643387235 | 0.049778620099614 |
| | (2, 1, 3) | (1, 1, 4) | |
| $\alpha_{00}$ | −0.016876767465088 | −0.000487412601086 | |
| $\alpha_{10}$ | 0.090336905956256 | 0.000746501625707 | |
| $\alpha_{20}$ | −0.147748560871734 | −0.000121099674506 | |
| $\alpha_{30}$ | 1.304890989044986 | 0.632041452259285 | |
| $\alpha_{02}$ | 0.074179246592458 | 0.000197747463660 | |

Table 5. The computed enclosures of the Abelian integrals for the seven limit-cycle configurations.

| Configuration | Branch | $h$ | $I(h)$ |
|---|---|---|---|
| (2, 3, 2) | Left | −0.002510 | $[\ 7.56, 7.60]\times 10^{-7}$ |
| (2, 3, 2) | Left | −0.001444 | $[-14.3, -7.42]\times 10^{-9}$ |
| (2, 3, 2) | Left | −0.001401 | $[\ 1.85, 77.6]\times 10^{-10}$ |
| (2, 3, 2) | Right | −0.002900 | $[-2.34, -2.12]\times 10^{-8}$ |
| (2, 3, 2) | Right | −0.001950 | $[\ 1.83, 5.93]\times 10^{-9}$ |
| (2, 3, 2) | Right | −0.001497 | $[-13.7, -8.61]\times 10^{-9}$ |
| (2, 3, 2) | Right | −0.001401 | $[\ 5.57, 11.2]\times 10^{-9}$ |
| (2, 3, 2) | Out | −0.001360 | $[\ 2.05, 6.10]\times 10^{-8}$ |
| (2, 3, 2) | Out | −0.001050 | $[-13.4, -8.98]\times 10^{-8}$ |
| (2, 3, 2) | Out | −0.000800 | $[\ 1.98, 2.44]\times 10^{-7}$ |
| (1, 3, 3) | Left | −0.002555 | $[\ 6.32, 6.38]\times 10^{-7}$ |
| (1, 3, 3) | Left | −0.001470 | $[-7.47, -6.12]\times 10^{-8}$ |
| (1, 3, 3) | Right | −0.002845 | $[-3.04, -2.80]\times 10^{-8}$ |
| (1, 3, 3) | Right | −0.001850 | $[\ 2.33, 6.71]\times 10^{-9}$ |
| (1, 3, 3) | Right | −0.001495 | $[-6.85, -1.65]\times 10^{-9}$ |
| (1, 3, 3) | Right | −0.001401 | $[\ 9.96, 15.7]\times 10^{-9}$ |
| (1, 3, 3) | Out | −0.001399 | $[-2.58, -1.82]\times 10^{-8}$ |
| (1, 3, 3) | Out | −0.001340 | $[\ 6.03, 13.9]\times 10^{-9}$ |
| (1, 3, 3) | Out | −0.001146 | $[-3.21, -2.38]\times 10^{-8}$ |
| (1, 3, 3) | Out | −0.000980 | $[\ 6.38, 7.24]\times 10^{-8}$ |

Table 6. The computed enclosures of the Abelian integrals for the 6 limit cycle configurations.

| Configuration | Branch | $h$ | $I(h)$ |
|---|---|---|---|
| (4, 0, 2) | Left | −0.002946 | $[-7.47, -6.21]\times 10^{-10}$ |
| (4, 0, 2) | Left | −0.002238 | $[\ 1.60, 1.92]\times 10^{-9}$ |
| (4, 0, 2) | Left | −0.001689 | $[-7.82, -3.10]\times 10^{-10}$ |
| (4, 0, 2) | Left | −0.001463 | $[\ 1.24, 1.82]\times 10^{-9}$ |
| (4, 0, 2) | Left | −0.001401 | $[-5.24, -4.57]\times 10^{-9}$ |
| (4, 0, 2) | Out | $-1.000\times 10^{-4}$ | $[-3.01, -2.89]\times 10^{-7}$ |
| (4, 0, 2) | Out | $-2.798\times 10^{-5}$ | $[\ 5.91, 6.99]\times 10^{-8}$ |
| (4, 0, 2) | Out | $-1.000\times 10^{-6}$ | $[-10.3, -9.22]\times 10^{-8}$ |
| (3, 2, 1) | Left | −0.002856 | $[\ 2.88, 3.18]\times 10^{-8}$ |
| (3, 2, 1) | Left | −0.002147 | $[-2.12, -1.46]\times 10^{-8}$ |
| (3, 2, 1) | Left | −0.001658 | $[\ 2.14, 3.07]\times 10^{-8}$ |
| (3, 2, 1) | Left | −0.001401 | $[-3.35, -3.21]\times 10^{-7}$ |
| (3, 2, 1) | Right | −0.002646 | $[-6.23, -6.10]\times 10^{-7}$ |
| (3, 2, 1) | Right | −0.001580 | $[\ 1.09, 1.31]\times 10^{-7}$ |
| (3, 2, 1) | Right | −0.001401 | $[-2.46, -2.22]\times 10^{-7}$ |
| (3, 2, 1) | Out | −0.001314 | $[-1.08, -1.04]\times 10^{-6}$ |
| (3, 2, 1) | Out | −0.000100 | $[\ 3.09, 3.10]\times 10^{-5}$ |
| (3, 1, 2) | Left | −0.002809 | $[\ 6.74, 7.10]\times 10^{-8}$ |
| (3, 1, 2) | Left | −0.001975 | $[-5.86, -5.05]\times 10^{-8}$ |
| (3, 1, 2) | Left | −0.001496 | $[\ 3.66, 4.83]\times 10^{-8}$ |
| (3, 1, 2) | Left | −0.001401 | $[-6.09, -4.69]\times 10^{-8}$ |
| (3, 1, 2) | Right | −0.002659 | $[-4.88, -4.74]\times 10^{-7}$ |
| (3, 1, 2) | Right | −0.001507 | $[\ 2.80, 3.05]\times 10^{-7}$ |
| (3, 1, 2) | Out | −0.001399 | $[\ 6.21, 9.82]\times 10^{-8}$ |
| (3, 1, 2) | Out | −0.001345 | $[-9.16, -5.47]\times 10^{-8}$ |
| (3, 1, 2) | Out | −0.001000 | $[\ 1.99, 2.04]\times 10^{-6}$ |
| (2, 1, 3) | Left | −0.002521 | $[\ 1.14, 1.16]\times 10^{-6}$ |
| (2, 1, 3) | Left | −0.001515 | $[-1.29, -1.13]\times 10^{-7}$ |
| (2, 1, 3) | Left | −0.001401 | $[\ 9.49, 11.4]\times 10^{-8}$ |
| (2, 1, 3) | Right | −0.002627 | $[\ 5.39, 5.58]\times 10^{-7}$ |
| (2, 1, 3) | Right | −0.001482 | $[-4.74, -4.40]\times 10^{-7}$ |
| (2, 1, 3) | Out | −0.001399 | $[-1.90, -1.40]\times 10^{-7}$ |
| (2, 1, 3) | Out | −0.001340 | $[\ 6.27, 11.4]\times 10^{-8}$ |
| (2, 1, 3) | Out | −0.000431 | $[-5.85, -5.79]\times 10^{-6}$ |
| (2, 1, 3) | Out | $-3.000\times 10^{-6}$ | $[\ 5.29, 5.36]\times 10^{-6}$ |
| (1, 1, 4) | Left | −0.002288 | $[\ 2.18, 2.19]\times 10^{-6}$ |
| (1, 1, 4) | Left | −0.001401 | $[-4.07, -4.06]\times 10^{-7}$ |
| (1, 1, 4) | Right | −0.002304 | $[-2.17, -2.16]\times 10^{-6}$ |
| (1, 1, 4) | Right | −0.001401 | $[\ 4.79, 4.81]\times 10^{-7}$ |
| (1, 1, 4) | Out | −0.001399 | $[\ 6.87, 6.98]\times 10^{-8}$ |
| (1, 1, 4) | Out | −0.001137 | $[-9.89, -8.61]\times 10^{-9}$ |
| (1, 1, 4) | Out | −0.000648 | $[\ 1.51, 1.69]\times 10^{-8}$ |
| (1, 1, 4) | Out | −0.000143 | $[-2.98, -2.75]\times 10^{-8}$ |
| (1, 1, 4) | Out | $-1.000\times 10^{-6}$ | $[\ 5.44, 5.68]\times 10^{-8}$ |

which corresponds to the following polynomial one-form $\omega$, where we have normalized $\omega$ by setting $\alpha_{12} = -1$:

$$\omega = \left( (\alpha_{00} + \alpha_{10}x + \alpha_{20}x^2 + \alpha_{30}x^3)y + \alpha_{02}\frac{y^3}{3} - \frac{xy^3}{3} \right) \mathrm{d}x. \tag{19}$$

Using the method described above to generate candidate coefficients, we get the result listed in Tables 3 and 4 for the configurations with seven and six limit cycles, respectively.

To prove that the perturbations with these coefficients yield the claimed bifurcations, we proceed as in the Lienard case, and enclose the values of the corresponding Abelian integrals at intermediate ovals. As is shown in Tables 5 and 6, for the configurations with seven and six limit cycles, respectively; the claimed number of limit cycles bifurcate from the hyper-elliptic Hamiltonian, using the generated perturbations.

## 6. Conclusions

We have applied the method developed in [11] to study a special case of a generic hyper-elliptic Hamiltonian of degree five, and proved the existence of several different configurations of limit cycles that can bifurcate from it. The approach we have used illustrates how one can employ a validated approach to determine the possible configurations of limit cycles bifurcating from a given Hamiltonian system. We believe that such studies can be very useful for any application where a specific polynomial Hamiltonian is used, in order to explore what configurations of limit cycles that can appear when the system is perturbed by some given family of polynomial vector fields.

## Notes

1. A hyper-elliptic Hamiltonian is of the form $H(x, y) = (y^2/2) + f(x)$, where $f(x)$ is a polynomial of degree at least five.
2. Classically, $g(x) = x$, the studied case is sometimes called the generalized Lienard equation.

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