Differential and Integral Equations Volume 16, Number 12, December 2003, Pages 1409–1439

SOME COUNTEREXAMPLES FOR THE SPECTRAL-RADIUS CONJECTURE

Irina Mitrea
Department of Mathematics, Cornell University
593 Malott Hall, Ithaca, NY, 14853

Warwick Tucker
Department of Mathematics, Uppsala University
Box 480, S-751 06 Uppsala, Sweden

(Submitted by: Gustavo Ponce)

Abstract. The goal of this paper is to produce a series of counterexamples for the L^p spectral radius conjecture, $1 < p < \infty$, for double-layer potential operators associated to a distinguished class of elliptic systems in polygonal domains in $\mathbb{R}^2$. More specifically the class under discussion is that of second-order elliptic systems in two dimensions whose coefficient tensor (with constant real entries) is *symmetric* and *strictly positive definite*. The general techniques employed are those of the Mellin transform and Calderón-Zygmund theory. For the case $p \in (1, 4)$, we construct a computer-aided proof utilizing validated numerics based on interval analysis.

1. Introduction

A classical approach to solving the L^p Dirichlet problem for the Laplacian

$$u \in \mathcal{C}^2(\Omega), \quad \triangle u = 0 \quad \text{in} \quad \Omega, \quad u\Big|_{\partial\Omega} = f \in L^p(\partial\Omega), \tag{1.1}$$

for $1 < p < \infty$, is to write

$$u(X) = \mathcal{D}[(I + K)^{-1} f](X), \quad X \in \Omega, \tag{1.2}$$

where I is the identity operator, $\mathcal{D}$ is the double-layer potential operator associated with the Laplacian, and K is its boundary version. Let Γ stand for twice the standard fundamental solution of $\triangle$; i.e., $\Gamma(X) = -\frac{1}{\pi} \ln |X|$. The operators $\mathcal{D}$ and K in (1.2) are defined by formally setting, for $f : \partial\Omega \longrightarrow \mathbb{R}$,

$$\mathcal{D}f(X) := \int_{\partial\Omega} \frac{\partial \Gamma(Q - X)}{\partial N(Q)} f(Q)\, d\sigma(Q), \tag{1.3}$$

Accepted for publication: May 2003.
AMS Subject Classifications: 45E05, 47A10; 35J25, 42B20.

$$Kf(P) := p.v. \int_{\partial\Omega} \frac{\partial\Gamma(Q-P)}{\partial N(Q)} f(Q)\, d\sigma(Q),$$

where $X \in \mathbb{R}^n \setminus \partial\Omega$ and $P \in \partial\Omega$. The seemingly nonexplict part of (1.2) is the inverse $(I+K)^{-1}$, and a natural question is whether

$$\rho\left(K; L^p(\partial\Omega)\right) < 1, \tag{1.4}$$

for any bounded, Lipschitz domain $\Omega \subset \mathbb{R}^n$. Here, for a linear and continuous operator T mapping the Banach space $\mathcal{X}$ into itself, we denote by $\sigma(T;\mathcal{X})$ and by $\rho(T;\mathcal{X})$ the spectrum and, respectively, the spectral radius of T on $\mathcal{X}$; i.e., $\sigma(T;\mathcal{X}) := \{w \in \mathbb{C};\, wI - T \text{ is not invertible on } \mathcal{X}\}$ and $\rho(T;\mathcal{X}) := \sup\{|w|;\, w \in \sigma(T;\mathcal{X})\}$. In particular, (1.4) would allow one to expand the inverse in (1.2) as the strongly convergent Neumann series $(I+K)^{-1} = \sum_{j=0}^{\infty}(-K)^j$; cf., e.g., [33].

The question whether (1.4) holds is referred to in the literature as the *spectral-radius conjecture*. While this conjecture (cf. C. Kenig [15] and G. Verchota [4]) remains open at the moment, progress has been made in a number of related directions. In [6], L. Escauriaza, E. Fabes, and G. Verchota show that for any Lipschitz domain Ω one has

$$\sigma\left(K; L^2(\partial\Omega)\right) \cap \mathbb{R} \subseteq (-1, 1]. \tag{1.5}$$

Based on this, (1.4) holds when $p = 2$, whenever Ω is bounded, convex domain in $\mathbb{R}^n$ (see [11]). The crux of the matter in this scenario is the fact that K is a positive operator, and therefore, cf. [14], the spectral radius belongs to the spectrum. Recently in [7], this result has been extended to all $p \in [2, \infty)$ in the case of bounded, convex domains. Another direction of progress is the case of bounded polyhedra in $\mathbb{R}^3$. In this case, see [8], J. Elschner proves (1.4) for the case $p = 2$. When Ω is a polygonal domain in two dimensions, in [32] it has been shown that

$$\rho\left(K; \mathcal{C}(\partial\Omega)\right) < 1, \tag{1.6}$$

where $\mathcal{C}(\partial\Omega)$ stands for the space of continuous functions on $\partial\Omega$. For related issues see also [34], [21], [16], and the references therein.

Similar issues can be raised in the case when one deals with second-order, constant-coefficient, $m \times m$ elliptic systems in $\Omega \subset \mathbb{R}^n$, such as

$$\vec{u} \in \left(\mathcal{C}^2(\Omega)\right)^m, \quad \mathcal{L}\vec{u} = \vec{0} \quad \text{in} \quad \Omega, \quad \vec{u}\Big|_{\partial\Omega} = \vec{f} \in \left(L^p(\partial\Omega)\right)^m, \tag{1.7}$$

where $1 < p < \infty$. Here

$$(\mathcal{L}\vec{u})^\alpha := a_{ij}^{\alpha\beta} \partial_i \partial_j u^\beta, \tag{1.8}$$

with $a_{ij}^{\alpha\beta} \in \mathbb{R}$, for all $i, j = 1, \ldots, n$ and $\alpha, \beta = 1, \ldots, m$. Above, repeated indices denote summation, and the coefficient tensor $A := (a_{ij}^{\alpha,\beta})_{\alpha,\beta,i,j}$ is assumed to be symmetric and satisfy the Legendre-Hadamard ellipticity condition (see Section 2). When A is also *strictly positive definite*, (1.7) behaves much like (1.2). For instance, in this scenario, the well-posedness range of the boundary-value problem (1.7) is $2 - \epsilon < p < \infty$ due to the fact that, for such p, the operator $I + K_A$ is invertible on $(L^p(\partial\Omega))^m$. Here,

$$K_A \vec{f}(P) := p.v. \int_{\partial\Omega} \Big[\frac{\partial \Gamma}{\partial N_A}(P - \cdot) \Big]^t (Q) \vec{f}(Q)\, d\sigma(Q), \quad P \in \partial\Omega. \tag{1.9}$$

In (1.9) the superscript t stands for transposition of matrices, the matrix-valued function Γ is twice the fundamental solution of the operator $\mathcal{L}$ given in (1.8), and the conormal derivative $\frac{\partial}{\partial N_A}$ defined in (2.5) is applied to the columns of Γ; i.e.,

$$\frac{\partial \Gamma}{\partial N_A}(X - Q) = \Big(N_i(Q) a_{ij}^{\alpha\beta} \partial_j \Gamma^{\beta\gamma}(X - Q) \Big)_{\alpha,\gamma}, \ \alpha, \gamma \in \{1, \ldots, m\}, \tag{1.10}$$

where $N(Q) = (N_1(Q), \ldots, N_n(Q))$ is the outward unit-normal vector which exists at almost every $Q \in \partial\Omega$. It is therefore natural to investigate whether

$$\rho\,(K_A; (L^p(\partial\Omega))^m) < 1, \tag{1.11}$$

provided A is symmetric and strictly positive definite, at least when $p \geq 2$.

This is true in a number of important cases such as double-layer potential operators associated to the Lamé system on curvilinear polygons. Here

$$\mathcal{L}\vec{u} := \mu \triangle \vec{u} + (\lambda + \mu) \nabla \mathrm{div}\, \vec{u} \tag{1.12}$$

is the system of linear elastostatics with Lamé moduli μ and λ which are assumed to satisfy $\mu > 0$ and $\mu + \lambda \geq 0$. The operator $\mathcal{L}$ can be represented in the form of (1.8) with

$$a_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta}(r) := \mu\, \delta_{ij}\delta_{\alpha\beta} + (\mu + \lambda - r)\delta_{i\alpha}\delta_{j\beta} + r\, \delta_{i\beta}\delta_{j\alpha}. \tag{1.13}$$

Above, $r \in \mathbb{R}$ is arbitrary, δ_{ij} is the Kronecker symbol, and $\alpha, \beta, i, j \in \{1, 2\}$. Straightforward algebraic manipulations show that the symmetric coefficient tensor A introduced in (1.13) is strictly positive definite if and only if $r \in (-\mu, \mu)$. The special choice $r_0 := \mu(\mu + \lambda)/(3\mu + \lambda) \in (-\mu, \mu)$ gives rise in (1.9) to the pseudostress double-layer potential operator denoted here by K_{r_0}. In [22] it has been shown that the inequality (1.11) holds for the operator K_{r_0} for all $2 \leq p < \infty$ whenever Ω is a curvilinear polygon in $\mathbb{R}^2$.

More specifically, when Ω is the domain consisting of the interior of an angle of aperture $\theta \in (0, 2\pi)$, the following holds:

$$\rho(K_{r_0}; (L^p(\partial\Omega))^2) = \frac{|\frac{\upsilon}{p} \sin\theta \cos(\frac{\pi-\theta}{p}) + \sin(\frac{\pi-\theta}{p})\sqrt{1 - \upsilon^2 \frac{\sin^2\theta}{p^2}}|}{\sin(\frac{\pi}{p})}, \qquad (1.14)$$

where $\upsilon := (\mu + \lambda)/(3\mu + \lambda)$, and $2 \leq p < \infty$. In the special case $\upsilon = 0$ (see, e.g., [34]) the identity (1.14) holds for all $p \in (1, \infty)$.

In general, let K_r be the double-layer potential operator (1.9) associated to (1.13) for $r \in (-\mu, \mu)$. A partial result in this setting is that (see [23] and [24]), given Ω a curvilinear polygon in $\mathbb{R}^2$, there exists $p(r) \in [2, \infty)$ such that (1.11) holds for the operator K_r whenever $p \in [p(r), \infty)$. In the meantime, substantial numerical evidence supports the case $p = 2$ (and therefore, by interpolation, (1.11) for $p \in [2, \infty)$).

In the light of this discussion, it comes as a surprise that (1.11) *fails* for more general systems, even though the symmetry and strict positivity conditions for the coefficient tensor A are enforced. Indeed, the main result of this paper—which answers a question posed to us by C. Kenig—reads as follows.

Theorem. *For any $p \in (1, \infty)$ there exists a polygon Ω in $\mathbb{R}^2$ and a second-order elliptic operator $\mathcal{L}$ as in* (1.8) *with the coefficient matrix A symmetric and strictly positive definite such that the L^p spectral-radius conjecture* (1.11) *fails. A similar negative result holds for the case of continuous functions.*

It is worth pointing out that all our counterexamples are second-order elliptic systems in canonical form, which are of the *second kind* (see the discussion at the beginning of Section 3). Such a class does not include the Lamé operator (1.12), which is of the first kind, in the classification alluded to above.

Our proof proceeds in several stages. The simplest case is $1 < p < 2$ when, if Ω is a sector in the plane of aperture $\theta \in (0, \pi]$, we have

$$\rho(K; L^p(\partial\Omega)) = \frac{|\sin((\pi - \theta)/p)|}{\sin(\pi/p)} > 1 \quad \text{if} \quad 1 < p < \frac{2\pi - \theta}{\pi}. \qquad (1.15)$$

Recall here that K, the double-layer potential associated to the Laplacian, was introduced in (1.3). The spectral-radius formula (1.15) can be obtained for instance by formally making $\upsilon = 0$ in (1.14). Finally, if $p \in (1, 2)$, the last condition in (1.15) is satisfied if θ is small enough.

Next, if $p \geq 4$, we consider Ω to be a sector in the plane of aperture $\pi/2$ with sides along the positive coordinate axes. Starting from the description of the spectrum of the double-layer potential operators associated to a specific (carefully chosen) second-order elliptic systems in canonical form,

$$\sigma\left(K_A;(L^p(\partial\Omega))^2\right)=\bigcup_{i=1}^{4}\Sigma_i(p), \tag{1.16}$$

where $\Sigma_i(p)$ denotes a certain (closed) curve in the plane, associated with the tensor A and the integrability exponent p, we explicitly identify a point $w_0 \in \sigma(K_A;(L^p(\partial\Omega))^2)\cap\mathbb{R}$ such that $|w_0|>1$. This once again yields (1.11).

Finally, when $2 \leq p < 4$, on the same domain as above, we employ interval-analysis techniques to prove the existence of a point $w \in \sigma(K_A;(L^p(\partial\Omega))^2)$ such that $|w|>1$. Somewhat more specifically, in this situation our previous candidate point in the spectrum, w_0, no longer satisfies $|w_0|>1$. Nonetheless, extensive numerical experiments indicate that the search for $w \in \Sigma_i(p)$ with $|w|>1$ can be restricted to a specific compact set. These computations, however, yield only approximate values, and therefore do not constitute a rigorous proof. In practice, it is virtually impossible to predict the accumulative effect of the roundoff errors in any nontrivial floating-point computation. To overcome this difficulty, we use techniques from interval analysis, which automatically take all computational errors into account. The existence of $w \in \Sigma_i(p)$ such that $|w|>1$ is rigorously guaranteed by a search algorithm on the compact set alluded to above. This algorithm utilizes the computer's internal representation of floating-point numbers, as well as its rounding procedures.

The paper is organized in seven sections. In Section 2 we present some basic definitions and results. Next, in Section 3 we introduce the family of second-order elliptic systems with real, constant coefficients in canonical form in planar domains and their associated layer potentials. Section 4 contains the Mellin analysis of the operators from Section 3 in the case of a sector. In Section 5 we construct the L^p, $1<p<\infty$, counterexamples to the spectral-radius conjecture. Included is an analytical treatment when $p \in [4,\infty)$. Section 6 contains, after some introductory results of interval analysis, the rigorous numerical algorithm guaranteeing the validity of our counterexamples for $1<p<4$. Finally, Section 7 presents a counterexample to the spectral-radius conjecture on the space of continuous functions.

Acknowledgments. The first-named author would like to thank Professor Carlos Kenig for several discussions that have triggered this research.

The second-named author was partially supported by NSF Award No. DMS-0107242.

2. Preliminaries

Here we collect some basic definitions and results useful for our exposition. Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{C}$. The function f is called *Lipschitz* provided there exists $M > 0$ so that $|f(x) - f(y)| \leq M|x - y|$ for any $x, y \in D$.

Definition 2.1. A bounded domain $\Omega \subset \mathbb{R}^2$ is called Lipschitz if for any $X_0 \in \partial\Omega$ there exist $r,\ h > 0$ and a coordinate system $\{x_1, x_2\}$ in $\mathbb{R}^2$ (isometric to the canonical one) with origin at X_0 along with a function $\varphi : \mathbb{R} \to \mathbb{R}$ which is Lipschitz and so that the following holds. If $C(r, h)$ denotes the cylinder $\{(x_1, x_2);\ |x_j| < r \text{ all } j\} \times (0, h) \subset \mathbb{R}^2$, then

$$\begin{aligned} \Omega \cap C(r,h) &= \{X = (x_1, x_2);\ |x_j| < r \text{ all } j \text{ and } x_2 > \varphi(x_1)\}, \\ \partial\Omega \cap C(r,h) &= \{X = (x_1, x_2);\ |x_j| < r \text{ all } j \text{ and } x_2 = \varphi(x_1)\}. \end{aligned} \tag{2.1}$$

Consider now a 2×2 second-order system on $\mathbb{R}^2$ with real, constant coefficients given by

$$(\mathcal{L}\vec{u})^\alpha := a_{ij}^{\alpha\beta} \partial_i \partial_j u^\beta, \qquad a_{ij}^{\alpha\beta} \in \mathbb{R}, \quad \forall\, \alpha, \beta, i, j = 1, 2, \tag{2.2}$$

where $\vec{u} = (u^1, u^2)$, with $u^\beta : \mathbb{R}^2 \longrightarrow \mathbb{R}$ for all $\beta = 1, 2$. Hereafter, repeating indices in the same expression denote summation. Whenever $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$ for all $\alpha, \beta, i, j \in \{1, 2\}$ we call $A = (a_{ij}^{\alpha\beta})_{\alpha,\beta,i,j}$ in (2.2) *symmetric*. The tensor A is said to satisfy the *Legendre-Hadamard ellipticity condition* provided there exists $c > 0$ such that

$$a_{ij}^{\alpha\beta} \xi_i \xi_j \eta^\alpha \eta^\beta \geq c|\xi|^2|\eta|^2 \text{ for any } \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \quad \text{and} \quad \eta = (\eta^1, \eta^2) \in \mathbb{R}^2. \tag{2.3}$$

Also, the tensor A is called *strictly positive definite* provided there exists $c > 0$ such that

$$a_{ij}^{\alpha\beta} \zeta_i^\alpha \zeta_j^\beta \geq c|\zeta|^2 \quad \text{for any} \quad \zeta = (\zeta_i^\alpha)_{\alpha,i} \in \mathbb{R}^4. \tag{2.4}$$

It is well known that whenever A is symmetric and satisfies the Legendre-Hadamard condition (2.3), then the operator $\mathcal{L}$ in (2.2) has a fundamental solution (cf., e.g., [28]), which is symmetric and its rows and columns satisfy the system (2.2) in $\mathbb{R}^2 \backslash \{0\}$. Denote by $\Gamma(X) = (\Gamma^{\alpha\beta}(X))_{\alpha,\beta},\ \alpha, \beta = 1, 2$ twice the fundamental solution referred to above.

Definition 2.2. Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and let $N(P) = (N_1(P), \ldots, N_n(P))$ be the outward unit-normal vector that exists at almost every

$P \in \partial\Omega$. For a fixed tensor $A = (a_{ij}^{\alpha\beta})_{\alpha,\beta,i,j}$ with $\alpha, \beta = 1, 2$ and $i, j = 1, 2$, given a vector function $\vec{u} = (u^1, u^2)$, we define its *conormal* derivative at almost every $P \in \partial\Omega$ by

$$\Big(\frac{\partial \vec{u}}{\partial N_A}\Big)^{\alpha}(P) := N_i(P) a_{ij}^{\alpha\beta} \partial_j u^{\beta}(P). \tag{2.5}$$

The double-layer potential operators associated with the system (2.2) whose tensor of coefficients A is symmetric and satisfies the Legendre-Hadamard ellipticity condition (2.3) are introduced next. For $\vec{f} : \partial\Omega \longrightarrow \mathbb{R}^2$ we formally set

$$K_A \vec{f}(P) = p.v. \int_{\partial\Omega} \Big[\frac{\partial \Gamma}{\partial N_A}(P - \cdot)\Big]^t(Q) \vec{f}(Q)\, d\sigma(Q), \quad P \in \partial\Omega. \tag{2.6}$$

In (2.6) the superscript t stands for transposition of matrices and the conormal derivative $\frac{\partial}{\partial N_A}$ defined in (2.5) is applied to the columns of Γ; i.e.,

$$\frac{\partial \Gamma}{\partial N_A}(P - Q) = \Big(N_i(Q) a_{ij}^{\alpha\beta} \partial_j \Gamma^{\beta\gamma}(P - Q)\Big)_{\alpha,\gamma}, \qquad \alpha, \gamma \in \{1, 2\}. \tag{2.7}$$

In light of the results in [3], for any $1 < p < \infty$, we have

$$K_A : (L^p(\partial\Omega))^2 \longrightarrow (L^p(\partial\Omega))^2 \quad \text{boundedly}. \tag{2.8}$$

The rest of the section contains further notation and preliminaries on the Mellin transform together with the rudiments on the spectral analysis of Hardy kernels on $(L^p(\mathbb{R}_+))^2$ where $\mathbb{R}_+ := [0, \infty)$. To this end, let $\mathcal{C}_0^{\infty}(\mathbb{R}_+)$ be the space of infinitely many times differentiable functions, compactly supported on $[0, \infty)$. The Mellin transform of a function $f \in \mathcal{C}_0^{\infty}(\mathbb{R}_+)$ is defined as

$$\mathcal{M}f(z) := \int_0^{\infty} x^{z-1} f(x)\, dx, \quad z \in \mathbb{C}. \tag{2.9}$$

Note that $\mathcal{M}f(z)$ in (2.9) is an entire function. For any $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, define the strip $H_{\alpha,\beta} := \{z \in \mathbb{C}\, ; \alpha < \operatorname{Re} z < \beta\}$, and let $H_{\alpha} := \{z \in \mathbb{C}\, ; \operatorname{Re} z = \alpha\}$. If f is measurable on $\mathbb{R}_+$ and the integral in (2.9) converges absolutely for all z in some strip $H_{\alpha,\beta}$, we shall call the integral $\mathcal{M}f(z)$ the *Mellin transform* of the function f. Note that $\mathcal{M}f$ is a holomorphic function in the strip $H_{\alpha,\beta}$.

The Hardy kernels on $L^p(\mathbb{R}_+)$, $1 < p < \infty$, are introduced next.

Definition 2.3. Let $k(x, y)$ be a measurable function defined on $\mathbb{R}_+ \times \mathbb{R}_+$. Then k is a Hardy kernel on $L^p(\mathbb{R}_+)$, $1 < p < \infty$, provided that $k(x, y)$

is homogeneous of degree -1 (i.e., for any $\lambda > 0$ we have $k(\lambda x, \lambda y) = \lambda^{-1} k(x,y)$), and there holds

$$\int_0^\infty |k(1,y)|\, y^{-1/p}\, dy \Big(= \int_0^\infty |k(x,1)|\, x^{1/p-1}\, dx \Big) < \infty.$$

Also, a matrix $k = (k_{ij})_{i,j=1,2}$ of measurable functions on $\mathbb{R}_+ \times \mathbb{R}_+$ is called a Hardy kernel for $(L^p(\mathbb{R}_+))^2$, provided that each individual entry k_{ij} is a Hardy kernel for $L^p(\mathbb{R}_+)$.

Consider now $k = (k_{ij})_{i,j=1,2}$ a Hardy kernel for $(L^p(\mathbb{R}_+))^2$, $1 < p < \infty$, and for each $\vec{f} \in (L^p(\mathbb{R}_+))^2$ let

$$K\vec{f}(x) := \int_0^\infty k(x,y)\vec{f}(y)\, dy, \qquad x \in \mathbb{R}_+. \tag{2.10}$$

To state the characterization of the spectrum of operators K as in (2.10) on $(L^p(\partial\Omega))^2$ we need more notation. Let $\mathcal{X}$ be a Banach space and $T : \mathcal{X} \longrightarrow \mathcal{X}$ be a linear and continuous operator. We denote by $\sigma(T;\mathcal{X})$ the spectrum of the operator; i.e.,

$$\sigma(T;\mathcal{X}) := \{w \in \mathbb{C}\,;\, wI - T \text{ is not invertible on } \mathcal{X}\}. \tag{2.11}$$

Also, the spectral radius of T is

$$\rho(T;\mathcal{X}) := \sup\{|w|\,;\, w \in \sigma(T;\mathcal{X})\}, \tag{2.12}$$

i.e., the radius of the smallest closed, circular disc centered at the origin which contains $\sigma(T,\mathcal{X})$.

We include next an explicit description of the spectrum of K as an operator on $(L^p(\mathbb{R}_+))^2$, $1 < p < \infty$. This can be obtained by adapting the argument in [10] or [2] to the matrix setting described above.

Theorem 2.4. *If k is a Hardy kernel for $(L^p(\mathbb{R}_+))^2$, $1 < p < \infty$, then the operator K defined in* (2.10) *is a bounded operator on $(L^p(\mathbb{R}_+))^2$. The spectrum of K as an operator on $(L^p(\mathbb{R}_+))^2$ is the closure of the range of the Mellin transform $\mathcal{M}k(1/p + i\xi)$; i.e., it is the closure in the plane of the set of all points $w \in \mathbb{C}$ such that*

$$\det(wI - \mathcal{M}k)(1/p + i\xi) = 0 \quad \text{for some } \xi \in \mathbb{R}. \tag{2.13}$$

Above, I is the identity matrix operator and $\mathcal{M}k := (\mathcal{M}k_{ij})_{i,j=1,2}$.

We shall call the matrix $\mathcal{M}k$ the *matrix of the Mellin symbols* of the operator K on $(L^p(\mathbb{R}_+))^2$, $1 < p < \infty$. In the notation of [19], [20], and [9] of the algebra of pseudodifferential operators of Mellin type, the condition (2.13) reads $\det \mathrm{Smbl}^{1/p}(wI - K)(0, z) = 0$ for some $z = 1/p + i\xi$, $\xi \in \mathbb{R}$.

3. Elliptic systems and their associated layer potentials

In this section we will be concerned with presenting some concrete examples of elliptic systems (2.2) with the coefficient tensor A being symmetric and strictly positive definite (cf. (2.4)) and their corresponding double-layer potential operators as in (2.6). Consider

$$\mathcal{L} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\lambda}{k^2} \end{pmatrix} \frac{\partial^2}{\partial x^2} + \begin{pmatrix} 0 & \frac{\lambda - k^2}{k} \\ \frac{\lambda-1}{k} & 0 \end{pmatrix} \frac{\partial^2}{\partial x \partial y} + \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial y^2}, \tag{3.1}$$

with $0 < k \leq 1$, $\lambda > 0$, and $\lambda \neq 1, k^2$. If $k = 1$ the system is called *of the first kind* and if $0 < k < 1$ the system is called *of the second kind*. If one multiplies (3.1) on the left by the matrix

$$M := \begin{pmatrix} \left|\frac{k}{\lambda - k^2}\right| & 0 \\ 0 & \left|\frac{k}{\lambda-1}\right| \end{pmatrix}, \tag{3.2}$$

the coefficient matrix of the mixed partial derivatives in (3.1) becomes

$$\text{either} \quad \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \mp 1 \\ \mp 1 & 0 \end{pmatrix}. \tag{3.3}$$

In the first case, that is, when $\lambda > 1$ or $\lambda < k^2$, the system has been transformed into a *symmetric* system. When $k^2 < \lambda < 1$ we are left with a *nonsymmetric* system. For the remaining part of the paper we assume that

$$\text{either} \quad \lambda > 1 \quad \text{or} \quad \lambda < k^2. \tag{3.4}$$

Using Fourier-transform methods one can compute twice the fundamental solution of the operator $\mathcal{L}$, i.e., $\Gamma(x, y)$ such that $\mathcal{L}(\Gamma(x, y)) = 2\delta(x, y)I$, where I is the 2×2 identity matrix. We have

$$\begin{aligned}
\Gamma^{11}(x, y) &= \frac{1}{2\pi\lambda(a+b)} \left(a \log(x^2 + y^2) + kb \log(k^2x^2 + y^2) \right), \\
\Gamma^{22}(x, y) &= \frac{1}{2\pi\lambda(a+b)} \left(bk^2 \log(x^2 + y^2) + ka \log(k^2x^2 + y^2) \right), \\
\Gamma^{12}(x, y) &= \frac{ak}{\pi\lambda(a+b)} \tan^{-1} \left[\frac{xy(k-1)}{kx^2 + y^2} \right], \\
\Gamma^{21}(x, y) &= \frac{-kb}{\pi\lambda(a+b)} \tan^{-1} \left[\frac{xy(k-1)}{kx^2 + y^2} \right],
\end{aligned} \tag{3.5}$$

where

$$a := \lambda - k^2 \ \text{ and } \ b := 1 - \lambda. \tag{3.6}$$

Recall the writing of the operator $\mathcal{L}$ in (3.1) as in (2.2). A particular choice of the coefficient matrix A accomplishing this is

$$A := \left(a_{ij}^{\alpha\beta}\right)_{\alpha,\beta,i,j=1,2} = \begin{pmatrix} 1 & 0 & 0 & \frac{\lambda-k^2}{\lambda+k} \\ 0 & \lambda & \frac{\lambda(\lambda-k^2)}{k(\lambda+k)} & 0 \\ 0 & \frac{\lambda(\lambda-1)}{k(\lambda+k)} & \frac{\lambda}{k^2} & 0 \\ \frac{\lambda-1}{\lambda+k} & 0 & 0 & 1 \end{pmatrix}. \tag{3.7}$$

Here $A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}$ where $A^{\alpha\beta} := \begin{pmatrix} a_{11}^{\alpha\beta} & a_{12}^{\alpha\beta} \\ a_{21}^{\alpha\beta} & a_{22}^{\alpha\beta} \end{pmatrix}$ for any $\alpha, \beta \in \{1,2\}$. Notice that K_A coincides with the classical double-layer potential associated with the Laplacian when formally making $\lambda = k = 1$. Also, K_A coincides with the elastostatic double-layer potential corresponding to the pseudostress conormal derivative when $k = 1$ and λ is an explicit combination of the Lamé moduli. For the interested reader we point out that invertibility properties of $I + K_A$ on L^p spaces, $1 < p < \infty$, have been studied in [5] in the symmetric case and [35] in the nonsymmetric setting.

For $m, n \in \mathbb{R}$ with $m, n \neq 0$, introduce

$$\mathcal{L}_{m,n} := \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \mathcal{L}, \tag{3.8}$$

where $\mathcal{L}$ is as in (3.1). A quick inspection shows that the operator $\mathcal{L}_{m,n}$ can be written in the notation (2.2) using the coefficients $A(m,n) := \tilde{A}$, with

$$\tilde{A} := \left(\tilde{a}_{ij}^{kl}\right)_{i,j,k,l=1,2}, \quad \text{where } \ \tilde{a}_{ij}^{kl} := \begin{cases} m \cdot a_{ij}^{\alpha\beta}, & \alpha = 1, \\ n \cdot a_{ij}^{\alpha\beta}, & \alpha = 2. \end{cases} \tag{3.9}$$

Let $\tilde{\Gamma}$ be twice the matrix-valued fundamental solution of the operator $\mathcal{L}_{m,n}$ in (3.8); that is, $\mathcal{L}_{m,n}\tilde{\Gamma} = 2\delta I$, where I is the 2×2 identity matrix. We denote by $K_{\tilde{A}}$ the double-layer potential operator given in (2.6), associated to the operator $\mathcal{L}_{m,n}$ from (3.8) and the coefficient matrix $\tilde{A}$ in (3.9). We have

Lemma 3.1. *Twice a fundamental solution for the operator $\mathcal{L}_{m,n}$ is given by*

$$\tilde{\Gamma} = \Gamma \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{n} \end{pmatrix}, \tag{3.10}$$

where Γ *is twice the fundamental solution of* $\mathcal{L}$ *as given in* (3.5). *Also, for any* $w \in \mathbb{C}$ *we have*

$$(wI - K_{\tilde{A}}) = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \cdot (wI - K_A) \cdot \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{n} \end{pmatrix}. \tag{3.11}$$

Proof. A straightforward calculation shows that $\mathcal{L}_{m,n}\tilde{\Gamma} = \mathcal{L}\Gamma = 2\delta I$. Next, denoting by $\tilde{\Gamma}^i$ and Γ^i, $i = 1, 2$, the i-th column in the matrix $\tilde{\Gamma}$ and respectively Γ, we have

$$\tilde{\Gamma}^1 = \frac{1}{m}\Gamma^1, \quad \text{and} \quad \tilde{\Gamma}^2 = \frac{1}{n}\Gamma^2. \tag{3.12}$$

Let $k = (k_{ij})_{i,j=1,2}$ be the matrix-valued kernel of the double-layer potential operator K_A and $\tilde{k} = (\tilde{k}_{ij})_{i,j=1,2}$ be the kernel of $K_{\tilde{A}}$. The first column in $\tilde{k}$ is given by

$$\begin{pmatrix} \tilde{k}_{11} \\ \tilde{k}_{21} \end{pmatrix} = N_i \tilde{a}_{ij}^{\alpha\beta} \partial_j \tilde{\Gamma}^{\beta 1} = \frac{1}{m}(\delta_{1\alpha} m + \delta_{2\alpha} n) N_i a_{ij}^{\alpha\beta} \partial_j \Gamma^{\beta 1} = \begin{pmatrix} k_{11} \\ \frac{n}{m} k_{21} \end{pmatrix}. \tag{3.13}$$

The second equality in (3.13) follows from (3.9) and (3.12). Similarly, $\begin{pmatrix} \tilde{k}_{12} \\ \tilde{k}_{22} \end{pmatrix} = \begin{pmatrix} \frac{m}{n} k_{12} \\ k_{22} \end{pmatrix}$. This implies

$$\tilde{k} = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \cdot k \cdot \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{n} \end{pmatrix}. \tag{3.14}$$

Finally, (3.11) follows from (3.14). The proof of Lemma 3.1 is now complete. $\square$

Next we study the properties of the coefficient matrix $\tilde{A}$ in (3.9) when $m = |\frac{k}{\lambda - k^2}|$ and $n = |\frac{k}{\lambda - 1}|$. Recall (3.4) and (2.4). We have

Lemma 3.2. *Let* $m = |\frac{k}{\lambda - k^2}|$ *and* $n = |\frac{k}{\lambda - 1}|$, *and recall* $\tilde{A}$ *from* (3.9). *Then, for any* $0 < k \leq 1$, $\lambda > 0$ *satisfying* (3.4), *the matrix* $\tilde{A}$ *is symmetric and strictly positive definite.*

Proof. A straightforward computation based on (3.7) and (3.9) gives

$$\begin{aligned} \tilde{a}_{ij}^{kl} \zeta_i^k \zeta_j^l &= m \left(\zeta_1^1\right)^2 + m\lambda \left(\zeta_2^1\right)^2 + \frac{n\lambda}{k^2} \left(\zeta_1^2\right)^2 \\ &+ \Big(\frac{n\lambda(\lambda - 1)}{k(\lambda + k)} + \frac{m\lambda(\lambda - k^2)}{k(\lambda + k)}\Big) \zeta_2^1 \zeta_1^2 + \Big(\frac{m(\lambda - k^2)}{\lambda + k} + \frac{n(\lambda - 1)}{\lambda + k}\Big) \zeta_1^1 \zeta_2^2 + n(\zeta_2^2)^2. \end{aligned} \tag{3.15}$$

This is

$$\tilde{a}_{ij}^{kl}\zeta_i^k\xi_j^l = m\left(\zeta_1^1\right)^2 + m\lambda\left(\zeta_2^1\right)^2 + \frac{n\lambda}{k^2}\left(\zeta_1^2\right)^2 + n\left(\zeta_2^2\right)^2 \qquad (3.16)$$
$$+ \frac{2k\,\mathrm{sign}(\lambda-k^2)}{\lambda+k}\Big[\frac{\lambda}{k}\zeta_2^1\zeta_1^2 + \zeta_1^1\zeta_2^2\Big],$$

as $\mathrm{sign}(\lambda-k^2) = \mathrm{sign}(\lambda-1)$ and $m(\lambda-k^2)+n(\lambda-1) = 2k\,\mathrm{sign}(\lambda-k^2)$. Clearly there holds $(\lambda-k^2)(\lambda-1) < (\lambda+k)^2$. This implies $\frac{k^2}{(\lambda+k)^2} < mn$, with m and n as in the hypothesis. By continuity, there exists $\varepsilon(\lambda,k) > 0$ such that

$$\frac{k^2}{(\lambda+k)^2} < (m-\varepsilon(\lambda,k))(n-\varepsilon(\lambda,k)). \qquad (3.17)$$

Now, regarding the expression

$$E(\zeta_1^1,\zeta_2^2) := (m-\varepsilon(\lambda,k))\left(\zeta_1^1\right)^2 \pm \frac{2k}{\lambda+k}\zeta_1^1\zeta_2^2 + (n-\varepsilon(\lambda,k))\left(\zeta_2^2\right)^2$$

as a polynomial of degree two in ζ_1^1, its discriminant $\triangle_1$ is given by

$$\triangle_1 = 4\left(\zeta_2^2\right)^2\Big[\frac{k^2}{(\lambda+k)^2} - (m-\varepsilon(\lambda,k))(n-\varepsilon(\lambda,k))\Big]. \qquad (3.18)$$

Using (3.17) we obtain that $\triangle_1 < 0$, and therefore $E(\zeta_1^1,\zeta_2^2) \geq 0$ for any $\zeta_1^1,\zeta_2^2 \in \mathbb{R}$. This immediately gives that

$$m\left(\zeta_1^1\right)^2 \pm \frac{2k}{\lambda+k}\zeta_1^1\zeta_2^2 + n\left(\zeta_2^2\right)^2 > \varepsilon(\lambda,k)\left(\left(\zeta_1^1\right)^2 + \left(\zeta_2^2\right)^2\right). \qquad (3.19)$$

Next we regard

$$E(\zeta_2^1,\zeta_1^2) := (m-\varepsilon(\lambda,k))\lambda\left(\zeta_2^1\right)^2 \pm \frac{2\lambda}{\lambda+k}\zeta_2^1\zeta_1^2 + (n-\varepsilon(\lambda,k))\frac{\lambda}{k^2}\left(\zeta_1^2\right)^2$$

as a polynomial of degree two in ζ_2^1. Its discriminant $\triangle_2$ is

$$\triangle_2 = 4\frac{\lambda^2}{k^2}\left(\zeta_1^2\right)^2\Big[\frac{k^2}{(\lambda+k)^2} - (m-\varepsilon(\lambda,k))(n-\varepsilon(\lambda,k))\Big]. \qquad (3.20)$$

Appealing again to (3.17) we obtain that $\triangle_2 < 0$, and hence $E(\zeta_2^1,\zeta_1^2) \geq 0$ for any $\zeta_2^1,\zeta_1^2 \in \mathbb{R}$. In turn this gives

$$m\lambda(\zeta_2^1)^2 \pm \frac{2\lambda}{\lambda+k}\zeta_2^1\zeta_1^2 + n\frac{\lambda}{k^2}(\zeta_1^2)^2 > \varepsilon(\lambda,k)\lambda(\zeta_2^1)^2 + \varepsilon(\lambda,k)\frac{\lambda}{k^2}(\zeta_1^2)^2. \qquad (3.21)$$

Now, based on (3.16), (3.19), and (3.21) we can conclude that in this case

$$\tilde{a}_{ij}^{kl}\zeta_i^k\zeta_j^l \geq \gamma\left|\zeta\right|^2, \qquad (3.22)$$

with

$$\gamma := \min\left\{\varepsilon(\lambda,k), \varepsilon(\lambda,k)\lambda, \varepsilon(\lambda,k)\frac{\lambda}{k^2}\right\} > 0, \tag{3.23}$$

and the coefficient matrix $\tilde{A}$ is therefore *strictly positive definite.*

Finally, straightforward computations give

$$\begin{aligned} &\tilde{a}_{12}^{11} = \tilde{a}_{21}^{11} = 0, \quad \tilde{a}_{11}^{21} = \tilde{a}_{11}^{12} = 0, \quad \tilde{a}_{22}^{21} = \tilde{a}_{22}^{12} = 0, \\ &\tilde{a}_{12}^{12} = \tilde{a}_{21}^{21} = \pm\frac{k}{\lambda + k}, \quad \tilde{a}_{21}^{12} = \tilde{a}_{12}^{21} = \pm\frac{\lambda}{\lambda + k}, \end{aligned} \tag{3.24}$$

where the sign $+$ in (3.24) corresponds to the case $\lambda > 1$ while the sign $-$ corresponds to $\lambda < k^2$. This shows that $\tilde{A}$ is symmetric for $\lambda < k^2$ or $\lambda > 1$. This finishes the proof of Lemma 3.2. □

Our next goal is to compute explicitly the kernel of the operator K_A introduced in (2.6). Note that this is the same as the conormal derivative $\frac{\partial\Gamma}{\partial N_A}$ where Γ is twice the matrix-valued fundamental solution of the operator $\mathcal{L}$ given in (3.5) and A is our particular choice of the coefficient matrix as in (3.7). To this end, let us denote by $k^{ij}(x,y)$ the ij entry in the matrix $\frac{\partial\Gamma}{\partial N_A}(x,y)$, for $i,j = 1,2$. Recall a and b from (3.6). Straightforward but tedious computations that we omit based on (2.5) and (3.5) lead to

$$\begin{aligned} k^{11}(x,y) &= \frac{\langle (x,y), N\rangle}{\pi(a+bk)}\Big(\frac{a}{x^2+y^2} + \frac{bk^2}{k^2x^2+y^2}\Big), \\ k^{22}(x,y) &= \frac{\langle (x,y), N\rangle}{\pi(a+bk)}\Big(\frac{bk}{x^2+y^2} + \frac{ak}{k^2x^2+y^2}\Big), \\ k^{12}(x,y) &= \frac{-ak}{\pi(a+bk)}\Big(\frac{-N_2x+N_1y}{x^2+y^2} - \frac{-k^2N_2x+N_1y}{k^2x^2+y^2}\Big), \\ k^{21}(x,y) &= \frac{b}{\pi(a+bk)}\Big(\frac{-N_2x+N_1y}{x^2+y^2} - \frac{-k^2N_2x+N_1y}{k^2x^2+y^2}\Big). \end{aligned} \tag{3.25}$$

We end this section by pointing out that $k = (k^{ij})_{i,j=1,2}$ as in (3.25) is a Hardy kernel in the sense of Definition 2.3.

4. Mellin transform techniques in the case of a sector

Recall the elliptic operator $\mathcal{L}_{m,n}$ in (3.8) and the choice of the coefficient tensor $\tilde{A}$ from (3.9). The main goal of this section is to obtain an explicit characterization of the spectra of the operator $K_{\tilde{A}}$ on L^p spaces, $1 < p < \infty$, for

$$m := \frac{k}{|\lambda - k^2|} \quad \text{and} \quad n := \frac{k}{|\lambda - 1|}, \tag{4.1}$$

when Ω is the domain consisting of the interior of an angle of aperture $\theta \in (0, 2\pi)$ and orientation $\phi \in (0, 2\pi)$.

Throughout this section Ω is a sector of opening θ and orientation ϕ. That is,

$$\Omega = \{(x, y); x = r\cos(\phi + \beta),\ y = r\sin(\phi + \beta), 0 < r < \infty,\ 0 < \beta < \theta\}. \tag{4.2}$$

We have $\partial\Omega = (\partial\Omega)_1 \cup (\partial\Omega)_2$ with

$$\begin{aligned}(\partial\Omega)_1 &= \{(\tau, \rho); \tau = r\cos\phi, \rho = r\sin\phi, r > 0\},\\ (\partial\Omega)_2 &= \{(\tau, \rho); \tau = r\cos(\phi + \theta), \rho = r\sin(\phi + \theta), r > 0\}.\end{aligned} \tag{4.3}$$

The exterior unit-normal vector to $(\partial\Omega)_1$ is $\vec{N} = (\sin\phi, -\cos\phi)$, and the exterior unit-normal vector to $(\partial\Omega)_2$ is $\vec{N} = (-\sin(\phi+\theta), \cos(\phi+\theta))$. Next, we write the kernels $\tilde{k}^{ij}(x, y)$, $i, j = 1, 2$, in the above coordinates (that is, for $(x, y) \in (\partial\Omega)_l$, $l = 1, 2$). Then we compute the Mellin transforms $\mathcal{M}(\tilde{k}^{ij}(\cdot, 1))$. We have

Proposition 4.1. *For any $1 < p < \infty$, the matrix of the Mellin symbols of $wI - K_{\tilde{A}}$ on $(L^p(\partial\Omega))^2$ is*

$$(wI - \mathcal{M}\tilde{k})(z) = \begin{pmatrix} w & 0 & D(z) & -\frac{mb}{n}C(z) \\ 0 & w & \frac{nka}{m}C(z) & B(z) \\ D(-z) & -\frac{mb}{n}C(-z) & w & 0 \\ \frac{nka}{m}C(-z) & B(-z) & 0 & w \end{pmatrix}, \tag{4.4}$$

where

$$\begin{aligned} D(z) &:= \frac{a\sin[(\pi-\theta)z] + kbk^{-z}R^z\sin[(\pi-\alpha)z]}{(a+kb)\sin(\pi z)},\\ B(z) &:= \frac{kb\sin[(\pi-\theta)z] + ak^{-z}R^z\sin[(\pi-\alpha)z]}{(a+kb)\sin(\pi z)},\\ C(z) &:= \frac{\cos[(\pi-\theta)z] - k^{-z}R^z\cos[(\pi-\alpha)z]}{(a+kb)\sin(\pi z)}, \end{aligned} \tag{4.5}$$

and $z = 1/p + iy$, $y \in \mathbb{R}$. In (4.5) above we take

$$R := \left|\frac{k\cos(\phi+\theta) + i\sin(\phi+\theta)}{k\cos\phi + i\sin\phi}\right|, \quad \textit{and}$$

$$\alpha := \operatorname{Arg}\Big(\frac{k\cos(\phi+\theta)+i\sin(\phi+\theta)}{k\cos\phi+i\sin\phi}\Big). \tag{4.6}$$

Proof. Recall that k stands for the kernel of the double-layer potential operator K_A associated to (3.1) for the choice of coefficient tensor A given in (3.7). The first step is to write the kernels k^{ij}, $i,j=1,2$, in the coordinates introduced in (4.3). We consider $P,Q\in\partial\Omega$, $P-Q=(x,y)$. When P and Q belong to the same side of the angle we have that $k^{ij}(x,y)=0$, as $\langle P-Q, N(Q)\rangle$ is a factor for all k^{ij}, $i,j=1,2$ (see, e.g., (3.25)). Let us now consider the case $P\in(\partial\Omega)_1$ and $Q\in(\partial\Omega)_2$. We have

$$P=(s\cos\phi, s\sin\phi),\ s\geq 0,\quad Q=(t\cos(\phi+\theta), t\sin(\phi+\theta)),\ t\geq 0. \tag{4.7}$$

A direct computation gives

$$\begin{aligned}
&\langle (x,y),N\rangle = \langle P-Q, N(Q)\rangle = -s\sin\theta, \qquad (4.8)\\
&x^2+y^2 = (s\cos\phi - t\cos(\phi+\theta))^2 + (s\sin\phi - t\sin(\phi+\theta))^2,\\
&k^2x^2+y^2 = k^2(s\cos\phi - t\cos(\phi+\theta))^2 + (s\sin\phi - t\sin(\phi+\theta))^2,\\
&k^2N_2x - N_1y = sk^2\cos(\phi+\theta)\cos\phi - t[k^2\cos^2(\phi+\theta)-\sin^2(\phi+\theta)]\\
&\qquad\qquad + s\sin\phi\sin(\phi+\theta).
\end{aligned}$$

Corresponding to the case $P\in(\partial\Omega)_1$ and $Q\in(\partial\Omega)_2$, we introduce the operators with kernels $\frac{1}{x^2+y^2}$ and $\frac{k}{k^2x^2+y^2}$ given by

$$Uf(s) := \frac{1}{\pi}\int_0^\infty \frac{-s\sin\theta}{(s\cos\phi - t\cos(\phi+\theta))^2 + (s\sin\phi - t\sin(\phi+\theta))^2} f(t)\,dt, \tag{4.9}$$

and

$$U_kf(s) := \frac{1}{\pi}\int_0^\infty \frac{-sk\sin\theta}{k^2(s\cos\phi - t\cos(\phi+\theta))^2 + (s\sin\phi - t\sin(\phi+\theta))^2} f(t)\,dt. \tag{4.10}$$

Also, for $P\in(\partial\Omega)_1$ and $Q\in(\partial\Omega)_2$, the operators with kernels $\frac{-N_2x+N_1y}{x^2+y^2}$ and $\frac{-k^2N_2x+N_1y}{k^2x^2+y^2}$ are

$$Vf(s) := \frac{1}{\pi}\int_0^\infty \frac{-s\cos\theta + t}{(s\cos\phi - t\cos(\phi+\theta))^2 + (s\sin\phi - t\sin(\phi+\theta))^2} f(t)\,dt, \tag{4.11}$$

and

$$V_kf(s) := \frac{1}{\pi}\int_0^\infty \Big[\frac{t[k^2\cos^2(\phi+\theta)+\sin^2(\phi+\theta)]}{k^2(s\cos\phi - t\cos(\phi+\theta))^2 + (s\sin\phi - t\sin(\phi+\theta))^2}$$

$$- \frac{s[k^2\cos(\phi+\theta)\cos\phi+\sin\phi\sin(\phi+\theta)]}{k^2(s\cos\phi-t\cos(\phi+\theta))^2+(s\sin\phi-t\sin(\phi+\theta))^2}\Big] f(t)\,dt. \quad (4.12)$$

In a similar manner we introduce U^*, U_k^*, V^*, and V_k^* corresponding to the case when $P \in (\partial\Omega)_2$ and $Q \in (\partial\Omega)_1$. We have

$$U^*f(s) := \frac{1}{\pi}\int_0^\infty \frac{-s\sin\theta}{(s\cos(\phi+\theta)-t\cos\phi)^2+(s\sin(\phi+\theta)-t\sin\phi)^2} f(t)\,dt, \quad (4.13)$$

and

$$U_k^*f(s) := \frac{1}{\pi}\int_0^\infty \frac{-sk\sin\theta}{k^2(s\cos(\phi+\theta)-t\cos\phi)^2+(s\sin(\phi+\theta)-t\sin\phi)^2} f(t)\,dt. \quad (4.14)$$

Also

$$V^*f(s) := \frac{1}{\pi}\int_0^\infty \frac{s\cos\theta-t}{(s\cos(\phi+\theta)-t\cos\phi)^2+(s\sin(\phi+\theta)-t\sin\phi)^2} f(t)\,dt, \quad (4.15)$$

and

$$V_k^*f(s) := \frac{1}{\pi}\int_0^\infty \Big[\frac{-t[k^2\cos^2(\phi+\theta)+\sin^2(\phi+\theta)]}{k^2(s\cos(\phi+\theta)-t\cos\phi)^2+(s\sin(\phi+\theta)-t\sin\phi)^2}$$
$$+ \frac{s[k^2\cos(\phi+\theta)\cos\phi+\sin\phi\sin(\phi+\theta)]}{k^2(s\cos(\phi+\theta)-t\cos\phi)^2+(s\sin(\phi+\theta)-t\sin\phi)^2}\Big] f(t)\,dt. \quad (4.16)$$

Now, based on (3.25), (4.9)–(4.12), and (4.13)–(4.16) we can regard the operator K_A as a 4×4 matrix of operators (each entry being an operator on $L^p(\mathbb{R}_+)$). We have

$$K_A = \frac{1}{a+bk}\begin{pmatrix} 0 & 0 & aU+kbU_k & b(V-V_k) \\ 0 & 0 & -ak(V-V_k) & kbU+aU_k \\ aU^*+kbU_k^* & b(V^*-V_k^*) & 0 & 0 \\ -ak(V^*-V_k^*) & kbU^*+aU_k^* & 0 & 0 \end{pmatrix}. \quad (4.17)$$

Therefore, the Mellin transform of the kernel of the operator $wI - K_A$ is

$$(wI-\mathcal{M}k)(z) = \frac{1}{a+bk}\begin{pmatrix} wI & A(z) \\ A^*(z) & wI \end{pmatrix}, \quad (4.18)$$

where I is the identity 2×2 matrix and

$$A(z) := \begin{pmatrix} -a\mathcal{M}u(z)-kb\mathcal{M}u_k(z) & -b(\mathcal{M}v(z)-\mathcal{M}v_k(z)) \\ ak(\mathcal{M}v(z)-\mathcal{M}v_k(z)) & -kb\mathcal{M}u(z)-a\mathcal{M}u_k(z) \end{pmatrix}, \quad (4.19)$$

and

$$A^*(z) := \begin{pmatrix} -a\mathcal{M}u^*(z) - kb\mathcal{M}u_k^*(z) & -b(\mathcal{M}v^*(z) - \mathcal{M}v_k^*(z)) \\ ak(\mathcal{M}v^*(z) - \mathcal{M}v_k^*(z)) & -kb\mathcal{M}u^*(z) - a\mathcal{M}u_k^*(z) \end{pmatrix}. \quad (4.20)$$

In (4.19)–(4.20) we denote by u, u_k, v, and v_k the kernels of the operators U, U_k, V, and V_k respectively. Similarly u^*, u_k^*, v^*, v_k^* stand for the kernels of U^*, U_k^*, V^*, and V_k^* respectively. We have

$$u(s,1) = -\frac{1}{\pi} \cdot \frac{s\sin\theta}{s^2 - 2s\cos\theta + 1} = -\frac{1}{\pi} \cdot \frac{s\sin(\pi-\theta)}{s^2 + 2s\cos(\pi-\theta) + 1}. \quad (4.21)$$

Then we use that (see, e.g, formulas 2.60 and 2.62 on page 24 in [30]) for

$$f(x) = \frac{x\sin\gamma}{x^2 + 2ax\cos\gamma + a^2} \quad \text{and} \quad g(x) = \frac{x\cos\gamma + a}{x^2 + 2ax\cos\gamma + a^2}, \quad (4.22)$$

with $-\pi < \gamma < \pi$, we have that

$$\mathcal{M}f(z) = \pi a^{z-1}\frac{\sin(\gamma z)}{\sin(\pi z)} \quad \text{and} \quad \mathcal{M}g(z) = \pi a^{z-1}\frac{\cos(\gamma z)}{\sin(\pi z)}, \quad (4.23)$$

for $-1 < \mathrm{Re} z < 1$. Using (4.23) together with (4.21) for $\gamma = \pi - \theta$ we obtain

$$\mathcal{M}u(z) := \mathcal{M}(u(\cdot,1))(z) = -\frac{\sin((\pi-\theta)z)}{\sin(\pi z)}. \quad (4.24)$$

Based on (4.9) and (4.13) it is easy to see that actually $U^* = U$ and therefore $u^*(s,1) = u(s,1)$. This implies

$$\mathcal{M}u^*(z) := \mathcal{M}(u^*(\cdot,1))(z) = -\frac{\sin((\pi-\theta)z)}{\sin(\pi z)}. \quad (4.25)$$

Next we compute the Mellin transform of the kernels of the operators V and V^*. Using (4.11) and (4.15) we have that $V = -V^*$ and

$$v(s,1) = \frac{1}{\pi} \cdot \frac{-s\cos\theta + 1}{s^2 - 2s\cos\theta + 1} = \frac{1}{\pi} \cdot \frac{s\cos(\pi-\theta) + 1}{s^2 + 2s\cos(\pi-\theta) + 1}. \quad (4.26)$$

Therefore, appealing to (4.23) we obtain that $\mathcal{M}v(z) := \mathcal{M}(v(\cdot,1))(z)$ and $\mathcal{M}v^*(z) := \mathcal{M}(v^*(\cdot,1))(z)$ are given by

$$\mathcal{M}v(z) = \frac{\cos((\pi-\theta)z)}{\sin(\pi z)} \quad \text{and} \quad \mathcal{M}v^*(z) = -\frac{\cos((\pi-\theta)z)}{\sin(\pi z)}. \quad (4.27)$$

Recall now R and α from (4.6). We have

$$R = \sqrt{\frac{k^2\cos^2(\phi+\theta) + \sin^2(\phi+\theta)}{k^2\cos^2\phi + \sin^2\phi}}. \quad (4.28)$$

Straightforward manipulations of (4.6) give

$$\sin\alpha = \frac{k\sin\theta}{\sqrt{k^2\cos^2(\phi+\theta)+\sin^2(\phi+\theta)}\cdot\sqrt{k^2\cos^2\phi+\sin^2\phi}} \tag{4.29}$$

and

$$\cos\alpha = \frac{k^2\cos\phi\cos(\phi+\theta)+\sin\phi\sin(\phi+\theta)}{\sqrt{k^2\cos^2(\phi+\theta)+\sin^2(\phi+\theta)}\cdot\sqrt{k^2\cos^2\phi+\sin^2\phi}}. \tag{4.30}$$

From (4.10), (4.23), (4.29)–(4.30), and elementary algebraic manipulations it follows that the kernel of the operator U_k and its Mellin transform $\mathcal{M}u_k(z) := \mathcal{M}(u_k(\cdot,1))(z)$ are, respectively, given by

$$\begin{aligned} u_k(s,1) &= -\frac{R}{\pi}\cdot\frac{s\sin(\pi-\alpha)}{s^2+2sR\cos(\pi-\alpha)+R^2}, \quad \text{and} \\ \mathcal{M}u_k(z) &= -R^z\frac{\sin((\pi-\alpha)z)}{\sin(\pi z)}. \end{aligned} \tag{4.31}$$

Similarly,

$$\begin{aligned} u_k^*(s,1) &= -\frac{1}{R\pi}\cdot\frac{s\sin(\pi-\alpha)}{s^2+2sR^{-1}\cos(\pi-\alpha)+R^{-2}}, \quad \text{and} \\ \mathcal{M}u_k^*(z) &= -R^{-z}\frac{\sin((\pi-\alpha)z)}{\sin(\pi z)}. \end{aligned} \tag{4.32}$$

In a similar fashion we obtain the Mellin transforms of the kernels of the operators V_k and V_k^*. We have

$$\begin{aligned} v_k(s,1) &= \frac{R}{\pi}\cdot\frac{s\cos(\pi-\alpha)+R}{s^2+2sR\cos(\pi-\alpha)+R^2}, \\ v_k^*(s,1) &= \frac{-1}{R\pi}\cdot\frac{s\cos(\pi-\alpha)+R^{-1}}{s^2+2sR^{-1}\cos(\pi-\alpha)+R^{-2}}. \end{aligned} \tag{4.33}$$

Using (4.23) in (4.33) we obtain the following formulas for $\mathcal{M}v_k(z) := \mathcal{M}(v_k(\cdot,1))(z)$ and $\mathcal{M}v_k^*(z) := \mathcal{M}(v_k^*(\cdot,1))(z)$, respectively:

$$\mathcal{M}v_k(z) = R^z\frac{\cos((\pi-\alpha)z)}{\sin(\pi z)} \quad \text{and} \quad \mathcal{M}v_k^*(z) = -R^{-z}\frac{\cos((\pi-\alpha)z)}{\sin(\pi z)}. \tag{4.34}$$

Finally, the conclusion of Proposition 4.1 follows from (3.14) in Lemma 3.1, (4.17), and the formulas for $\mathcal{M}u(z)$, $\mathcal{M}u_k(z)$, $\mathcal{M}u_k^*(z)$, $\mathcal{M}v(z)$, $\mathcal{M}v_k(z)$, and $\mathcal{M}v_k^*(z)$ given in (4.24), (4.31), (4.32), (4.27), and, respectively, (4.34). $\square$

Introduce

$$M(z) := -2k\,ab\,C(z)C(-z) + B(z)B(-z) + D(z)D(-z),$$

$$N(z) := \left(D(z)B(z) + k\,ab\,C(z)^2\right)\left(D(-z)B(-z) + k\,ab\,C(-z)^2\right). \quad (4.35)$$

In this notation by a straightforward computation, using (4.4) from Proposition 4.1, we obtain

$$\det(wI - \mathcal{M}\tilde{k})(z) = w^4 - M(z)w^2 + N(z). \quad (4.36)$$

Now we are going to use Theorem 2.4 to conclude

Theorem 4.2. *Let Ω be as in* (4.2)*. Then the following holds.*

$$\sigma(K_{\tilde{A}}; (L^p(\partial\Omega))^2) = \Big\{ w \in \mathbb{C}\,;\, \det(wI - \mathcal{M}\tilde{k})(z) = 0, z = \frac{1}{p} + iy \text{ for some } y \in \mathbb{R} \Big\} \cup \{0\}. \quad (4.37)$$

In particular, using (4.36) *this gives*

$$\sigma(K_{\tilde{A}}; (L^p(\partial\Omega))^2) = \Big\{ w \in \mathbb{C}; w^4 - M(z)w^2 + N(z) = 0,\ z = \frac{1}{p} + iy \text{ for some } y \in \mathbb{R} \Big\} \cup \{0\}. \quad (4.38)$$

Note that as a consequence of Theorem 4.2 we have

$$\sigma\left(K_{\tilde{A}}; (L^p(\partial\Omega))^2\right) = \bigcup_{i=1}^{4} \Sigma_i(\theta,\phi,p) =: \Sigma(\theta,\phi,p), \quad (4.39)$$

where $\Sigma_i(\theta,\phi,p)$, $i = 1,\dots,4$, are the curves in the plane (parametrized by $y \in \mathbb{R}$) given by a specific choice of $+$ or $-$ below

$$\pm\frac{M(z) \pm \sqrt{M(z)^2 - 4N(z)}}{2}, \qquad z = 1/p + iy. \quad (4.40)$$

Finally, much as in [22], based on Theorem 4.2, we provide a characterization of the spectrum of the operator $K_{\tilde{A}}$ on L^p $(1 < p < \infty)$ spaces of the boundary of bounded curvilinear polygons. More specifically we have

Theorem 4.3. *Consider $\Omega \subseteq \mathbb{R}^2$ a bounded, simply connected curvilinear polygon with angles θ_i with orientations ϕ_i, $i = 1,\dots,n$, and let $p \in (1,\infty)$. For each $1 \le i \le n$ consider the curve $\Sigma(\theta_i,\phi_i,p)$, as in* (4.39)*, associated with the angle θ_i, the orientation ϕ_i and the integrability exponent p. Set $\Sigma\widehat{(\theta_i,\phi_i},p)$ for the closure of its interior. Then*

$$\sigma\left(K_{\tilde{A}}; (L^p(\partial\Omega))^2\right) = \Big(\bigcup_{1\le i\le n} \Sigma\widehat{(\theta_i,\phi_i},p)\Big) \bigcup \{\lambda_j\}_j, \quad (4.41)$$

where $\{\lambda_j\}_j \subseteq (-1,1]$ *consists of finitely many eigenvalues of the operator* $K_{\tilde{A}}$ *on* $(L^p(\partial\Omega))^2$*. For* $w \in \mathbb{C}$*,* $w \in \bigcup_{1\le i\le n} \Sigma(\theta_i,\phi_i,p)$*, the operator* $wI - K_{\tilde{A}}$ *is not Fredholm on the space* $(L^p(\partial\Omega))^2$*. For* $w \in \mathbb{C}$*,* $w \notin \bigcup_{1\le i\le n} \Sigma(\theta_i,\phi_i,p)$ *the operator* $wI - K_{\tilde{A}}$ *is Fredholm on* $(L^p(\partial\Omega))^2$*. Moreover, its index is given by*

$$\mathit{index}\left(wI - K_{\tilde{A}}; (L^p(\partial\Omega))^2\right) = \sum_{i=1}^{n} W(w, \Sigma(\theta_i,\phi_i,p)), \tag{4.42}$$

where $W(w,\Sigma(\theta_i,\phi_i,p))$ *stands for the sum of the winding numbers of the point* $w \notin \Sigma(\theta_i,\phi_i,p)$ *with respect to each one of the four closed curves* $\Sigma_k(\theta_i,\phi_i,p)$ *,* $k = 1,\dots,4$ *constituting* $\Sigma(\theta_i,\phi_i,p)$*.*

5. L^p COUNTEREXAMPLES FOR THE SPECTRAL-RADIUS CONJECTURE

In this section, for any given $1 < p < \infty$, we construct an elliptic systems whose coefficient tensor is symmetric and strictly positive definite (see (2.4)) and Ω a polygon in $\mathbb{R}^2$ such that the associated double-layer potential operator has spectral radius on the L^p space of the boundary strictly bigger than one. More specifically, in the light of Theorem 4.3, given $p \in (1,\infty)$ it suffices to find $\lambda > 0$ and $0 < k < 1$ as in (3.4) such that the operator $K_{\tilde{A}}$ satisfies

$$\rho\left(K_{\tilde{A}}; (L^p(\partial\Omega))^2\right) > 1, \tag{5.1}$$

when Ω is the domain consisting of the interior of an angle of aperture $\frac{\pi}{2}$ with orientation 0 (see (4.2)). Here $K_{\tilde{A}}$ is the double-layer potential given in (2.6) associated with the elliptic system $\mathcal{L}_{m,n}$ with m and n as in (4.1) and the coefficient tensor $\tilde{A}$ as in (3.9). Recall that $\tilde{A}$ is *symmetric* and *strictly positive definite* (see Lemma 3.2).

Throughout the rest of this section Ω is the first quadrant, i.e., as in (4.2) with $\theta = \frac{\pi}{2}$ and $\phi = 0$. Recall α and R from (4.6). It is immediate that now we have

$$R = \frac{1}{k} \qquad \text{and} \qquad \alpha = \frac{\pi}{2}. \tag{5.2}$$

With an eye toward employing Theorem 4.2 in the search for λ and k such that (5.1) holds, we note the following identities:

$$D(1/p) = \frac{\sin(\frac{\pi}{2p})[a + kbk^{-\frac{2}{p}}]}{(a+bk)\sin(\frac{\pi}{p})} = \frac{B(-1/p)}{k^{\frac{2}{p}}},$$

$$B(1/p) = \frac{\sin(\frac{\pi}{2p})[kb + ak^{-\frac{2}{p}}]}{(a+bk)\sin(\frac{\pi}{p})} = \frac{D(-1/p)}{k^{\frac{2}{p}}}, \tag{5.3}$$

$$C(1/p) = \frac{\cos(\frac{\pi}{2p})[1 - k^{-\frac{2}{p}}]}{(a+bk)\sin(\frac{\pi}{p})} = \frac{C(-1/p)}{k^{\frac{2}{p}}}.$$

In (5.3) the functions D, B, and C are as introduced in (4.5). Recall now M and N from (4.35). According to (4.39) and (4.40) we infer that

$$\rho(K_{\tilde{A}}; (L^p(\partial\Omega))^2) \geq \mathcal{W}(1/p) := \left|\frac{M(1/p) + \sqrt{M^2(1/p) - 4N(1/p)}}{2}\right|. \tag{5.4}$$

Based on (4.35), (5.3), and straightforward calculations we have

$$M^2(\frac{1}{p}) - 4N(\frac{1}{p}) = -\frac{4k^{\frac{2}{p}}k\,ab}{(a+bk)^4\sin^2(\frac{\pi}{p})}[1-k^{-\frac{2}{p}}]^2\Big[a^2 + k^2b^2 + akb(k^{\frac{2}{p}} + k^{-\frac{2}{p}})\Big]. \tag{5.5}$$

Recall a and b from (3.6). Since $\lambda, k > 0$ and $k < 1$ are as in (3.4), we have $ab < 0$. Therefore, (5.5) gives

$$M^2(\frac{1}{p}) - 4N(\frac{1}{p}) \geq 0 \quad \Longleftrightarrow \quad E(\lambda,k,p) := a^2 + k^2b^2 + akb\left(k^{\frac{2}{p}} + k^{-\frac{2}{p}}\right) \geq 0. \tag{5.6}$$

Now, (4.35) gives that

$$M(1/p) = 2k^{\frac{2}{p}}\left(B(1/p)D(1/p) - k\,ab\,C^2(1/p)\right), \tag{5.7}$$

and (5.5) further implies

$$\mathcal{W}(1/p) = \frac{1}{(a+bk)^2}\Big[\frac{E(\lambda,k,p)}{4\cos^2(\frac{\pi}{2p})} - k\,ab\frac{(k^{-\frac{2}{p}} - k^{\frac{2}{p}})^2}{4\sin^2(\frac{\pi}{2p})} + \sqrt{-k\,abE(\lambda,k,p)}\frac{k^{-\frac{2}{p}} - k^{\frac{2}{p}}}{\sin(\frac{\pi}{p})}\Big], \tag{5.8}$$

whenever $E(\lambda,k,p) \geq 0$, where $\mathcal{W}(1/p)$ has been introduced in (5.4).

We now record the following result useful in the sequel.

Proposition 5.1. *For any $\lambda > 0$ and $0 < k < 1$ satisfying (3.4) we have*

$$\frac{\partial E}{\partial p}(\lambda,k,p) = \frac{2akb\ln k}{p^2}\left(k^{-\frac{2}{p}} - k^{\frac{2}{p}}\right) > 0. \tag{5.9}$$

In particular, $E(\lambda,k,p)$ is increasing in p. Also, for any $p\in(1,\infty)$ and $k\in(0,1)$ we have

$$\frac{k^{-\frac{1}{p}}-k^{\frac{1}{p}}}{\sin(\frac{\pi}{2p})}\geq\frac{-4\ln k}{\pi}\quad \text{and}\quad \frac{k^{-\frac{1}{p}}-k^{\frac{1}{p}}}{\sin(\frac{\pi}{p})}\geq\frac{-2\ln k}{\pi}. \tag{5.10}$$

Proof. The inequality (5.9) follows immediately by differentiating with respect to p in (5.6) since $ab<0$, $\ln k<0$, and $k^{-\frac{2}{p}}-k^{\frac{2}{p}}>0$ for λ and k as in the statement. Next, with an eye toward proving (5.10) consider the functions $F_\gamma, G:\big(0,\frac{\pi}{2}\big)\longrightarrow\mathbb{R}$ given by

$$F_\gamma(x):=\frac{\gamma^{-x}-\gamma^{x}}{\sin x}\quad\text{and}\quad G(x):=\ln(\frac{1}{\gamma})(\gamma^{-x}+\gamma^{x})\tan x-(\gamma^{-x}-\gamma^{x}), \tag{5.11}$$

for $\gamma\in(0,1)$ fixed. Note that

$$G'(x)=\ln^2(\frac{1}{\gamma})(\gamma^{-x}-\gamma^{x})\tan x+\ln(\frac{1}{\gamma})(\gamma^{-x}+\gamma^{x})(\frac{1}{\cos^2 x}-1)\geq 0, \tag{5.12}$$

since $\gamma\in(0,1)$. Since $G(0)=0$, this entails $G(x)\geq 0$ on $\big(0,\frac{\pi}{2}\big)$. Straightforward calculations give $F_\gamma'(x)=\cos x\frac{G(x)}{\sin^2 x}$, and $\lim_{x\to 0}F_\gamma(x)=-2\ln\gamma$, and hence

$$F_\gamma(x)\geq -2\ln\gamma\quad\text{for}\quad x\in(0,\frac{\pi}{2}). \tag{5.13}$$

Note now that (5.10) follows from (5.13) for the particular choice $\gamma=k^{\frac{2}{\pi}}$ and $x=\frac{\pi}{2p}$, and $\gamma=k^{\frac{1}{\pi}}$ and $x=\frac{\pi}{p}$, respectively. $\square$

Now we are ready to present

Theorem 5.2. *Let Ω be the domain consisting of the interior of the first quadrant. Then for any $1<p<\infty$ there exist $\lambda>0$ and $k\in(0,1]$ satisfying (3.4) such that $\tilde{A}$ is symmetric and strictly positive definite and*

$$\rho\left(K_{\tilde{A}};(L^p(\partial\Omega))^2\right)>1. \tag{5.14}$$

Moreover, when $p\in[4,\infty)$ one can choose $\lambda=240$ and $k=.4$, while $\lambda=10.1$ and $k=0.1$ will give (5.14) for $p\in(1,4)$. The latter case is established by a computer-aided proof.

Proof. Let $k:=.4$ and $\lambda:=240$. According to (5.9) in Lemma 5.1, for any $p\in[4,\infty)$ we have $E(\lambda,k,p)\geq E(\lambda,k,4)>0$, where E has been introduced in (5.6) (actually a direct calculation shows $E(\lambda,k,4)\geq 15900$). Therefore,

$$\frac{E(\lambda,k,p)}{4(a+bk)^2\cos^2(\frac{\pi}{2p})}\geq\frac{E(\lambda,k,4)}{4(a+bk)^2}\geq 0.1900, \tag{5.15}$$

where a and b are as in (3.6). Going further, using (5.10) and $ab<0$ we conclude that for any $p\in[3,\infty)$ we have

$$-\frac{k\,ab}{4(a+bk)^2}\frac{(k^{-\frac{1}{p}}-k^{\frac{1}{p}})^2}{\sin^2(\frac{\pi}{2p})}\geq-\frac{k\,ab}{4(a+bk)^2}(\frac{4\ln k}{\pi})^2\geq 0.37. \tag{5.16}$$

Appealing again to (5.10) we get

$$\frac{\sqrt{-k\,abE(\lambda,k,p)}}{(a+bk)^2}\cdot\frac{k^{-\frac{2}{p}}-k^{\frac{2}{p}}}{\sin(\frac{\pi}{p})}\geq\frac{\sqrt{-k\,abE(\lambda,k,4)}}{(a+bk)^2}\left(\frac{-2\ln k}{\pi}\right)\geq 0.53. \tag{5.17}$$

Finally (5.8) and (5.15)–(5.17) give that $\mathcal{W}(1/p)\geq 1.08$, and the conclusion of Theorem 5.2 follows from (5.4) for $p\in[4,\infty)$. The remaining case $p\in(1,4)$ is treated in the next section. $\square$

6. Validated numerics for the case $1<p<4$

For the remaining case $p\in(1,4)$, the choice $z=1/p$ no longer guarantees that its associated point $w\in\sigma(K_{\tilde{A}};(L^p(\partial\Omega))^2)$ is such that $|w|>1$. Instead, we must consider values of $z=\frac{1}{p}+iy$ with $y>0$. In this situation, the corresponding w has a nontrivial imaginary part, which makes the previous type of analytic approach much harder to handle. Nevertheless, extensive numerical experiments seem to suggest that Theorem 5.2 is valid for $p\in(1,4)$. In what follows, we will present a technique that allows us to verify the correctness of the numerical computations.

6.1. **Interval analysis.** Let $\mathbb{IR}$ denote the set of all closed intervals of the real line. For any element $[a]\in\mathbb{IR}$, we adopt the notation $[a]=[\underline{a},\bar{a}]$, and we allow for degenerate (*thin*) intervals $[a]$ with $\underline{a}=\bar{a}$. We define binary arithmetic operations on elements of $\mathbb{IR}$ in the following set-theoretic manner:

Definition 6.1. If $\odot$ is one of the operators $+$, $-$, $\times$, and $\div$, we define arithmetic operations on elements of $\mathbb{IR}$ by $[a]\odot[b]=\{a\odot b\colon a\in[a],b\in[b]\}$, with the exception that $[a]\div[b]$ is undefined if $0\in[b]$.

Working exclusively with closed intervals, it is possible to describe the resulting interval in terms of the endpoints of the operands:

$$[a]+[b]=[\underline{a}+\underline{b},\bar{a}+\bar{b}],\quad [a]-[b]=[\underline{a}-\bar{b},\bar{a}-\underline{b}],$$

$$[a] \times [b] = [\min\{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}],$$
$$[a] \div [b] = [a] \times [1/\bar{b}, 1/\underline{b}], \quad \text{if } 0 \notin [b].$$

The resulting arithmetic is called *interval arithmetic*, and constitutes the core of validated numerics; see [26], [27], [17], and [1].

Using interval arithmetic, we can easily construct interval versions of rational functions by replacing all occurrences of the real variable x with the interval variable $[x]$, and all operations by interval operations. It is also possible to construct interval versions of the standard functions: $\sin x$, $\cos x$, e^x, $\log x$, x^n, $|x|$, etc. As an example, we have $e^{[x]} = [e^{\underline{x}}, e^{\bar{x}}]$ for all $[x] \in \mathbb{IR}$. Thus, any elementary function, made up by a finite combination of standard functions, arithmetic operations, and the composition operator, can also be extended to the interval realm.

Definition 6.2. Given a fixed representation of an elementary function $f\colon D \to \mathbb{R}$, the interval-valued function $F\colon D \cap \mathbb{IR} \to \mathbb{IR}$, formed by replacing all standard functions, operations, and variables with their interval versions, is called the *natural interval extension* of f.

One of the main reasons for extending functions to the interval domain lies in the key feature of interval arithmetic, namely that of *inclusion monotonicity*:

Theorem 6.3. *If $[a] \subseteq [a']$, $[b] \subseteq [b']$, and $\odot \in \{+, -, \times, \div\}$, then $[a] \odot [b] \subseteq [a'] \odot [b']$, where we demand that $0 \notin [b']$ for division.*

This theorem follows directly from Definition 6.1. More generally, it is possible to ensure that any elementary interval function F is inclusion monotonic; i.e., $[x] \subseteq [x'] \Rightarrow F([x]) \subseteq F([x'])$. We will consider only such interval functions in what follows.

Interval analysis provides us with a powerful means of enclosing the range of an elementary function: $R(f; D) = \{f(x)\colon x \in D\}$. This is known as *the fundamental theorem of interval analysis;* see [26] and [27].

Theorem 6.4. *If F is a natural interval extension of an elementary function f, then the range of f over $[x]$ satisfies $R(f; [x]) \subseteq F([x])$.*

Note that, by $F([x])$, we mean the interval resulting from evaluating F in interval arithmetic. This set is trivial to compute, whereas $R(f; [x])$ is generally very hard to obtain. Of course, the set $F([x])$ may very well overestimate the actual range $R(f; [x])$, but as we shall see, this overestimation can often be controlled.

By the the inclusion monotonicity of F, it follows that by splitting the domain $[x]$ into smaller pieces $[x^{(1)}], \ldots, [x^{(k)}]$, we have

$$\begin{aligned} R(f;[x]) &= R(f; \bigcup_{i=1}^{k}[x^{(i)}]) = \bigcup_{i=1}^{k} R(f;[x^{(i)}]) \subseteq \bigcup_{i=1}^{k} F([x^{(i)}]) \\ &\subseteq F(\bigcup_{i=1}^{k}[x^{(i)}]) = F([x]). \end{aligned}$$

It turns out that by subdividing $[x]$ into many small pieces, we can approximate $R(f;[x])$ to any desired accuracy, provided that F is Lipschitz; i.e., there is a $K > 0$ such that for all $[x]$, we have $W(F([x])) \leq KW([x])$. Here we are using $W([x])$ to denote the *width* of an interval; i.e., $W([x]) = \bar{x} - \underline{x}$. We make this statement precise in the following theorem due to Moore; see [25] and [29].

Theorem 6.5. *Consider a Lipschitz elementary function $f\colon D \to \mathbb{R}$, and let $F\colon D \cap \mathbb{IR} \to \mathbb{IR}$ be an inclusion monotonic, Lipschitz, natural interval extension of f. Given an interval $[x] \in D \cap \mathbb{IR}$, there exists a positive real number K, depending on F and $[x]$, such that if $[x] = \cup_{i=1}^{k}[x^{(i)}]$, then*

$$\begin{aligned} & R(f;[x]) \subseteq \bigcup_{i=1}^{k} F([x^{(i)}]) \quad \textit{and} \qquad (6.1) \\ & W\Big(\bigcup_{i=1}^{k} F([x^{(i)}])\Big) \leq W(R(f;[x])) + K \max_{i=1,\ldots,k} W([x^{(i)}]). \end{aligned}$$

Real interval arithmetic has natural generalizations to higher dimensions ($\mathbb{IR}^n$) as well as to the complex setting ($\mathbb{IC}^n$); see [26], [27], [17], [1], and [13]. Also, for computer applications, we remark that, if the basic interval arithmetic operations are implemented using *outward rounding*, then the computed enclosures are mathematically guaranteed, despite rounding errors due to the finite precision of the computer arithmetic; see e.g. [31], [12], [17], and [13].

6.2. **Domain decomposition.** With this introductory material covered, let us proceed toward our main goal. Consider an elementary analytic function $g\colon [z] \to \mathbb{C}$, where $[z]$ denotes a closed rectangle $[z] = [x] + i[y]$ in the complex plane. One of our goals is to find a subset of $[z]$ on which $|g(z)|$ is strictly greater than one. In other words, we want to discard those parts of $[z]$ where $|g(z)| \leq 1$. For the actual problem at hand, we have $[z] = [\frac{1}{4}, 1] + [0, 1]i$, and g is the positive branch of (4.40); i.e.,

$$g(z) = \frac{M(z) + \sqrt{M(z)^2 - 4N(z)}}{2}, \tag{6.2}$$

with $\theta = \frac{\pi}{2}$, $\phi = 0$, $\lambda = 10.1$, and $k = 0.1$. To finish the proof of Theorem 5.2 for $p \in (1, 4)$, it suffices to show the following holds:

$$\forall\, x \in [\tfrac{1}{4}, 1],\ \exists\, y \in [0, 1] \ \text{ such that } \ |g(z)| > 1, \ \text{ where } \ z = x + iy. \tag{6.3}$$

This goal can be achieved by adaptively decomposing $[z]$ into smaller rectangles $[z_1], \ldots, [z_n]$, all of which are treated according to four criteria depending on $[r_i] = F([z_i])$, where $F\colon \mathbb{IC} \to \mathbb{IR}$ is the complex interval extension of $f(z) = |g(z)|$. Recall that the interval $[r_i]$ is guaranteed to contain the range $R(|g|; [z_i])$. In essence, we will

- further decompose $[z_i]$ if $1 \in [r_i]$ and $w([z_i]) > \texttt{TOL}$;
- store $[z_i]$ in `unverifiedList` if $1 \in [r_i]$ and $w([z_i]) \leq \texttt{TOL}$;
- store $[z_i]$ in `verifiedList` if $1 < \underline{r}_i$;
- store $[z_i]$ in `removedList` if $\bar{r}_i < 1$.

Here, `TOL` is a predefined tolerance, which determines the size of the elements of `unverifiedList`. The decomposition of a $[z_i]$ is carried out by bisecting the wider of the real and imaginary part of $[z_i]$. In case of a tie, the real part is bisected. Note that a positive value of `TOL` guarantees that the algorithm terminates within a finite time. At this point, we can use any elements of `verifiedList` in our attempt to prove (6.3).

In our proof, we succeed to prove the following: on termination, the algorithm produces a collection of complex rectangles $[z_1], \ldots, [z_n]$, all belonging to `verifiedList`, whose union contains the two horizontal strips $s_1 = \{x + iy\colon x \in [\frac{1}{4}, \frac{13}{20}], \quad y \in [\frac{9}{100}, \frac{11}{100}]\}$ and $s_2 = \{x + iy\colon x \in [\frac{1}{2}, 1], \quad y \in [\frac{29}{100}, \frac{31}{100}]\}$. The crucial point is that the projection of $s_1 \cup s_2$ onto the real line completely covers the interval $[\frac{1}{4}, 1]$. This clearly proves (6.3).

The computer program implementing the outlined algorithm was programmed in the `C-XSC` language described in [13]. This is a `C++`-based language with many additional features such as real and complex interval arithmetic. Below, we have included a verbatim copy of the implementation of the main algorithm. Here `CRECT` is a complex rectangle, the functions `Inf` and `Sup` return the lower and upper bound of an interval, respectively, and the function `Func` denotes our interval function F. The remaining details of the code should be more or less self-explanatory. For the problem under consideration, the variable `level` was set to 1.

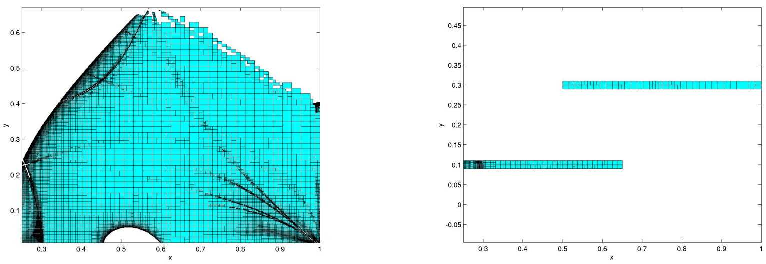

FIGURE 1. (a) The elements of `verifiedList` with `TOL` $= 2^{-10}$. (b) The strips s_1 and s_2.

```
//////////////////////////////////////////

void greaterThanLevel(List<CRECT> &verifiedList,
                      List<CRECT> &unverifiedList,
                      List<CRECT> &removedList,
                      const CRECT &domain,
                      const real  &level,
                      const real  &TOL)
{
  List<CRECT> refineList;
  refineList += domain;
  while( !IsEmpty(refineList) ) {
    CRECT    localX = First(refineList);
    interval localY = Func(localX);
    RemoveCurrent(refineList);
    if ( (Inf(localY) <= level) && (level <= Sup(localY)) )
      if ( max(diam(Re(localX)), diam(Im(localX))) > TOL )
        splitAndStore(localX, refineList);
      else
        unverifiedList += localX;
    else if ( level < Inf(localY) )
      verifiedList += localX;
    else
      removedList += localX;
  }
}

//////////////////////////////////////////
```

The entire program is available from `http://www.math.uu.se/~warwick/main/papers.html`. This finishes the proof of Theorem 5.2. □

7. A counterexample on the space of continuous functions

In this section we produce a counterexample for the spectral radius conjecture on the space of continuous functions. To this end, recall the second-order elliptic operator $\mathcal{L}_{m,n}$ from (3.8) with $m = |\frac{k}{\lambda-k^2}|$ and $n = |\frac{k}{\lambda-1}|$, where $\lambda > 0$ and $k \in (0,1]$ satisfy (3.4). Note that $\mathcal{L}_{m,n}\vec{u}$ admits the writing $\tilde{a}_{ij}^{\alpha\beta}\partial_i\partial_j u^\beta$, $\alpha,\beta,i,j = 1,2$, where the coefficient tensor $\tilde{A} = (\tilde{a}_{ij}^{\alpha\beta})_{\alpha,\beta,i,j}$ given in (3.9) is symmetric and strictly positive definite (see Lemma 3.2). The double-layer potential operator $K_{\tilde{A}}$ is as introduced in (2.6) with $A = \tilde{A}$. In this notation our main result is

Theorem 7.1. *Let Ω be the square with vertices $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$. Then*

$$K_{\tilde{A}} : (\mathcal{C}(\partial\Omega))^2 \longrightarrow (\mathcal{C}(\partial\Omega))^2 \quad \textit{boundedly}, \tag{7.1}$$

and if $k := .4$ and $\lambda := 240$ we have that $\tilde{A}$ is symmetric and strictly positive definite and

$$\rho\left(K_{\tilde{A}}; (\mathcal{C}(\partial\Omega))^2\right) > 1. \tag{7.2}$$

Before proceeding with the proof of Theorem 7.1 let us record the following result from [34] and [18] important in the sequel. Let Ω be a bounded, Lipschitz domain in $\mathbb{R}^2$, and consider the operator

$$K\vec{f}(P) := \int_{\partial\Omega} G\left(\frac{P-Q}{|P-Q|}\right)\frac{\langle P-Q, N(Q)\rangle}{|P-Q|^2}\vec{f}(Q)\,d\sigma(Q), \tag{7.3}$$

where $N(Q)$ is the outward unit-normal vector which exists almost everywhere at $Q \in \partial\Omega$. In (7.3), for any $P \in \mathbb{R}^2$, $G(P)$ is a 2×2 matrix whose elements are continuous, even functions on the unit sphere $S^1 \subseteq \mathbb{R}^2$. Then the following holds:

Theorem 7.2. *Let $\Omega \subseteq \mathbb{R}^2$ be a polygon and K be as in (7.3) such that the matrix $\int_{S^1} G(Q)\,d\sigma(Q)$ is nonsingular. Then*

$$K : (\mathcal{C}(\partial\Omega))^2 \longrightarrow (\mathcal{C}(\partial\Omega))^2 \quad \textit{boundedly}. \tag{7.4}$$

Furthermore, the operator $wI - K$, $w \in \mathbb{C}$, is Fredholm on $(\mathcal{C}(\partial\Omega))^2$ if and only if for any vertex V of $\partial\Omega$ the following condition is satisfied:

$$\det(wI - \mathcal{M}k)(iy) \neq 0, \quad \forall\, y \in \mathbb{R}, \tag{7.5}$$

where $k(P,Q) := G(\frac{P-Q}{|P-Q|})\frac{\langle P-Q,N(Q)\rangle}{|P-Q|^2}$, *with* P *and* Q *belonging to the boundary of the infinite sector with vertex at* V *and with sides along those of the polygon* Ω *at* V.

We turn now to the proof of Theorem 7.1.

Proof. Recall (3.14) from the proof of Lemma 3.1. In the light of this identity matters can be reduced to showing that the double-layer potential operator K_A from (2.6) with the coefficient tensor A given in (3.7) has the form (7.3) and that the condition (7.5) is violated at the vertex $(0,0)$ of Ω. The kernel k of the operator K_A has been computed explicitly in (3.25). A straightforward computation gives that k can be written in the form

$$k^{11}(x,y) = c_1\frac{\lambda y^2+k^2x^2}{k^2x^2+y^2}\frac{\langle(x,y),N\rangle}{x^2+y^2}, \quad k^{12}(x,y) = c_2\frac{xy}{k^2x^2+y^2}\frac{\langle(x,y),N\rangle}{x^2+y^2}, \tag{7.6}$$

$$k^{21}(x,y) = c_3\frac{xy}{k^2x^2+y^2}\frac{\langle(x,y),N\rangle}{x^2+y^2}, \quad k^{22}(x,y) = c_4\frac{y^2+\lambda x^2}{k^2x^2+y^2}\frac{\langle(x,y),N\rangle}{x^2+y^2},$$

where

$$c_1 = \frac{k+1}{\pi(\lambda+k)},\ c_2 = \frac{ak(k+1)}{\pi(\lambda+k)},\ c_3 = \frac{-b(k+1)}{\pi(\lambda+k)},\ c_4 = \frac{k(k+1)}{\pi(\lambda+k)}. \tag{7.7}$$

In turn, this and (3.25) imply the representation of K_A in the form (7.3) with

$$G(s,t) = \begin{pmatrix} c_1\frac{\lambda t^2+k^2s^2}{k^2t^2+s^2} & c_2\frac{st}{k^2t^2+s^2} \\ c_3\frac{st}{k^2t^2+s^2} & c_4\frac{t^2+\lambda s^2}{k^2t^2+s^2} \end{pmatrix}. \tag{7.8}$$

A tedious direct calculations that we omit shows that $\int_{S^1} G(Q)\,d\sigma(Q)$ is nonsingular. Then, according to Theorem 7.2, the mapping property (7.1) from Theorem 7.1 follows.

Next, much as in Section 4 we have that the condition (7.5) at the vertex $(0,0)$ of Ω is equivalent to

$$w^4 - M(iy)w^2 + N(iy) \neq 0, \qquad \forall\, y \in \mathbb{R}, \tag{7.9}$$

where $M(z)$ and $N(z)$ are as in (4.35) with $\theta = \frac{\pi}{2}$, $\phi = 0$, $\lambda = 240$, and $k = .4$. Therefore, according to Theorem 7.2, we have

$$w_0 \in \mathbb{C} \text{ such that } w_0^4 - M(0)w_0^2 + N(0) = 0 \implies w_0 \notin \sigma(K_{\tilde{A}}; (\mathcal{C}(\partial\Omega))^2). \tag{7.10}$$

Passing to the limit as $z \to 0$ in (4.5), we obtain

$$B(0) = D(0) = \frac{1}{2} \quad \text{and} \quad C(0) = \frac{2\ln k}{\pi(a+kb)}, \tag{7.11}$$

where a and b are as in (3.6). Using (4.35) we conclude

$$M(0) = \frac{1}{2} - 2k\,ab\frac{4(\ln k)^2}{\pi^2(a+kb)^2}, \quad M^2(0) - 4N(0) = -4k\,ab\,C^2(0). \qquad (7.12)$$

Finally,

$$\omega_0 = \frac{M(0) + \sqrt{M^2(0) - 4N(0)}}{2} = \frac{\frac{1}{2} - \frac{8k\,ab|\ln k|^2}{\pi^2(a+kb)^2} + \frac{4|\ln k|\sqrt{-k\,ab}}{\pi(a+kb)}}{2} > 1, \qquad (7.13)$$

for $k = .4$ and $\lambda = 240$. Then $w_0 = \sqrt{\omega_0}$ satisfies $w_0^4 - M(0)w_0^2 + N(0) = 0$ and $w_0 > 1$. Consequently, (7.10) implies (7.2), and the proof of Theorem 7.1 is completed. □

REFERENCES

[1] G. Alefeld and J. Herzberger, "Introduction to Interval Computations," Academic Press, New York, 1983.

[2] D.W. Boyd, *Spectra of convolution operators*, Acta Scientiarum Mathematicarum, 35 (1973), 31–37.

[3] R.R. Coifman, A. McIntosh, and Y. Meyer, *L'integrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Annals of Math., 116 (1982), 361–387.

[4] W. Derrick, *Open problems in singular integral theory*, Journal of Integral Equations, 5 (1993), 23–28.

[5] L. Diomeda and B. Lisena, *The Dirichlet problem for elliptic systems in piecewise $\mathcal{C}^1$ plane domains*, Indiana University Mathematics Journal, 41 (1992), 649–670.

[6] L. Escauriaza, E.B. Fabes, and G. Verchota, *On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries*, Proc. Amer. Math. Soc., 115 (1992), 1069–1076.

[7] L. Escauriaza and M. Mitrea, *Transmission problems and spectral theory for singular integral operators on Lipschitz domains*, preprint.

[8] J. Elschner, *The double layer potential operator over polyhedral domains I: Solvability in weighted Sobolev spaces*, Applicable Analysis, 45 (1992), 117–134.

[9] J. Elschner, *Asymptotics of solutions to pseudodifferential equations of Mellin type*, Math. Nachr., 130 (1987), 267–305.

[10] E.B. Fabes, M. Jodeit, and J.E. Lewis, *On the spectra of a Hardy kernel*, Journal of Functional Analysis, 21 (1976), 187–194.

[11] E.B. Fabes, M. Sand, and J.K. Seo, *The spectral radius of the classical layer potential on convex domains*, IMA Volumes in Mathematics and its Applications, 42 (1992), 129–138.

[12] D. Goldberg, *What every computer scientist should know about floating-point arithmetic*, Computing Surveys, 23 (1991), No. 1, 5–48.

[13] R. Hammer, M. Hocks, U. Kulisch, and D. Ratz, "C++ Toolbox for Verified Computing," Springer-Verlag, New York, 1995.

[14] S. Karlin, *Positive operators*, J. Math. Mech., 8 (1959), 907–937.

[15] C.E. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, No. 83, AMS, Providence, RI, 1994.

[16] V.A. Kozlov, V.G. Maz'ya, and J. Rossmann, *Spectral problems associated with corner singularities of solutions to elliptic equations,* Mathematical Surveys and Monographs, 85, American Mathematical Society, Providence, RI, 2001.

[17] U.W. Kulisch and W.L. Miranker, "Computer Arithmetic in Theory and Practice," Academic Press, 1981.

[18] G.I. Kresin and V.G. Maz'ya, *The norm and the essential norm of the double layer elastic and hydrodynamic potentials in the space of continuous functions*, Math. Methods Appl. Sci., 18 (1995), 1095–1131.

[19] J.E. Lewis, *A symbolic calculus for layer potentials on $\mathcal{C}^1$ curves and $\mathcal{C}^1$ curvilinear polygons*, Proc. Amer. Math. Soc., 112 (1991), 419–427.

[20] J.E. Lewis and C. Parenti, *Pseudodifferential operators of Mellin type*, Comm. in PDE, 8 (1983), 477–544.

[21] V.G. Maz'ya and S.V. Poborchi, "Differentiable Functions on Bad Domains," World Scientific Publishing Co., Inc., River Edge, NJ, 1997.

[22] I. Mitrea, *On the spectra of elastostatic and hydrostatic layer potentials on curvilinear polygons*, J. Fourier Anal. Appl., 8 (2002), 443–487.

[23] I. Mitrea, *On the traction problem for the Lamé system on curvilinear polygons*, submitted.

[24] I. Mitrea, "Spectral Properties of Elliptic Layer Potentials on Non-Smooth Domains," Ph.D. Thesis, University of Minnesota, 2000.

[25] R.E. Moore, *The Automatic Analysis and Control of Error in Digital Computations Based on the Use of Interval Numbers*, in "Error in Digital Computing," Vol. 1, John Wiley & Sons Inc., New York, 1965.

[26] R.E. Moore, "Interval Analysis," Prentice-Hall, Englewood Cliffs, New Jersey, 1966.

[27] R.E. Moore, "Methods and Applications of Interval Analysis," SIAM Studies in Applied Mathematics, Philadelphia, 1979.

[28] C. Morrey, "Second Order Elliptic Systems of Differential Equations," Annals of Math. Studies, No. 33, 1954.

[29] A. Neumaier, "Introduction to Numerical Analysis," Cambridge University Press, Cambridge, 2001.

[30] F. Oberhettinger, "Tables of Mellin Transforms," Springer-Verlag, New York-Heidelberg, 1974.

[31] M.L. Overton, "Numerical Computing with IEEE Floating Point Arithmetic," SIAM, Philadelphia, 2001.

[32] A. Rathsfeld, *The invertibility of the double layer potential operator in the space of continuous functions defined on a polyhedron: The panel method*, Appl. Anal., 45 (1992), 135–177.

[33] W. Rudin, "Functional Analysis," second edition, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[34] V.Y. Shelepov, *On the index and spectrum of integral operators of potential type along Radon curves*, Math. USSR Sbornik, 70 (1991), 175–203.

[35] G.C. Verchota and A.L. Vogel, *Nonsymmetric systems on nonsmooth planar domains*, Trans. Amer. Math. Soc., 349 (1997), 4501–4535.